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New Approaches For Finding Roots Of Nonlinear Equations Utilizing Means

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Abstract:
This paper aims at combining Newton's method with an approach proposed by S. Weerakoon and T.G.I. Fernando, using the function evaluated at the arithmetic, harmonic, geometric and quadratic means to obtain different iterative formulations, the convergence of one of them is demonstrated, which is the arithmetic mean which is obtained as the order of cubic convergence. The resulting order of convergence is investigated in some examples, and it is found to be less or greater than the cubic convergence, and with some means, convergence is not achieved.
Background *There are various methods to find the root of a nonlinear equation. Closed methods include bisection, secant, and false position, where an initial interval containing the root is provided. On the other hand, open methods such as fixed-point iteration and Newton's method require an initial value to begin iterations. From these methods, variants have been developed, such as a hybrid approach combining two of them, and another based on Newton's method. These techniques are applied across various fields including exact sciences, engineering, health sciences, and socio-administrative sciences, among others.*
Materials and Methods*: The purpose of this article is to incorporate means into the method developed by S. Weerakoon and T.G.I. Fernando, which has a third-order convergence. This is done with the intention of creating new methods that maintain an equal or even lower order of convergence, aiming to find the simple root of a nonlinear equation*
Results *For functions that do not have maximums, minimums and inflection points given an initial value all the methods converge, while in the opposite case only two of them converge, that of the Arithmetic mean squared mean as can be observed in table no. 3.*
Conclusion: *The methods changing the formulation converge as long as they do not encounter a maximum or a minimum or inflection points, but the harmonic mean and the quadratic mean do converge, see table No. 3.*
Key Word*: Simple root; iterative methods; integral; means*

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I. Introduction

From secondary school onward, we encounter problems involving nonlinear equations. Typically, methods such as factorization and the general formula are taught to solve them. However, these methods can become complex when values are not integers or rationals, as calculators are required in such cases. For instance, calculating the square root of two, an irrational number, poses difficulties without the use of tools like a calculator. It is in these situations that numerical methods come into play. This work proposes algorithms that utilize various mathematical means and integrate them into the algorithm developed by[1]. Let $f: I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a function defined on an open interval I such that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, where $\alpha \in \mathbb{R}$. In Newton's method to approximate a root of a nonlinear equation $f(x) = 0$ the iterative form is given by equation (1)[1].

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (1)$$

The equation (1) exhibits quadratic convergence[2].

II. Material And Methods

In the literature, the first third-order convergence method based on numerical integration was developed by[2]. In this method, the integral appearing in equation (2) is numerically approximated.

$f(x) = f(x_n) + \int_{x_n}^{x} f(\lambda)d\lambda$ (2)

The trapezoidal rule is applied to the indefinite integral in equation (2).

$\int_{x_n}^{x} f(\lambda)d\lambda \approx \frac{1}{2}(x - x_n)(f'(x_n) + f'(x))$ (3)

The equation 3 is considered to obtain a variant[2], which exhibits cubic convergence[2]. Its iterative form is given by equations 4 and 5.

$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)+f'(y_n)}$ (4)

Where y_n is given in equation (5).

$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$ (5)

This developed method[2] consists of two steps: Next, the means are substituted into the derivative of the function. To do this, the relationship between means is used[3] as given by equation (6).

$0 < \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}} \le (\prod_{i=1}^{n} x_i)^{\frac{1}{n}} \le \frac{\sum_{i=1}^{n} x_i}{n} \le \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n}}$ (6)

A demonstration of (6) is carried out[4]. A generalization of means is given by equation (7) and is called the Hölder mean.

