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Strong Result for Level Crossings of Random Polynomials

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Abstract: *Let N_n be the number of real roots of the algebraic equation*

$f_n(x)=\sum_{k=0}^{n}\xi_k x^v=0$ *where* $\xi_k x^v$ *are independent random variables assuming real values only. Then there exists an integer n_0 such that for each $n>n_0$ the number of real roots of most of the equations f(x)=0 is at least en log n except for a set of measure at most* $\frac{\mu}{(\in_{n0}\log n_0)}$.

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I. Theorem

Let $f_n(x,w)$ be a polynomial of degree n whose coefficients are independent random variables with a common characteristics function exp $\left(-C|t|^{\alpha}\right)$, where α=1 and C is a positive constant. Take, $\{e_n\}$to be any sequence tending to zero such that e_n log n tends to infinity as n tends to infinity. Then there exists an integer n_0 such that for each $n>n_0$ the number of real roots of most of the equations f(x)=0 is at least en log n except for a set of measure at most $\frac{\mu}{(\in_{n0}\log n_0)}$.

II. Introduction

Let N_n be the number of real roots of the algebraic equation

$$f_n(x)=\sum_{k=0}^{n}\xi_k x^v=0$$

where $\xi_k x^v$ are independent random variables assuming real values only. Several authors have estimated bounds for N_n when the random variables satisfy different distribution laws. Littlewood and Offord [2] made the first attempt in this direction. They considered the cases when the $\xi_k x^v$ are normally distributed or uniformly distributed in (-1, +1) or assume only the values +1 and –1 with equal probability. They obtained in each case that

$$P_r\left(N_n>\frac{\mu\log n}{\log\log\log n}\right)>1-\frac{A}{\log n}$$

Samal [3] has considered the general case when the $\xi_k x^v$ have identical distribution, with exception zero, variance and third absolute moment finite and non-zero. He has shown that $N_n>s_n\log n$ outside an exceptional set whose measure tends to zero as n tends to infinity, where s_n tends to zero, but s_n log n tends to infinity.

Samal and Mishra [4] have considered the case the $\xi_k x^v$ have a common characteristics function exp $\left(-C|t|^{\alpha}\right)$ where C is a positive constant and $\alpha\geq 1$. They have shown that

$$N_n>\frac{\mu\log n}{\log\log n}$$

outside an exceptional set measure at most

$$\left\{\begin{array}{l}\dfrac{\mu'}{(\log\log n)(\log n)^{\alpha-1}}, \\ \dfrac{\mu \log\log n}{\log n}\end{array}\right\} \text{if } 1 \le \alpha < 2, \text{if } \alpha = 2$$

In all the above cases the exceptional set depends upon n. Evans [1], was the first to obtain 'strong result' for these bounds. In such case the exceptional set is independent of the degree n of the polynomial. We use the term 'strong result' in the following sense:

All the above results are of the form

$$P_r\left(\frac{N_n}{\Delta_n} > \mu\right) \to 1 \text{ as in tends to infinity}$$

whereas the theorem of Evans is of the form

$$P_r\left(\sup_{n>n_0}\frac{N_n}{\Delta_n} > \mu\right) \to 1 \text{ as } n_0 \text{ tends to infinity.}$$

Evans [1] has shown, in case of normally distributed coefficients, that there exists an integer n_0 such that for $n>n_0$.

$$N_n > \frac{\mu \log n}{\log\log n}$$

except for a set of measure at most $\dfrac{\mu' \log\log n_0}{\log n_0}$

Samal and Mishra [5] have shown in the case of characteristic function exp $\left(-C|t|^{\alpha}\right)$ that for $n>n_0$

$$N_n > \frac{\mu \log n}{\log\log n}$$

outside an exceptional set of measure at most

$$\frac{\mu'}{\left\{\log\left(\dfrac{\log n_0}{\log\log n_0}\right)\right\}^{\alpha-1}}$$

where $\alpha>1$.

