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The Transformative Impact Of Gauss And Riemann On Numerical Integration Calculus: Towards Efficient Computational Methods

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Abstract:
This article aims to expose the contributions of Carl Friedrich Gauss to the area of numerical analysis. While Gauss is recognized for his extensive contributions to various branches of mathematics, in this case we will focus on his impact on the development of numerical methods, particularly Gauss's method. Bernhard Riemann, in addition to his contributions in geometry, made fundamental contributions to the theory of functions of a complex variable, mathematical physics and number theory. He clarified the notion of integral, defining what is known today as the Riemann integral, allowing the calculation of integrals from their definition as a limit of sums.
Although the definition of the Riemann integral is not directly attributed to Gauss, his previous work laid the foundation for the development of this fundamental concept. Gauss's investigations into polynomial interpolation and approximation errors provided essential tools for the numerical calculation of integrals.
***Background**: Gauss-Legendre Quadrature and the Riemann Integral: A Journey Through Time, describes Carl Friedrich Gauss, as a titan of mathematics, who gave life to the Gauss-Legendre quadrature in 1814. His mathematical genius led him to employ a method based on continuous fractions, obtaining nodes and weights with a precision of 16 digits for a seventh degree polynomial.*
***Materials and Methods**: In the field of mathematics, definite integrals represent a fundamental concept for calculating areas under curves and solving physical problems. However, there are cases in which finding the primitive of a definite integral is a complex or even impossible task. This is where numerical analysis, as a branch of mathematics dedicated to the development of computational methods to approximate solutions to mathematical problems. Numerical analysis offers a wide range of tools to approximate the value of definite integrals, even those with no known primitive. Among the most common methods are the trapezoid rule, Simpson's rule and the Gauss quadrature method. These methods allow numerical results to be obtained with a high degree of precision, making them valuable tools for a wide variety of applications. In this analysis, the results obtained through analytical integration are compared with those obtained using the Gauss method with different configurations of nodes and weights.*
***Results**: In the field of calculus, the resolution of definite integrals using analytical methods can sometimes present a complex and tedious process to obtain the exact solution. In this article, the use of numerical methods is proposed as an alternative to address this type of problems. A specific numerical method will be used for three different applications. In each case, it was observed that by increasing the number of nodes and abscissa, the approximation to the area of the real solution became more precise. The results obtained in this study demonstrate the usefulness of numerical methods as complementary tools for solving definite integrals, particularly in those cases where the application of analytical methods is complex or impractical.*
***Conclusion:** Although the Gauss-Legendre technique has proven to be an effective method for solving definite integrals in various applications, it is important to recognize that there are certain limitations that must be considered when implementing it. In the specialized literature, cases have been reported where up to one hundred values of the nodes and weights necessary for the application of the method have been found. However, for applications 1, 2 and 3 described in the article, this strategy is not viable, since the same instability problem*

would arise. In general, it is recommended to be cautious when applying the Gauss-Legendre technique with trigonometric functions that have a long period, as well as with combinations of these functions together with powers raised to a considerably large number. In such scenarios, the approximation to the integral can be significantly affected.

***Key Word**: Carl Friedrich Gauss, Bernhard Riemann, numerical methods, complex variable, integral.*

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I. Introduction

Gauss-Legendre Quadrature and the Riemann Integral: A Journey Through Time, describes Carl Friedrich Gauss, as a titan of mathematics, who gave life to the Gauss-Legendre quadrature in 1814. His mathematical genius led him to employ a method based on continuous fractions, obtaining nodes and weights with a precision of 16 digits for a seventh degree polynomial.

Later, Carl Gustav Jacobi discovered a deep connection between this technique and the orthogonal set of Legendre polynomials. However, for almost a century, these values remained the only ones available.

In parallel, Georg Friedrich Bernhard Riemann introduced a fundamental concept in integral calculus: the Riemann integral. This tool, of great relevance in engineering, science and even social sciences, allows evaluating the definite integral of a continuous function f on an interval $[a, b]$.

The Riemann integral is based on the concept of addition. The interval is divided into subintervals and the lower and upper sums of f in each subinterval are calculated. The value of the integral approaches the limit of these sums as the number of subintervals increases and their size decreases.

The Gauss-Legendre quadrature and the Riemann integral are two pillars of numerical analysis. The former offers an accurate and efficient way to approximate integrals, while the latter provides a fundamental theoretical framework for understanding integration.

