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An Efficient Numerical Integration Schemes For Triangular Domain Integrals

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Abstract:
A domain integral is frequently encountered in a variety of domains, including geometric modeling, robotics, computer-aided design (CAD), computer-aided engineering (CAE), computer-aided manufacturing (CAM), and in FEM solution procedures for boundary value problems. It is well known that any arbitrary domain can be conveniently divided into triangles, allowing triangular domain integral formulas to be used to compute integrals for such domains. The purpose of this research is to present unique and effective numerical integration formulas for triangular domain integrals. These formulas also allow for the efficient computation of integrals for functions with singularities at the triangle vertices. Furthermore, this note aims to include a complete FORTRAN code based on the developed formulas, assessing its accuracy and efficiency across a wide range of real-world applications.
Keywords*: Triangular domain Integral; Boundary Value Problem; Linear triangle; Cubic triangle*

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I. Introduction

Most domain integrals in science and engineering either cannot be evaluated analytically or their evaluation requires tedious, lengthy calculations. The finite element method (FEM) has recently acquired popularity because of its ability to solve field problems with complex domains that would otherwise be intractable using other numerical methods[1-3, 9, 14–15, 18, 20, 22–24].

Among the stages of the FEM solution procedure, the evaluation of domain integrals is a pivotal task that requires more computing time. More details on the complexities of such computations can be seen in[5-7, 10–12, 19, 25, 27,29]. The employment, in FEM, of linear elements gives rise to the simple form of domain integrals to form element matrices. However, the use of higher-order as well as deformed finite elements generates a large number of rational integrals. Thus, in each of these situations, to compute element matrices, we need to evaluate numerically a significant number of integrals using costly numerical integration techniques[8, 13, 16, 17].

Among all the numerical integration techniques, Gaussian quadrature procedures are widely used to evaluate these integrals because of their correctness and computing efficiency[1, 20, 22, 23]. However, in order to achieve the necessary precision for triangle domain integrals, the existing Gaussian quadrature formulas, such as the 7-point and 13-point formulas, are inadequate[13]. A thorough analysis of this Gaussian quadrature rule limitation has also been done in works[4, 26, 28].

The adaptability of triangular (lower- and higher-order) elements is widely known. The increased use of triangle elements necessitates further development of numerical integration formulas for triangular domain integrals. It is to be noted that high-order Gaussian quadrature formulas exist for square domain integrals; extending these to triangle domains is extremely difficult. Translating triangle domain integrals into square-domain integrals leverages existing Gaussian quadrature to evaluate such triangular domain integrals[20, 21]. However, this technique results in time-consuming and laborious calculations for evaluating these resulting rational integrals.

Therefore, the task of this note is to present simple numerical techniques for which the resulting integrals will remain in the same form and can be computed efficiently with the desired accuracy. Furthermore, since any arbitrary domain can be easily discretized by triangles, any domain integral can be evaluated using the developed formulas for triangular domain integrals. As a result, such developed formulas will have wide-ranging applications in science and engineering, as well as in the Finite Element Method (FEM) for dealing with boundary value problems

II. Numerical Integration Formula Using Linear Triangles

Consider the triangular domain integral $\iint_{\Delta} F(x,y)dxdy$. The isoparametric transformations for both the domain and the integrand from the global (x,y) space in to the local (ξ,η) spaces are as the following:

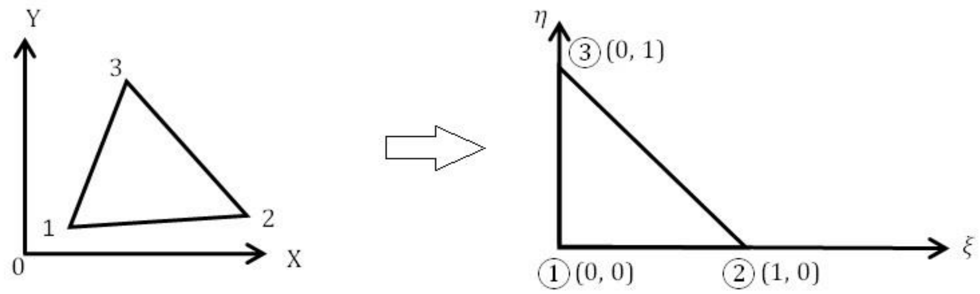
$$x=\sum_{i=1}^{3} x_i\, T_i(\xi,\eta),\quad y=\sum_{i=1}^{3} y_i\, T_i(\xi,\eta)\quad \text{and}\quad F(x,y)=\sum_{i=1}^{3} F_i\, T_i(\xi,\eta) \tag{1.1}$$