In [7], they have considered the 'strong result' in the general case. Assuming that the random variables (not necessarily identically distributed) have exception zero, variance and third absolute moment non-zero finite, they have shown that for $n \ge n_0$.

$$N_n > (\mu \log n)/\log\{(K_n/t_n)\log n\}$$

outside a set of measure at most

$$\frac{\mu'}{\left\{\log\left(\dfrac{\log n_0}{\log\left(\dfrac{K_{n0}}{t_{n0}}\log n_0\right)}\right)\right\}}$$

provided $\lim_{n\to\infty}\frac{P_n}{t_n}$ is finite and log $\left(\frac{K_{n0}}{t_{n0}}\log n_0\right)$ =0 (logn) where $K_n=\max_{0\le v\le n}\sigma_v, t_n=\max_{0\le v\le n}\sigma_v$ and $\max_{0\le v\le n}\tau_v\,\sigma^2{}_v, \tau^3{}_v$ being the variance and third absolute moment respectively of ξ_v.

Our object is to improve the 'strong result' for lower bound in case of characteristic function exp $\left(-C|t|^\alpha\right)$. We have shown that for $n>n_0$.

$N_n>\in_n \log n$

Outside an exceptional set of measure at most $\left(\frac{\mu'}{\in_{n0}\log n_0}\right)$ where $\in_n\to 0, \text{but } \in_{n0}\log n\to\infty$.

The result of Evans [1] is a special case of ours and is obtained by taking α=2 and $\in_n=(\log\log n)^{-1}$ in our theorem 1. The result of Samal and Mishra [5] is also a special case of our theorem 1. On the other hand our exceptional set is smaller.

All authors who have estimated bounds for N_n have used one kind of basic technique originally used by Littlewood and Offord [2].

We shall denote μ for positive constants which may have different values in different occurrences. We suppose always that n is large so that any inequalities true when n is large may be taken as satisfied.

Throughout the paper, [x] will denote the greatest integer not exceeding x.

It may be noted that although Evans [1] is a special case of ours, a much better estimate for the lower bound with smaller exceptional set can be derived from our theorem 1. For example, if we take α=2, $\in_n=(\log\log n)^{-p}$ where 0<p<1, then for $n>n_0$.

$$N_n>\frac{\log n}{(\log\log n)^p}$$

outside an exceptional set of measure at most

$$\frac{\mu(\log\log n_0)^p}{\log n_0}$$

Lemma 1.2.

If a random variable ζ has characteristic function exp $\left(-C|t|^\alpha\right)$, then for every e>0

$$P_r\left\{|\xi|>\in\right\}\le\frac{2^{1+\alpha}C}{1+\alpha}\frac{1}{\in^2}.$$

This lemma is due to Samal and Mishra [4].

Proof of the Theorem

Take constant A and B such that 0<B<1 and A>1. Choose β_m such that β_m and $\frac{\log m}{\log\beta_m}$ both tend to infinity as m tends to infinity. Let

$$\lambda_m=m^{2/\alpha}\beta_n, M_n=\left[2^\alpha\beta_n{}^\alpha\left(\frac{Ae}{B}\right)\right]+1. \qquad (1.1)$$

So $\mu_1\beta_n{}^\alpha\le M_n\le\mu_2\beta_n{}^\alpha$

We define

$$\Phi(X)=x^{[\log x]+x}$$

Let k be the integer determined by

$$\varphi(8k+7)M^{8k+7}{}_{n} \leq n < \varphi(8k+11)M^{8k+11} \qquad (1.2)$$

The first inequality gives $k \leq \dfrac{\log n}{\log \beta_m}$. The second inequality gives

$$\text{llog } \leq \{\log(8k+11)\}^2 + (8k+11)\log(8k+11) + (8k+11)\log M_n$$

$$< 2(8k+11) + (8k+11)^2 + (8k+11)\log M_n$$

$$< \mu k^2 \log M_n$$

So

$$k > \mu\sqrt{\frac{\log n}{\log M_n}} > \mu'\sqrt{\frac{\log n}{\log \beta_m}}.$$