Together, these tools allow scientists and engineers to address a wide range of problems in diverse fields, from physics and chemistry to economics and social sciences.

The works of Gauss, Jacobi and Riemann have left an indelible mark on the world of mathematics. His ideas remain essential to solve complex problems and advance various fields of knowledge.

$$\int_a^b f(x)dx = \lim_{n\to\infty} f(a + i\Delta x)\Delta x \qquad (1)$$

In the parabolas shown as Figures 1 and 2, the lower and upper sums of Riemann Georg Friedrich Bernhard are observed. Another geometric representation of the Riemann sum for a positive function is found in Jean-Paul Truc, (2019)[1].

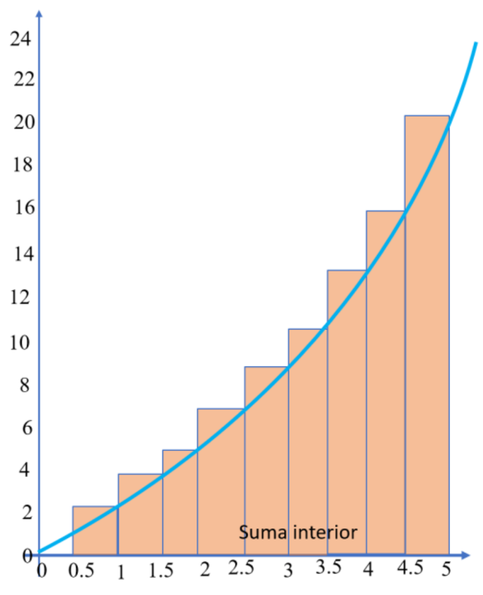

Figure 1: Show Riemann sums with upper height

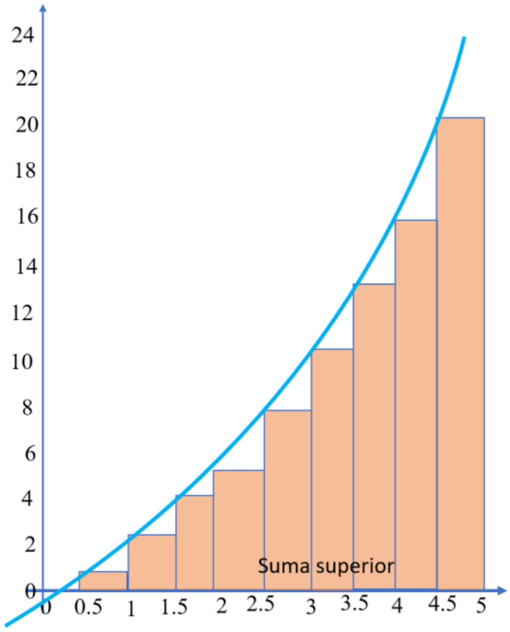

Figure 2: Shows Riemann sums with lower height

In his work "The fundamental theorem of calculus and its history" David M. Bressoud (2011) [2], provides a fascinating historical overview of the fundamental theorem of integral calculus, from its sketches in the 17th century to its definitive formalization in the 19th century. XIX and its subsequent incorporation into textbooks of the 20th century.

$\frac{d}{dx}\int_a^x f(t)dt = f(x)$ for $a < x < b$ (2)

And, if $F(x)$ for all $x \in [a, b]$, then

$\int_a^b f(t)dt = F(b) - F(a)$ (3)

Equation (1) is known as the primitive or antiderivative of the fundamental theorem of integral calculus, because it shows how to use the definite integral to construct an antiderivative. Equation (3) is known as the evaluation part of the fundamental theorem of integral calculus, it shows how to use the antiderivative to evaluate the definite integral.

In 1815 Carl Friedrich Gauss published Methodus nova integralium values per approximationem inveniendi (New method of finding the values of integrals by approximation), in which he introduced the quadrature rules that today bear his name. For a further explanation, J. M. Sanz-Serna (2019) and (2018) [3-4] expand. Gaussian quadrature comes from the use of mathematics to approximate the area of a function using small quadrilaterals of irregular shape. Nowadays, talking about quadrature is synonymous with integration.