The Jacobean matrix of the transformation is: $\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$,

where, (x_i,y_i) = Coordinate of ith node,

$T_i(\xi,\eta)$ = Linear shape functions for triangular element,

$F_i=F(x_i,y_i)$, Functional value at node i.



(a) Triangle in (x,y) space (b) Mapped triangle in (ξ,η) space

Figure- 1.1: Transformation of a triangle in (x,y) space in to a unit triangle in (ξ,η) space.

We have three global nodes $1:(x_1,y_1)$, $2:(x_2,y_2)$, $3:(x_3,y_3)$ and the linear shape functions for unit triangle are $T_1(\xi,\eta)=1-\xi-\eta$, $T_2(\xi,\eta)=\xi$, $T_3(\xi,\eta)=\eta$. Then the transformation equations for x and y

$$x=\sum_{i=1}^{3} x_i T_i\,,\qquad y=\sum_{i=1}^{3} y_i T_i$$

Give rise

$$x=x_1+\xi(x_2-x_1)+\eta(x_3-x_1) \tag{1.2}$$

$$y=y_1+\xi(y_2-y_1)+\eta(y_3-y_1) \tag{1.3}$$

Similarly, the iso-parametric transformation for the integrand yields

$$F\ =F_1+\xi(F_2-F_1)+\eta(F_3-F_1) \tag{1.4}$$

The Jacobean matrix $J=\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}=\begin{bmatrix} (x_2-x_1) & (y_2-y_1) \\ (x_3-x_1) & (y_3-y_1) \end{bmatrix}$

And the determinant of the Jacobean matrix is

$|J|=\big((x_2-x_1)(y_3-y_1)-(y_2-y_1)(x_3-x_1)\big)$; [$\Delta_{123}$= Area of triangle in xy space]

Then, the domain integral under consideration can be written as:

$$\iint_{\Delta} F(x,y)dxdy=|J|\int_0^{\eta}\int_0^{1-\xi} F d\xi d\eta$$

$=|J|\int_0^{\eta}\int_0^{1-\xi}\big(F_1+\xi(F_2-F_1)+\eta(F_3-F_1)\big)d\xi d\eta$ [using $\int_0^{\eta}\int_0^{1-\xi}\xi^p\eta^q\, d\xi\, d\eta=\frac{p!q!}{(p+q+2)!}$]

$$=|J|\left(\frac{1}{2}F_1+\frac{1}{6}(F_2-F_1)+\frac{1}{6}(F_3-F_1)\right)$$

$$\iint_{\Delta} F(x,y)dxdy=\frac{|J|}{6}(F_1+F_2+F_3) \tag{1.5}$$

This is the main (generating) formula to obtain the general formula for the triangular domain integrals. That means the formula in Eq. (1.5) can be used when the domain is subdivided by one or more triangles. We followed the technique that the subdivision can be done by subdividing each of the sides of the triangle into m parts. Accordingly, we derived the general formula for the said domain integral.

Case-1: Now, if we subdivide each side of the triangle in to two ($m = 2$) parts as shown in the Fig.1.2 then we have, the determinant of Jacobean for each new triangle [e] is $|J_e| = \frac{1}{4}|J|$, total number of triangles $=$ 4, total number of nodes $= 6$.