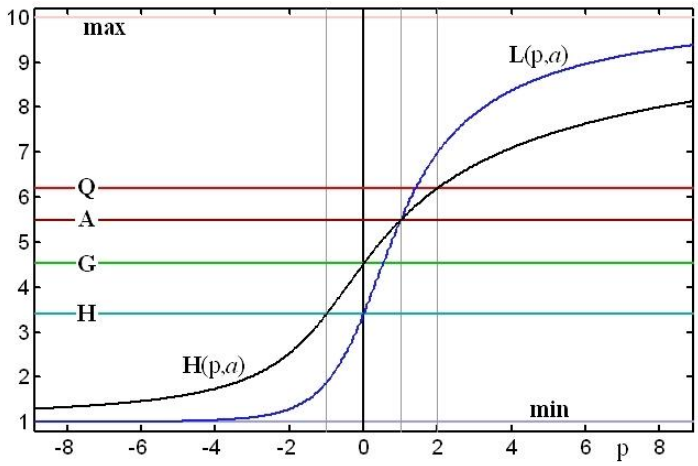
$\mathrm{H}(p, \boldsymbol{a}) = \left[\frac{1}{n}\sum_{k=1}^{n}(a_k)^p\right]^{\frac{1}{p}} \quad \boldsymbol{a}=a_1, a_2 \ldots, a_n \in \mathbb{R} - 0, \quad p \in \mathbb{Z}^+ - 0$ (7)

A description of classical averages, as well as their properties of these averages defined in the ℝ-0 used in the proof of the inequalities of these[5,6]. Derrick Henry Lehmer's mean is defined as.

$\mathrm{Lp}(x) = \frac{\sum_{k=1}^{n} x_k^p}{\sum_{k=1}^{n} x_k^{p-1}}$ (8)

Figure no.1. Shows the Hölder and Lehmer mean as a function.

Figure no. 1. Hölder mean $H(p, a)$ and Lehmer mean $L(p, a)$ as functions of p for the set $a = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$



The two lines trace the **Hölder path** (black) and the **Lehmer path** (blue). The horizontal lines indicate, for the same set *a*, the *minimum* (lowermost), the *harmonic mean* **H**, the *geometric mean* **G**, the *arithmetic mean* **A**, the *quadratic mean* **Q** and the *maximum* (uppermost)[9].

From equation (6), the harmonic, geometric, arithmetic, and quadratic means are obtained. Additionally, there exist $\alpha, \beta, \gamma, \delta > 0$such that this equation can be represented as equation (9).

$0 < \alpha\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}} = \beta(\prod_{i=1}^{n} x_i)^{\frac{1}{n}} = \gamma\frac{\sum_{i=1}^{n} x_i}{n} = \delta\sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n}} \quad x_i > 0$ (9)

From equation (9), another relationship can be found[5,6]. For $a, b > 0$, equation (10) holds:

$\frac{a+b}{2} + \sqrt{a * b} < \sqrt{\frac{a^2+b^2}{2}} + \frac{2ab}{a+b}$ (10)

In equation (9) there exist $\alpha, a, b \in \mathbb{R}^+ - 0, a \neq b, b, a \neq 0$ such that equation (11) follows[6].

$\frac{a+b}{2}+\sqrt{a*b}=\alpha\left(\sqrt{\frac{a^2+b^2}{2}}+\frac{2ab}{a+b}\right)$ (11)

The equation (11) is used in equation (7), which represents the arithmetic mean of the evaluation of the function's derivative at two points, as shown in equation (12).

$f'\left(\frac{x+x_n}{2}\right)=\frac{f'(x)+f'(x_n)}{2}$ (12)

By applying equations (6) and (7), we obtain an equivalent equation for Newton's method, namely equation (13).

$x_{n+1}=x_n-\frac{f(x_n)}{f'\left(\frac{x_n+x_{n+1}^*}{2}\right)}$ (13)

Algorithm 1.

For an initial value x_0, the root x_{n+1} can be approximated by performing the following iterative process given by equations (14), (15) and (16) until reaching a stopping criterion given by $|x_{n+1}-x_n|\leq\varepsilon$ ó $|f(x_{n+1})|\leq\varepsilon$.

$x_{n+1}=x_n-\frac{f(x_n)}{f'(y_n)}$ (14)

$z_n=x_n-\frac{f(x_n)}{f'(x_n)}$ (15)

$y_n=\frac{x_n+z_n}{2}$ (16)

Algorithm 2. Using the arithmetic mean the process involves equations (17) and (18).