Carl Friedrich Gauss showed that the integral given a polynomial function of degree 2n-1 is expressed as a sum of n terms, see article Ruohong Li eat[5] To derive the rule of Quadrature uses a set of orthogonal functions that forms a basis for the entire set of algebraic polynomials given by equation (1), if the functions $P_i(x)$ and $P_j(x)$ are orthogonal in $[a, b]$ with respect to the function weight $w(x)$if the product $P_i(x)\ P_j(x)\ w(x)$ is integrable in $[a, b]$, satisfies, if $P_i(x)\ P_j(x)$ are algebraic polynomials satisfies that

$$\int_a^b P_i(x)P_j(x)\omega(x)dx = \begin{cases} non-zero\ if\ i \neq j \\ 0\ if\ i = j \end{cases}$$

$f(x) \in \{1, x, x^2, \ldots, x^n\}$ (4)

Gauss-Legendre, and did so by a calculation with continuous fractions in 1814 the orthogonal set of the Legendre polynomials given by the following recurrence function, we have that the Gaussian Quadrature is a linear combination of the function $f(x)$ evaluated in the roots of the nth Legendre of the polynomial given by Equation (2).

$$\left\{P_{n+1}(x) = \frac{1}{2^{n+1}\,(n+1)!}\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^{n+1}\right\} \quad (5)$$

Figure 3, represents the Legende polynomials.

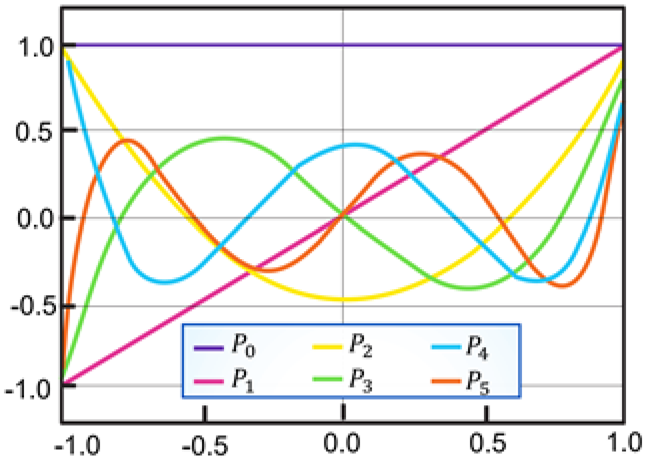

Figure 3: Legendre polynomials
(http://www.sc.ehu.es/sbweb/fisica3/especial/legendre/legendre.html)

These are defined in the interval $[-1,1]$. To calculate the nodes and weights, the polynomials are integrated in this interval. Carl Gustav Jacob Jacobi discovered a connection between the quadrature rule and the orthogonal family of polynomials of Legendre Equation (5). As there is no closed form formula for quadrature weights and nodes, for many decades people could only use them by hand for small n, to use it they made reference to a table containing the weight and values of the nodes. In 1942, these values were only known up to n = 16, Sanz [2]. The more constant uses of computers have had progress for the algorithms used for problems that have not been able to be solved analytically, Gauss quadrature is given by Equation (6).

$$G_n(f) = \sum_{i=1}^{n} w_i^{(n)}\, f\left(x_i^{(n)}\right) \approx \int_{-1}^{1} f(x)dx \quad (6)$$

The truncation error for the Gauss numerical integration method can be found in his work by R. Zafar Iqbal, M. O. Ahmad, (2016) [6], for two nodes it is given by:

$$E_2(f) = \frac{f^4(c)}{135}\;, c \in [-1,1] \quad (7)$$

This is only for the calculation of an area with two weights and two nodes. Performing a variable change from the interval $[-1,1]$ to $[a,b]$ given by Equation (8).

$$x = \frac{b-a}{2}t + \frac{b+a}{2} \quad (8)$$

So, the integral of Equation (6) becomes

$$I = \int_a^b f(x)dx = \int_{-1}^{1}\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)\frac{b-a}{2}dt \quad (9)$$

The approximate area of Equation (9) is given by equation (10)

$$A_1 = \sum_{i=1}^{n} f\left(\frac{(b-a)x_i + (b+a)}{2}\right) w_i \quad (10)$$

Where n is the number of points that will be used, w_i are the weights, x_i are the nodes that are calculated from the respective polynomials.

When using the variable change, the error in this case is affected by $\frac{b-a}{2}$.

$$I(j) - G_n(f) = \frac{(b-a)^{2n+1}(n!)^4}{(2n+1)[(2n)!]} f^{(2n)}(\eta_c) \; a < \eta_c < b \quad (11)$$

The approximate truncation error can be written as follows.

$$E = \frac{(b-a)^{2n+1}(n!)^4 f^{(2n)}(\xi_n)}{(2n+1)[(2n)!]^3} \quad a < \xi_n < b \quad (12)$$

The nodes and weights Carl H. Love. obtains abscissa and weights of order N=2 to 100, and N=125,150,125,200, (1966)[7].