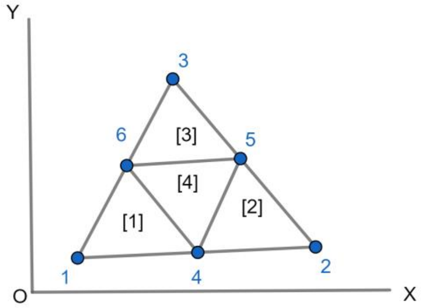


Figure 1.2: Triangular domain divided by 4 linear triangular elements

If we apply the formula given in Eq. (1.5) for each of the triangle then we obtain,
For element [1]:

$$\iint_{[1]} F(x,y)dxdy = |J_1|\frac{1}{6}(F_1 + F_4 + F_6) = |J|\frac{1}{24}(F_1 + F_4 + F_6) \qquad (1.5(a))$$

For element [2]:

$$\iint_{[2]} F(x,y)dxdy = |J_2|\frac{1}{6}(F_4 + F_2 + F_5) = |J|\frac{1}{24}(F_4 + F_2 + F_5) \qquad (1.5(b))$$

For element [3]:

$$\iint_{[3]} F(x,y)dxdy = |J_3|\frac{1}{6}(F_6 + F_5 + F_3) = |J|\frac{1}{24}(F_6 + F_5 + F_3) \qquad (1.5(c))$$

For element [4]:

$$\iint_{[4]} F(x,y)dxdy = |J_4|\frac{1}{6}(F_4 + F_5 + F_6) = |J|\frac{1}{24}(F_4 + F_5 + F_6) \qquad (1.5(d))$$

Adding $(1.5(a) - (d))$ for all the elements we get,

$$\iint_{\Delta} F(x,y)dxdy = \sum_{e=1}^{4} \iint_{[e]} F(x,y)dxdy$$

$$= \frac{|J|}{24}\left(F_1 + F_2 + F_3 + 3(F_4 + F_5 + F_6)\right) \qquad (1.6)$$

Case-2: Now, if we subdivide each side of the triangle into three ($m = 3$) parts as shown in the Fig.1.3 then we have, the determinant of Jacobean for each new triangle [e] is $|J_e| = \frac{1}{9}|J|$, number of elements $=$ 9 and number of nodes $=$ 10.

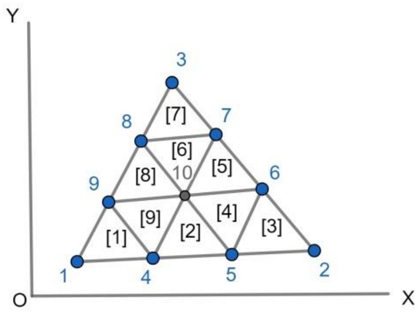


Figure 1.3: Triangular domain divided by 9 linear triangular elements

If we apply the previous concept here we get the following,

For element [1]: $\iint_{[1]} F(x,y)dxdy = \frac{|J|}{54}(F_1 + F_4 + F_9)$ $(1.5(e))$

For element [2]: $\iint_{[2]} F(x,y)dxdy = \frac{|J|}{54}(F_4 + F_5 + F_{10})$ $(1.5(f))$

For element [3]: $\iint_{[3]} F(x,y)dxdy = \frac{|J|}{54}(F_5 + F_2 + F_6)$ $(1.5(g))$

For element [4]: $\iint_{[4]} F(x,y)dxdy = \frac{|J|}{54}(F_5 + F_6 + F_{10})$ $(1.5(h))$

For element [5]: $\iint_{[5]} F(x,y)dxdy = \frac{|J|}{54}(F_{10} + F_6 + F_7)$ $(1.5(i))$

For element [6]: $\iint_{[6]} F(x,y)dxdy = \frac{|J|}{54}(F_{10} + F_7 + F_8)$ $(1.5(j))$

For element [7]: $\iint_{[7]} F(x,y)dxdy = \frac{|J|}{54}(F_8 + F_7 + F_3)$ $(1.5(k))$

For element [8]: $\iint_{[8]} F(x,y)dxdy = \frac{|J|}{54}(F_9 + F_{10} + F_8)$ $(1.5(l))$

For element [9]: $\iint_{[9]} F(x,y)dxdy = \frac{|J|}{54}(F_4 + F_{10} + F_9)$ $(1.5(m))$

Adding $(1.5(e) - (m))$ for all the elements we get,

$$\iint_{\Delta} F(x,y)dxdy = \sum_{e=1}^{9} \iint_{[e]} F(x,y)dxdy$$

$$\iint_{\Delta} F(x,y)dxdy = \frac{|J|}{54}(F_1 + F_2 + F_3 + 3(F_4 + F_5 + F_6 + F_7 + F_8 + F_9) + 6F_{10}) \quad (1.7)$$