$x_{n+1}^*=x_n-\frac{f(x_n)}{f'(x_n)}$ (17)

$x_{n+1}=x_n-\frac{f(x_n)}{f'\left(\frac{x_n+x_{n+1}^*}{2}\right)}$ (18)

In equation (18) using the property of convex functions[7,8] , is obtained equation (19)

$f'\left(\frac{x_n+x_{n+1}^*}{2}\right)=\frac{f'(x_n)+f'(x_{n+1})}{2}$ (19)

The same conditions as in Algorithm 1 are used. Now, the harmonic mean shown in equation (19) will be used.

$H=\frac{2*x_{n+1}^**x_n}{(x_{n+1}^*+x_n)}$ (20)

Substituting equation (20) into the derivative of equation (8), equation (21) is obtained.

$x_{n+1}=x_n-\frac{f(x_n)}{f'(H)}$ (21)

The equation (22) shows how x_{n+1} is obtained.

$x_{n+1}=x_n-\frac{f(x_n)}{f'\left(\frac{2*x_{n+1}^**x_n}{(x_{n+1}^*+x_n)}\right)}$ (22)

Algorithm. 2.

The same conditions as in Algorithm 1 are used.

$x_{n+1}^*=x_n-\frac{f(x_n)}{f'(x_n)}$ (23)

$H=\frac{2*x_{n+1}^**x_n}{(x_{n+1}^*+x_n)}$ (24)

$x_{n+1}=x_n-\frac{f(x_n)}{f'(H)}$ (25)

Algorithm 3. Using the geometric mean.

$G=\sqrt{x_n*x_{n+1}^*}$ (26)

Replacing equation (26) in the derivative of equation (8).

$x_{n+1}=x_n-\frac{f(x_n)}{f'(\sqrt{x_n+x_{n+1}^*}\ )}$ (27)

The quadratic mean is obtained using equation (28).

$Q=\sqrt{\frac{x_n^2+\left(x_{n+1}^*\right)^2}{2}}$ (28)

The equation (1) is substituted into the derivative of equation (8) to obtain equation (29).

$x_{n+1}=x_n-\frac{f(x_n)}{f'\left(\sqrt{\frac{x_n^2+\left(x_{n+1}^*\right)^2}{2}}\right)}$ (29)

Algorithm. 4. Using the same procedure, but now using the mean square.

Using the quadratic mean, equations (30), (31), and (32) are obtained.

$x_{n+1}^{*} = x_n - \frac{f(x_n)}{f'(x_n)}$ (30)

$Q = \sqrt{\frac{x_n^2 + \left(x_{n+1}^{*}\right)^2}{2}}$ (31)

$x_{n+1} = x_n - \frac{f(x_n)}{f'(Q)}$ (32)

The iteration for n=0,1,2,.., concludes when the stopping criterion $|x_{n+1} - x_n| \leq \varepsilon$ is satisfied, that is $\alpha \approx x_{n+1}$, and equation (33) is fulfilled.

$f(x_{n+1}) \approx 0.$ (33)

To demonstrate the order of convergence of the method given by algorithm 1. Equations (14-16.) are used, for algorithm 2. equations (23-25), in algorithm 3 equation (26) changing in only one part of algorithm 1. and algorithm 4. (20-32), in this work only the convergence of algorithm one will be evaluated. For the others, a computational comparison will be made.

III. Convergence Analysis.

Theorem. Let $\alpha \epsilon I$, $f \epsilon C^{n+1}(I)$ where $f: I \subset \mathbb{R} \to \mathbb{R}$ on an open interval I. Then, the three steps of the iterative method given by equations (4) (5), (14-16), and (17) (18) result in a method of cubic convergence.

Proof. Let α be a simple root of $f$such that $f(\alpha) = 0,$ and $f'(\alpha) \neq 0$. Taking $e_n = x_n + \alpha$, and since f is of class $C^{n+1}(I)$, with this condition, $f(x_n)$ can be expanded into a Taylor series around α, as shown in equation (33).