Thus

$$\mu'\sqrt{\frac{\log n}{\log \beta_m}} < k \leq \mu\sqrt{\frac{\log n}{\log \beta_m}}. \qquad (1.3)$$

By the condition imposed on β_n it follows that k tends to infinity as n tends to infinity. We have $f(x) = U_m + R_m$ at the points

$$X_m = \left\{1 - \frac{1}{\varphi(4m+1)M^{4m}{}_{n}}\right\}^{1/\alpha} \qquad (1.4)$$

for m= $[k/2]+1, [k/2]+2, ...k \quad where$

$$U_m = \sum_1 \xi \nu X \nu, R_m = \left(\sum_2 + \sum_3\right)\xi \nu X \nu$$

the index v ranging from $\varphi(4m+1)M^{4m-1}{}_{n} + 1$ to $\varphi(4m+3)M^{4m+3}{}_{n} \text{ in } \sum_1$, from 0 to $\varphi(4m+1)M^{4m-1}{}_{n}$ and from $\varphi(4m+3)M^{4m+3}{}_{n} + 1 \; to \; n \; in \sum_3$. We also have

$$f(x_{2m}) = U_{2m} + R_{2m}, f(x_{2m+1}) = U_{2m+1} + R_{2m+1} \qquad (1.5)$$

Obviously U_{2m} and U_{2m+1} are independent random variables. Again it follows from (1.3) that 2k+1<n for larger n. Also the maximum index in U_{2m+1} for m=k is $\varphi(8k+7)M^{8k+7}{}_{n}$, which, by (1.2) is consistent with (1.5).

Let $V_m = \left(\sum_1 x^{\alpha\nu}{}_{m}\right)^{1/2}$. Then

$$V^{\alpha}{}_{m} = \sum_1 x^{\alpha\nu}{}_{m} > \sum_{\varphi(4m-1)M^{4m-1}{}_{n}+1}^{\varphi(4m-1)M^{4m-1}{}_{n}+1} x^{\alpha\nu}{}_{m}$$

$$> \left\{\varphi(4m+1)M^{4m}{}_{n}\right\} - \varphi(4m-1)M^{4m-1}{}_{n}\, x^{2\phi(4m+1)M_n{}^{4m}}{}_{m}$$

$$> \left\{\varphi(4m+1)M^{4m}{}_{n}\right\}\left\{1 - \frac{\varphi(4m-1)}{\varphi(4m+1)}\frac{1}{M_n}\right\}(e^{-1}/A)$$

$$> \left\{\varphi(4m+1)M^{4m}{}_{n}\right\}(B/A)e^{-1} \qquad (1.6)$$

when n is large

Now we estimate

$$P = P_r\left\{(U_{2m} > V_{2m}, U_{2m+1} < -V_{2m+1}) \cup (U_{2m} < -V_{2m}, U_{2m+1} > V_{2m+1})\right\}$$

$$P_r\left\{(U_{2m} > V_{2m}, \Pr(U_{2m+1} < -V_{2m+1}) + \Pr(U_{2m} < -V_{2m}) \Pr(U_{2m+1} > V_{2m+1})\right\}$$

Since the characteristic function of ξ_v is $exp\left(-C|t|^{\alpha}\right)$, the characteristic function of U_{2m} is therefore

$$exp\left\{\left(-C|t|^{\alpha}\right)\sum_{4} x^{\alpha v}{}_{2m}\right\} = \exp\left(-C|t|^{\alpha} V^{\alpha}{}_{2m}\right).$$

where the index V ranges from $\varphi(8m-1)M^{m-1}{}_{n} + 1$ *to* $\varphi(8m+3)M_n{}^{8m+3}$ in $\sum_4$. Therefore the characteristic function of U_{2m}/V_{2m} *is exp* $\left\{\left(-C|t|^{\alpha}\right)\right\}$, *which* is similarly also the characteristic function U_{2m+1}/V_{2m+1}. Thus the characteristic function is dependent on m.