If we continue the process for m subdivisions of each side of the triangle then we get the formula as:

$$\iint_{\Delta} F(x,y)dxdy = \frac{|J|}{6m^2}\left(\sum_{i=1}^{3} F_i + 3\sum_{i=4}^{3m} F_i + 6\sum_{i=3m+1}^{\frac{(m+1)(m+2)}{2}} F_i\right) \quad (1.8)$$

Where for 'm' subdivisions of each side of the triangle we have,

- Number of triangular elements $= m^2$
- Total number of nodes $= \frac{1}{2}(m+1)(m+2)$
- Total mid side nodes with three vertices $= 3m$
- Total internal nodes $= \frac{1}{2}(m+1)(m+2) - 3m = \frac{1}{2}(m-1)(m-2)$

This is the general formula for triangular domain integral. Notice that this formula requires the weighted sum of functional values at node i, the number of triangles that node i connects, and a multiplier $\frac{|J|}{6m^2}$. So, this formula is memorable and manually usable. If the integrand has singularities at mid-side nodes or at internal nodes, the situation can be overcome by increasing the number of subdivisions.

Algorithm Development: This FORTRAN code calculates the numerical integration over triangular surfaces by splitting the domain into $m \times m$ sub-triangles.

1. Specify the integrand function $F(U,V)$ and for rational integrand, $F(U,V)$ and $G(U,V)$.
2. Define the coordinates of the vertices of the triangle as $(xt_i,\ yt_i) = (x_i,\ y_i), i = 1,2,3$.
3. Calculate the Jacobean of the triangle.
4. Input the number of trials (MT) to evaluate the integral.
5. Initialize trial counter MT and refinement counter M.

Loop over the trials:

a. If the function is undefined at any of the corner nodes, adjust the corner nodes using a derived formula:

If $G(xt_1, yt_1) = 0$ then $xt_1 = \frac{2(m-1)x_1+x_2}{2m}$ and $yt_1 = \frac{2(m-1)y_1+y_3}{2m}$
Else if $G(xt_2, yt_2) = 0$ then $xt_2 = \frac{2(m-1)x_2+x_1}{2m}$ and $yt_2 = \frac{2(m-1)y_2+y_3}{2m}$
Else if $G(xt_3, yt_3) = 0$ then $xt_3 = \frac{2(m-1)x_3+x_2}{2m}$ and $yt_3 = \frac{2(m-1)y_3+y_1}{2m}$

b. Calculate the sum of function values at the corner nodes as

$$sum = F(xt_1, yt_1) + F(xt_2, yt_2) + F(xt_3, yt_3)$$

c. Calculate the mid-nodes on the sides for $i = 1, \cdots, m-1$ as
Initially $sum1 = 0$ and

$x_{a_i} = \frac{(m-i)x_1 + ix_2}{m}$ and $y_{a_i} = \frac{(m-i)y_1 + iy_2}{m}$
$x_{b_i} = \frac{(m-i)x_2 + ix_3}{m}$ and $y_{b_i} = \frac{(m-i)y_2 + iy_3}{m}$
$x_{c_i} = \frac{(m-i)x_1 + ix_3}{m}$ and $y_{c_i} = \frac{(m-i)y_1 + iy_3}{m}$

$$sum1 = sum1 + 3(F(x_{a_i}, y_{a_i}) + F(x_{b_i}, y_{b_i}) + F(x_{c_i}, y_{c_i}))$$

d. If $m > 2$, calculate function values at nodes inside the triangle for $l = 2, \cdots, m-1$ *and* $j = 1, \cdots, l-1$ as
Initially $sum2 = 0$ and