$f(x_n) = f(\alpha + e_n) = f(\alpha) + \frac{f\prime(\alpha)}{1!}(x_n - \alpha) + \frac{f\prime\prime(\alpha)}{2!}(x_n - \alpha)^2 + \cdots + \frac{f^{(n)}(\alpha)}{n!}(x_n - \alpha)^n$ (34)

The result of substituting $f(\alpha) = 0$ into equation (13) is shown in equation (35).

$f(x_n) = 0 + \frac{f\prime(\alpha)}{1!} e_n + \frac{f\prime\prime(\alpha)}{2!} e_n^2 + \frac{f\prime\prime\prime(\alpha)}{3!} e_n^3 + O(e_n^4)$ (35)

Factorizing $f'(\alpha)$ from equation (14), equation (35) is obtained.

$f(x_n) = f'(\alpha)\left(e_n + \frac{f''(\alpha)}{2!f\prime(\alpha)} e_n^2 + \frac{f'''(\alpha)}{3!f\prime(\alpha)} e_n^3\right) + O(e_n^4)$ (36)

The constants c_k are defined in equation (36) and substituted into equation (14).

$c_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)}$, for $k = 2,3 \ldots$ (37)

Therefore, equation (15) can be written as shown in equation (37).

$f(x_n) = f'(\alpha)(e_n + c_2 e_n^2 + c_3 e_n^3) + O(e_n^4)$ (38)

Differentiating equation (16) with respect to e_n, equation (38) is obtained.

$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n e_n^2 + 3c_3 e_n^2 + O(e_n^3)]$ (39)

Dividing equation (16) by equation (17), we obtain equation (39) and equation (40).

$\frac{f(x_n)}{f'(x_n)} = [e_n + c_2 e_n^2 + c_3 e_n^3]\{1 - [2c_2 e_n + 3c_3 e_n^2 + O(e_n^3)]\}^{-1}$ (40)

$\frac{f(x_n)}{f'(x_n)} = [e_n + c_2 e_n^2 + c_3 e_n^3]\{1 - [2c_2 e_n + 3c_3 e_n^2 + O(e_n^3)]\}^{-1}$ (41)

By using the series shown in equation (40), we obtain equation (42) from equation (17).

$\frac{1}{1-x} = 1 - x + x^2 - x^3, \ldots$ which converges if $|x| < 1$ (46)

$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + (2c_2^2 - 2c_3)e_n^3 + O(e_n^4)$ (47)

In the equation (42) $e_n = x_n - \alpha$ Clearing$x_n = \alpha + e_n$ the equations are obtained (44), (45) y (46).

$x_{n+1}^{*} = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + e_n - \left(e_n - c_2 e_n^2 + (2c_2^2 - 2c_3)e_n^3 + O(e_n^4)\right)$ (48)

$x_{n+1}^{*} = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + O(e_n^4)$ (49)

$x_{n+1}^{*} = \alpha + c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + O(e_n^4)$ (50)

From equation (14), equation (45) is obtained.

$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(\frac{x_n + x_{n+1}^{*}}{2}\right)} = e_n - \left[e_n - \left(c_2^2 + \frac{3}{4}c_3\right)e_n^3 + O(e_n^4)\right]$ (51)

$e_{n+1} = \left(c_2^2 + \frac{3}{4}c_3\right)e_n^3 + O(e_n^4)$ (52)

From equation (52), it is demonstrated that this method has cubic convergence order.

IV. Sample Applications

$f(x) = x^2 - 2 = 0$ with initial value $x_0 = 1.0$

$f(x) = x^2 - x - 1 = 0$ with initial value $x_0 = 1.0$. The golden ratio (φ ≈ 1.618)

Using the following non-linear equations[9], an approximation with more decimals ($\alpha \approx$1.61803398874989)
$f(x) = x^3 - 5x = 0$ with initial value $x_0 = 1.0$.