Let F(x) be the common distribution function. Hence

$$P_r\left\{(U_{2m} > V_{2m}) = \Pr(U_{2m}/V_{2m} > 1) = 1 - \Pr(U_{2m}/V_{2m} \le 1) = 1 - F(1).\right\}$$

Thus P= $\{1 - F(1)\}F(-1) + F(-1)\{1 - F(1)\} = 2F(-1)\{1 - F(1)\} = \delta(\text{say})$.

Obviously $\delta > 0$.

1.2. We shall need the following lemmas.

Lemma 1.2.

$\left|\sum_3 \xi v x v_m\right| < V_m/2$ except for a set of measure at most

$\dfrac{2^{1+2\alpha} CAe}{B(1+\alpha)} \exp\left\{-(4m+1)^2 M^2{}_n\right\}$ for sufficiently large n.

Proof.

The characteristics function of

$$\left|\sum_3 \xi v x v_m\right| \text{is } \exp\left\{-C|t|^{\alpha} \sum_3 x^{\alpha v}{}_m\right\}$$

$$\le \frac{2^{1+2\alpha} C}{(1+\alpha) V^{\alpha}{}_m x^{\alpha v}{}_m}$$

But

$$\left|\sum_3 x v_m < \sum_{\varphi(4m+3)M^{4m+3+1}}^{\infty} X^{\alpha v}{}_m\right| = \frac{X^{\alpha\{\varphi(4m+3)M^{4m+3+1}\}}{}_m}{1 - x^{\alpha}{}_m}$$

$$= \varphi(4m+3)M^{4m+3+1}\left\{1 - \frac{1}{\varphi(4m+3)M^{4m+3+1}}\right\}^{\varphi(4m+3)M^{4m+3+1}}$$

Since

$$\varphi(4m+3)M^{4m+3+1} > (4m+3)^{[\log(4m+3)]+(4m+3)} M_n{}^{4m+3}$$

$$> (4m+1)^{[\log(4m+1)]+(4m+1)+2} M_n{}^{4m+1} M_n{}^2$$

$$> \varphi(4m+1)(4m+1) M_n{}^{4m} M_n{}^2$$

We have

$$\sum_3 x^{\alpha v}{}_m < \varphi(4m+1) M_n{}^{4m} \exp\left\{-(4m+1)^2 M_n{}^2\right\}$$

Hence using (1.6), we obtain

$$P_1 < \frac{2^{1+2\alpha} CAe}{B(1+\alpha)} \exp\left\{-(4m+1)^2 M_n{}^2\right\}$$

as required

Lemma 1.3.

$$\left|\sum_{2}\xi v x^{v}{}_{m}\right| < \lambda_{m}\left(\sum_{2}\xi v x^{v}{}_{m}\right)^{1/\alpha} \text{ except for a set of measure at most } \frac{2^{1+2\alpha}C}{(1+\alpha)\lambda^{\alpha}{}_{m}}$$

This follows directly from lemma 1.1.

1.3. Now

$$\lambda_{m}\left(\sum_{2}\xi v x^{v}{}_{m}\right)^{1/\alpha} < \lambda_{m}\left\{\varphi(4m-1)M_{n}{}^{4m-1}+1\right\}^{1/\alpha}$$

$$=\lambda_{m}\left\{\varphi(4m-1)M_{n}{}^{4m-1}\left(1+\frac{1}{\varphi(4m-1)M_{n}{}^{4m-1}}\right)^{1/\alpha}\right\}$$

$$< 2^{1/\alpha}\lambda_{m}\left\{\varphi(4m-1)M_{n}{}^{4m-1}\right\}^{1/\alpha}$$

$$= 2^{1/\alpha}\lambda_{m}\left\{(4m-1)^{[LOG(4M-1)+(4M-1)}M_{n}{}^{4m-1}\right\}^{1/\alpha}$$