$$x_t(l,j) = \frac{(l-j)x_{a_l} + jx_{c_l}}{l}$$

$$y_t(l,j) = \frac{(l-j)y_{a_l} + jy_{c_l}}{l}$$

$$sum2 = sum2 + 6\, F(x_t(l,j), y_t(l,j))$$

6. Finally, Then the integral value for m subdivision is (FORMULA):

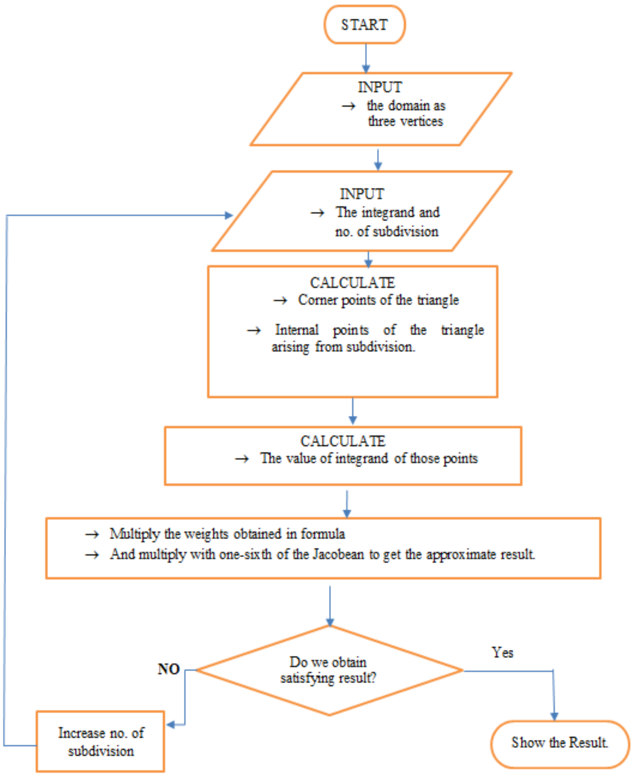
$$\iint_{\Delta} F(x,y)dxdy = \frac{|J|}{6m^2}\left[\sum_{i=1}^{3} F_i + 3\sum_{i=4}^{3m} F_i + 6 \sum_{i=3m+1}^{\frac{(m+1)(m+2)}{2}} F_i\right]$$

$$= \frac{|J|}{6m^2}[sum + sum1 + sum2]$$

7. Increment the refinement counter.

The algorithm starts by defining the triangular domain and iteratively refines the triangulation to improve the accuracy of the numerical integration.

Pseudo Code for Computer Code Implementation:



Computer Program: A complete FORTRAN program "AIOTS13.FOR" has been developed and which is appended in appendix.

III. Result

Verification of the Formula (Eq. (1.8)) with Application Examples: Consider the integral $\iint_{\Delta} F(x,y)dxdy$ with vertices $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ as shown in the following table. Computed results for different values of m (number of subdivisions) are listed in the following Table no 1.

Table no 1: Examples of triangular domain integrals evaluated by formula given by Eq. (1.8)

No of Sub-Division (m)	No of nodes $\frac{(m+1)(m+2)}{2}$	$F(x,y) = x+y$ (0,0), (1,0), (1,2)	$F(x,y) = 2-3x+xy$ (0,0), (1,0), (1,3)	$F(x,y) = (x+y)^{\frac{1}{2}}$ (0,0), (1,0), (0,1)	$F(x,y) = e^{\lvert x+y-1\rvert}$ (0,0), (1,0), (1,1)	$F(x,y) = (x^2+y^2)^{-\frac{1}{2}}$ (0,0), (1,0), (1,1)	$F(x,y) = (x+y)^{-\frac{1}{2}}$ (0,0), (1,0), (0,1)
1	3	1.33..	1.5	0.333333	1.072761	0.520220	0.50
2	6	1.33..	1.21875	0.385110	0.805370	0.668635	0.603553
3	10	1.33..	1.166667	0.393742	0.756858	0.732580	0.631282
4	15	1.33..	1.148438	0.396601	0.739955	0.767191	0.643283
5	21	1.33..	1.14	0.397878	0.732145	0.788791	0.649739
6	28	1.33..	1.135417	0.398554	0.727906	0.803536	0.653679
7	36	1.33..	1.132653	0.398953	0.725351	0.814236	0.656292
8	45	1.33..	1.130859	0.399208	0.723694	0.822354	0.658130
9	55	1.33..	1.12963	0.399381	0.722558	0.828721	0.659481
10	66	1.33..	1.12875	0.399503	0.721745	0.833849	0.660509
15	136	1.33..	1.126667	0.399786	0.719821	0.849418	0.663273
Exact		1.33…	1.1250	0.4	0.71828183	0.88137358	0.6666667