Table Num. 1: Shows the nodes and weights 2,3,4,5 and 8.

Number	Node	Weight
1	-0.577350269	1
2	0.577350269	1
Number	**Node**	**Weight**
1	-0.774596669	0.555555556
2	0	0.888888889
3	0.774596669	0.555555556
Number	**Node**	**Weight**
1	-0.861136312	0.347854845
2	-0.339981044	0.652145155
3	0.339981044	0.652145155
4	0.861136312	0.347854845
Number	**Node**	**Weight**
1	-0.906179846	0.236926885
2	-0.53846931	0.47862867
3	0	0.568888889
4	0.53846931	0.47862867
5	0.906179846	0.236926885
Number	**Node**	**Weight**
1	-0.960289856	0.101228536
2	-0.796666477	0.222381034
3	-0.52553241	0.313706646
4	-0.183434642	0.362683783
5	0.183434642	0.362683783
6	0.52553241	0.313706646
7	0.796666477	0.222381034
8	0.960289856	0.101228536

Table Num. 2: Shows the nodes and weight 16.

Number	Node	Weight
1	-0.989400935	0.027152459
2	-0.944575023	0.062253524
3	-0.865631202	0.095158512
4	-0.755404408	0.124628971
5	-0.617876244	0.149595989
6	-0.458016778	0.169156519
7	-0.281603551	0.182603415
8	-0.09501251	0.18945061
9	0.09501251	0.18945061
10	0.281603551	0.182603415
11	0.458016778	0.169156519
12	0.617876244	0.149595989
13	0.755404408	0.124628971
14	0.865631202	0.095158512
15	0.944575023	0.062253524
16	0.989400935	0.027152459

II. Methodology

In the field of mathematics, definite integrals represent a fundamental concept for calculating areas under curves and solving physical problems. However, there are cases in which finding the primitive of a definite integral is a complex or even impossible task. This is where numerical analysis, as a branch of mathematics dedicated to the development of computational methods to approximate solutions to mathematical problems.

Numerical analysis offers a wide range of tools to approximate the value of definite integrals, even those with no known primitive. Among the most common methods are the trapezoid rule, Simpson's rule and the Gauss quadrature method. These methods allow numerical results to be obtained with a high degree of precision, making them valuable tools for a wide variety of applications.

However, it is important to highlight that numerical analysis, despite its usefulness, is not without limitations. One of the main disadvantages is that numerical methods do not always provide information about the analytical behavior of the integral. In other words, while they can provide an accurate numerical approximation, it is not always possible to understand the exact form of the integral function from the numerical results.

Additionally, numerical methods can be computationally expensive, especially when high precision is required or when dealing with integrals with complex functions. In these cases, the use of numerical methods may require considerable computational resources, which may limit their applicability in practical contexts.

It is crucial to keep these limitations in mind when using numerical analysis to approximate primitive-free definite integrals. Although numerical methods offer a powerful tool to obtain numerical results, it is important to complement these results with a qualitative analysis of the integral in question, using tools of traditional calculus and mathematical intuition.

Numerical analysis provides a valuable contribution to the calculation of definite integrals, especially those with no known primitive. However, it is important to be aware of the limitations of these methods and use them in conjunction with qualitative analysis to gain a complete understanding.

III. Development

In this analysis, the results obtained through analytical integration are compared with those obtained using the Gauss method with different configurations of nodes and weights. The goal is to evaluate the accuracy of the Gaussian method compared to the exact solution provided by analytical integration.

1. $\int_0^{\pi} sin(x)dx = 2$
2. $\int_0^{\pi} sin(20x)dx = 0.1$
3. $\int_0^{1} x^{10000}dx = \frac{1}{10001} = 9.999000099990001e - 5$

Application 1.

Figure 4, shows the graph of the function in the interval $[a, b]$ in this case the area of $f(x) = sin(x)$ in the interval $[0, \pi]$.

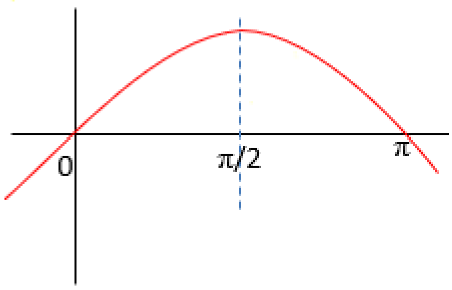

Figure 4: Represents the area under the curve $f(x) = sin(x)$

Table 3, shows the area obtained using 2, 3, 4, 5, node and the number of weights

Table Num. 3: Shows the area obtained using 2, 3, 4, 5 node and weights.