$$< \left\{\frac{2^{1/\alpha}\lambda_{m}\left\{(4m-1)^{[LOG(4M-1)+(4M-1)}M_{n}{}^{4m-1}\right\}}{\left\{(4m+1)^{2}M_{n}\right\}}\right\}^{1/\alpha}$$

$$< \left\{\frac{2\lambda^{\alpha}{}_{m}\varphi\left\{(4m+1)M_{n}{}^{4m}\right\}}{\left\{16m^{2}M_{n}\right\}}\right\}^{1/\alpha}$$

$$< \left\{\frac{2\lambda^{\alpha}{}_{m}\left(\frac{Ae}{B}\right)V_{n}{}^{4m}}{\left\{16m^{2}M_{n}\right\}}\right\}^{1/\alpha}$$

$$< \left\{\frac{2\lambda^{\alpha}{}_{m}\left(\frac{Ae}{B}\right)V_{n}{}^{4m}}{\left\{16m^{2}M_{n}\right\}}\right\}^{1/\alpha} < \left\{\frac{\beta^{\alpha}{}_{m}\left(\frac{Ae}{B}\right)V_{n}{}^{\alpha}}{\left\{M_{n}\right\}}\right\}^{1/\alpha}$$

$$< \frac{1}{2}V_{m}$$

The last two steps above follow from (1.1) and (1.6). Hence by using lemmas 1.2 and 1.3, we have $R_m < V_m$ for every sufficiently large n except for a set of measure at most

$$\mu\exp\left\{-(4m+1)^{2}M_{n}{}^{2}\right\}+\frac{\mu'}{\lambda^{\alpha}{}_{m}} \leq \mu\exp\left\{-(m^{2}M_{n}{}^{2})\right\}+\frac{\mu'}{\lambda^{\alpha}{}_{m}}$$

Thus we have

$$\left|R_{2m}\right| < V_{2m} \text{ and } \left|R_{2m+1}\right| < V_{2m+1}$$

for $m=m_0$, m_0+1……..k, where $m_0=[k/2]+1$

The measure of the exceptional set is at most

$$\mu \exp\left\{-(4m^2 M_n{}^2\right\} + \frac{\mu'}{\lambda^{\alpha}{}_{2m}} \leq \mu \exp\left\{-(2m+1)M_n{}^2)\right\} + \frac{\mu'}{\lambda^{\alpha}{}_{2m+1}}$$

$$< \mu \exp\left\{(-m^2 M_n{}^2)\right\} + \frac{\mu_2}{\lambda^{\alpha}{}_m} \qquad (1.7)$$

1.4. We define the events E_m and F_m as follows:

$E_m=\{U_{2m}>V_{2m}, U_{2m+1}<-V_{2m+1}\}$

$F_m=\{U_{2m}<V_{2m}, U_{2m+1}>-V_{2m+1}\}$

We have shown earlier that

$$P_r(E_m \cup F_m) = \delta > 0$$

Let η_m be a random variable such that it takes value 1 on Em Ufm and zero elsewhere. In other words

$$\eta_m \begin{cases} = 1, \text{ with probability} \delta \\ = 0, \text{ with probability } 1 - \delta \end{cases}$$

Let η_m are thus independent random variables with E $(\eta_m)=\delta$ and V $(\eta_m)= \delta - \delta^2 <1$.

We write

$$S_m \begin{cases} 0 \text{ if } |R_{2m}| < V_{2m} \text{ and } |R_{2m+1}| < V_{2m+1} \\ 1 \text{ otherwise} \end{cases}$$

III. Conclusion

By considering the polynomial $f_n(x) = \sum_{k=0}^{n} \xi_k x^v = 0$

where $\xi_k x^v$ are independent random variables assuming real values only we found that the number of zeros of the above polynomial of the equations f(x)=0 is at least (en log n) except for a set of measure at most for an integer $n>n_0$ the number of real roots of most $\frac{\mu}{(\in_{n0} \log n_0)}$

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