Result Discussion: Some important remarks from the Table no 1 are as follows:

- Integration formula can evaluate exactly the integral when the integrand is a polynomial by increasing the number subdivisions.
- Depending on the nature of the integrand the formula is slow in convergent.
- Formula can be applied for the integrals when the integrand has the singularity at any vertices of the triangle.

In the next section we wish to present the integration formulae that will be more efficient and applicable for evaluating the integral of the integrand with or without singularity at vertices.

IV. Numerical Integration Formula using Cubic Triangle

Here, we wish to derive the numerical integration formula by use of cubic triangles as show in the Fig. 2.1. Specifically, the main formula will be derived first and that will be implemented similarly as it is shown in the previous case.

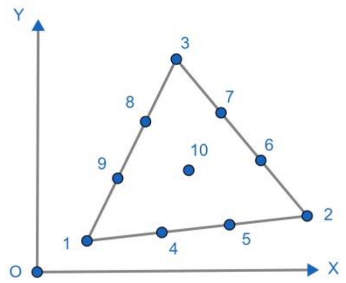


Figure 2.1: Triangular domain subdivided into one cubic element

Derivation of the Formula: Consider the triangular domain integral $\iint_{\Delta} F(x,y)dxdy$. Then isoparametric transformations for both the domain and the integrand from the global (x,y) space in to the local (ξ,η) spaces are as the following:

$$x = \sum_{i=1}^{10} x_i\, T_i(\xi,\eta), \quad y = \sum_{i=1}^{10} y_i\, T_i(\xi,\eta) \quad \text{and} \quad F(x,y) = \sum_{i=1}^{10} F_i\, T_i(\xi,\eta) \tag{2.1}$$

and the Jacobean matrix $\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$.

Where,

$T_i(\xi,\eta) =$ Cubic shape functions for a unit triangle.

$(x_i, y_i) =$ Nodal coordinate of ith node.

$F_i = F(x_i, y_i)$, Functional value of the integrand.

As the sides of the triangles are straight lines determinant of the Jacobean is, $|J| = 2\Delta_{123}$

and $\iint_{\Delta} F(x,y)dxdy = |J| \int_0^{\eta} \int_0^{1-\xi} F\, d\xi d\eta$ (2.2)

Shape functions for 10 noded triangles are,

$T_1(\xi,\eta) = \frac{1}{2}(1-3\xi-3\eta)(2-3\xi-3\eta)(1-\xi-\eta)$

$T_2\,(\xi,\eta) = \frac{1}{2}\xi(1-3\xi)(2-3\xi)$

$T_3\,(\xi,\eta) = \frac{1}{2}\eta(1-3\eta)(2-3\eta)$

$T_4\,(\xi,\eta) = \frac{9}{2}\xi(1-\xi-\eta)(2-3\xi-3\eta)$

$T_5\ (\xi,\eta) = -\frac{9}{2}\xi(1-3\xi)(1-\xi-\eta)$
$T_6\ (\xi,\eta) = -\frac{9}{2}\xi\eta(1-3\xi)$
$T_7\ (\xi,\eta) = -\frac{9}{2}\xi\eta(1-3\eta)$
$T_8\ (\xi,\eta) = -\frac{9}{2}\eta(1-3\eta)(1-\xi-\eta)$
$T_9\ (\xi,\eta) = \frac{9}{2}\eta(1-\xi-\eta)(2-3\xi-3\eta)$
$T_{10}\ (\xi,\eta) = 27\xi\eta(1-\xi-\eta)$