Number of nodes	Approximate area	Analytical solution	Percentage error
Area with two nodes	2.824107	2	41.20535
Area with three nodes	2.001389	2	0.06945
Area with four nodes	2.352066	2	17.6033
Area with five nodes	2	2	0

Application 2.

Figure 5, shows the graph of the function in the interval $[a, b]$ in this case the area of $f(x) = sin(20x)$ in the interval $[0, \pi]$.

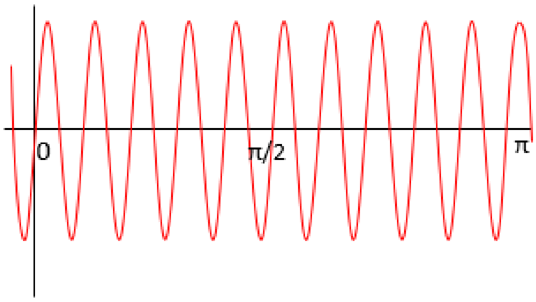

Figure 5: Represents the area under the curve $f(x) = sin(20x)$

Table 4, shows the area obtained using 2, 3, 4, 5, node and the number of weights

Table Num. 4: Shows the area obtained using 2, 3, 4, 5 node and weights.

Number of nodes	Approximate area	Analytical solution	Percentage error
Area with two nodes	-0.000004	0.1	5.0002
Area with three nodes	0.000001	0.1	4.99995
Area with four nodes	-0.006258	0.1	5.3129
Area with five nodes	-0.000001	0.1	5.00005

Application 3.

Figure 5, shows the graph of the function in the interval $[a, b]$ in this case the area of $f(x) = x^{10000}$ in the interval $[0,1]$.

Figure 6: Represents the area under the curve $f(x) = x^{10000}$

Table 6, shows the area obtained using 2, 3, 4, 5, 8, 16 node and the number of weights

Table Num. 5: Shows the area obtained using 2, 3, 4, 5,8,16 node and weights.

Number of nodes	Approximate area	Analytical solution	Percentage error
Area with two nodes	1.#INF00	1.00E-09	#¡VALOR!
Area with three nodes	1.#INF01	1.00E-09	#¡VALOR!
Area with four nodes	1.#INF02	1.00E-09	#¡VALOR!
Area with five nodes	1.#INF03	1.00E-09	#¡VALOR!

IV. Results And Discussion

In the field of calculus, the resolution of definite integrals using analytical methods can sometimes present a complex and tedious process to obtain the exact solution. In this article, the use of numerical methods is proposed as an alternative to address this type of problems.

A specific numerical method will be used for three different applications. In each case, it was observed that by increasing the number of nodes and abscissa, the approximation to the area of the real solution became more precise. This precision was evaluated by calculating the percentage of error compared to the value obtained by the analytical method.

In the first application, it was evident that as the nodes and weights increased, the numerical approximation became increasingly closer to the analytical result. In the second application, however, the program used failed to identify the presence of negative areas, which resulted in undesired values. Finally, in the third application, it was not possible to obtain an adequate approximation due to the magnitude of the power involved, which caused the values of the weights and nodes to tend to zero.

The results obtained in this study demonstrate the usefulness of numerical methods as complementary tools for solving definite integrals, particularly in those cases where the application of analytical methods is complex or impractical.

V. Conclusion

Although the Gauss-Legendre technique has proven to be an effective method for solving definite integrals in various applications, it is important to recognize that there are certain limitations that must be considered when implementing it.

In the specialized literature, cases have been reported where up to one hundred values of the nodes and weights necessary for the application of the method have been found. However, for applications 1, 2 and 3 described in the article, this strategy is not viable, since the same instability problem would arise.

In general, it is recommended to be cautious when applying the Gauss-Legendre technique with trigonometric functions that have a long period, as well as with combinations of these functions together with powers raised to a considerably large number. In such scenarios, the approximation to the integral can be significantly affected.

Despite these limitations, the Gauss-Legendre method continues to be a valuable tool for approximating definite integrals in a wide range of cases. In general, it offers accurate and efficient results, as long as the previously mentioned restrictions are taken into account.

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