Figure.No.2. Shows the graph of the equation of application 1.1

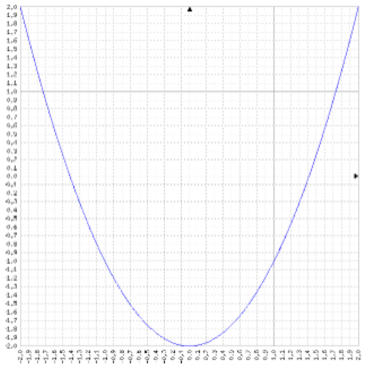


Figure no. 3. Shows the graph of application 1.2

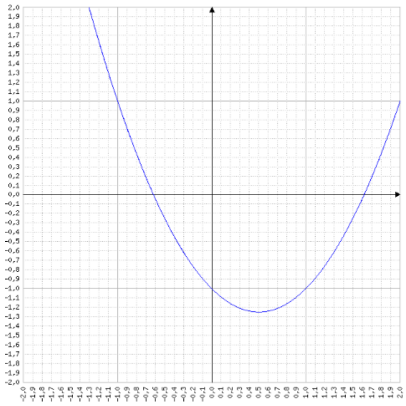


Figure no. 4. Shows the graph of application 1.3

Table no 1: Shows the solution of the different methods for the function $f(x) = x^2 - 2 = 0$ with initial value $x_0 = 1.0$ with an approximation error $\varepsilon = 0.00001$.
Table No.1

Table no 1: shows the results of the equation $f(x) = x^2 - 2 = 0$ with initial value $x_0 = 1.0$ with an approximation error $\varepsilon = 0.00001$

Method	Initital value	Iteration	Approximation	Error in last iteration
Newton	1.0	4	1.41421354	0.00000215
Fernando	1.0	3	1.414221354	0.00000036

Geometric mean	1.0	3	1.414221354	0.00000000
Aritmetic mean	1.0	3	1.414221354	0.00000000
Harmonic mean	1.0	3	1.414221354	0.00000036
Quadratic mean	1.0	3	1.414221354	0.00000203

Table no 2: Shows the solution of the different methods for the function $f(x) = x^2 - x - 1 = 0$ with initial value $x_0 = 1.0$ with an approximation error $\varepsilon = 0.00001$

Method	Initial value	Iteration	Approximation	Error in last iteration
Newton	1.0	5	1.61803401	0.00000048
Fernando	1.0	4	1.61803401	0.00000000
Geometric mean	1.0	4	1.61803401	0.00000000
Aritmetic mean	1.0	4	1.61803401	0.00000000
Harmonic mean	1.0	3	1.61803401	0.00000036
Quadratic mean	1.0	8	1.61803544	0.00000799

Table no 3: Shows the solution of the different methods for the function $f(x) = x^3 - 5x = 0$ with initial value $x_0 = 1.0$ with an approximation error $\varepsilon = 0.00001$.

Method	Initial value	Iteration	Approximation	Error in last iteration
Newton	1.0	It stays in a cycle		
Fernando	1.0	It stays in a cycle		
Geometric mean	1.0	It is undetermined		
Aritmetic mean	1.0	4	0	0.00000000
Harmonic mean	1.0	1	1 This is not a root of the equation	0.00000000
Quadratic mean	1.0	10	0.0000738 approaches the root of the equation	0.00000203

V. Result

Future research should focus on determining the order of convergence of the alternative formulations and on exploring the robustness of the methods when applied to functions that do not meet the theoretical convergence criteria. The preliminary results obtained in Table No. 3 suggest that the arithmetic mean and the square mean could present an interesting behavior in these cases

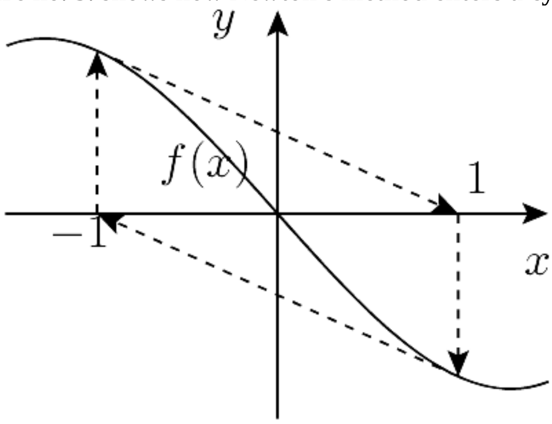
VI. Discussion

In the first table no. 1. of Fernando's method, all methods converged in the third iteration. The geometric mean and the arithmetic mean reached zero error, while Fernando's method and the harmonic mean presented an identical error (using the same stopping criterion). The squared mean exhibited the largest error.

In Table no. 2, it is observed that Fernando's method, the geometric mean, the arithmetic mean and the harmonic mean required eight iterations to converge.

In Table no. 3, only the arithmetic mean converged in four iterations, and the squared mean approached the root in ten iterations.

Figure no. 5. shows how Newton's method enters a cycle[10]



The figureno.5 shows how the iterations of this Newton method remain in a cycle but may have an overflow because the derivative becomes zero computationally.

VII. Conclusion

Future research should focus on determining the order of convergence of the alternative formulations and on exploring the robustness of the methods when applied to functions that do not meet the theoretical convergence criteria. The preliminary results obtained in Table no. 3 suggest that the arithmetic mean and the square mean could present an interesting behavior in these cases.

References

[1] S. Weerakoo, , & Fernando, T. G. I. (2000). A Variant Of Newton's Method With Accelerated Third-Order Convergence. Applied Mathematics Letters, 13(1), 97-103.
[2] Saba Akram, Q. Annu. (2015). Newton-Raphson Method. *International Journal Of Scientific & Engineering Research*, 6(7), 1748-1752.
[3] Ilhan M. Izmirli. (2013). An Elementary Proof Of The Mean Inequalities. *Advances In Pure Mathematics*, 3(3), 331-334. Doi: 10.4236/Apm.2013.33047 Https://Www.Scirp.Org/Html/4-5300350_31228.Htm
[4] Elena Castiñeira Holgado And Venancio Tomeo Perucha. (2019). Relaciones Entre Las Medias Estadísticas: Demostraciones Relationships Between Statistical Measures: Demonstrations. Universidad Complutense De Madrid. Retrieved From Https://Estudiosestadisticos.Ucm.Es/Data/Cont/Docs/12-2019-11-29-Ct04-2019.Pdf
[5] Stan Sýkora. (2009). Mathematical Means And Averages: Basic Properties. In *Stan's Library* (Vol. Iii, Pp. 1-22). Castano Primo, Italy: Ed. S. Sykora. Https://Www.Ams.Org/Journals/Bull/1948-54-05/S0002-9904-1948-08994-7/S0002-9904-1948-08994-7.Pdf
[6] Romero Meštrović. (2018). A Geometric Proof Of The Inequality Am + Gm < Qm + Hm In Two Variables Https://Www.Researchgate.Net/Publication/329701198
[7] E. F. Beckenbach. (1948). Convex Functions. *Bulletin Of The American Mathematical Society*, 54(5), 438-441.
[8] José R. Berrendero. Funciones Convexas Y Optimización Convexa. Universidad Autónoma De Madrid. Https://Verso.Mat.Uam.Es/~Joser.Berrendero/Cursos/Matematicas-Io/Io-Tema4-16.Pdf
[9] Birch Fett (2006). An In-Depth Investigation Of The Divine Ratio. *The Mathematics Enthusiast*, 3(2), 112-127. Doi: 10.54870/1551-3440.1047 Https://Scholarworks.Umt.Edu/Cgi/Viewcontent.Cgi?Article=1047&Context=Tme
[10] Knowino.(2012). Newton's Method. Retrieved From Https://Www.Theochem.Ru.Nl/~Pwormer/Knowino/Knowino.Org/Wiki/Newton's_Method.Html