Then, the domain integral under consideration can be written as (from Eq.2.1):

$$F = F_1T_1 + F_2T_2 + F_3T_3 + F_4T_4 + F_5T_5 + F_6T_6 + F_7T_7 + F_8T_8 + F_9T_9 + F_{10}T_{10}$$

$$\begin{aligned}
F(x,y) = {} & F_1\left(1-\frac{11}{2}\xi-\frac{11}{2}\eta+9\xi^2+18\xi\eta+9\eta^2-\frac{9}{2}\xi^3-\frac{27}{2}\xi^2\eta-\frac{27}{2}\xi\eta^2-\frac{9}{2}\eta^3\right)+F_2\left(\xi-\frac{9}{2}\xi^2+\frac{9}{2}\xi^3\right) \\
& + F_3\left(\eta-\frac{9}{2}\eta^2+\frac{9}{2}\eta^3\right)+F_4\left(9\xi-\frac{45}{2}\xi^2-\frac{45}{2}\xi\eta+\frac{27}{2}\xi^3+27\xi^2\eta+\frac{27}{2}\xi\eta^2\right) \\
& + F_5\left(-\frac{9}{2}\xi+18\xi^2+\frac{9}{2}\xi\eta-\frac{27}{2}\xi^3-\frac{27}{2}\ \xi^2\eta\right)+F_6\left(-\frac{9}{2}\xi\eta+\frac{27}{2}\xi^2\eta\right) \\
& + F_7\left(-\frac{9}{2}\xi\eta+\frac{27}{2}\xi\eta^2\right)+F_8\left(-\frac{9}{2}\eta\ +\frac{9}{2}\xi\eta+18\eta^2-\frac{27}{2}\xi\eta^2-\frac{27}{2}\eta^3\right) \\
& + F_9\left(9\eta-\frac{45}{2}\xi\eta-\frac{45}{2}\eta^2+\frac{27}{2}\xi^2\eta+27\xi\eta^2+\frac{27}{2}\eta^3\right)+F_{10}(27\xi\eta-27\xi^2\eta-27\xi\eta^2) \\
= {} & F_1+\xi\left(-\frac{11}{2}F_1+F_2+9F_4-\frac{9}{2}F_5\right)+\eta\left(-\frac{11}{2}F_1+F_3-\frac{9}{2}F_8+9F_9\right)+\xi^2\left(9F_1-\frac{9}{2}F_2-\frac{45}{2}F_4+18F_5\right) \\
& + \eta^2\left(9F_1-\frac{9}{2}F_3+18F_8-\frac{45}{2}F_9\right) \\
& + \xi\eta\left(18F_1-\frac{45}{2}F_4+\frac{9}{2}F_5-\frac{9}{2}F_6-\frac{9}{2}F_7+\frac{9}{2}F_8-\frac{45}{2}F_9+27F_{10}\right) \\
& + \xi^3\left(-\frac{9}{2}F_1+\frac{9}{2}F_2+\frac{27}{2}F_4-\frac{27}{2}F_5\right)+\eta^3\left(-\frac{9}{2}F_1+\frac{9}{2}F_3+\frac{27}{2}F_9-\frac{27}{2}F_8\right) \\
& + \xi^2\eta\left(-\frac{27}{2}F_1+27F_4-\frac{27}{2}F_5+\frac{27}{2}F_6+\frac{27}{2}F_9-27F_{10}\right) \\
& + \xi\eta^2\left(-\frac{27}{2}F_1+\frac{27}{2}F_4+\frac{27}{2}F_7-\frac{27}{2}F_8+27F_9\right. \\
& \left. -27F_{10}\right)
\end{aligned} \tag{2.3}$$

Using Eq. (2.3) in Eq. (2.2) we get the following,

$$|J|\int_0^{\eta}\int_0^{1-\xi}F d\xi d\eta = |J|\int_0^{\eta}\int_0^{1-\xi}\Big[F_1+\xi\left(-\frac{11}{2}F_1+F_2+9F_4-\frac{9}{2}F_5\right)+\eta\left(-\frac{11}{2}F_1+F_3-\frac{9}{2}F_8+9F_9\right)+ \xi^2\left(9F_1-\frac{9}{2}F_2-\frac{45}{2}F_4+18F_5\right)+\eta^2\left(9F_1-\frac{9}{2}F_3+18F_8-\frac{45}{2}F_9\right)+\xi\eta\left(18F_1-\frac{45}{2}F_4+\frac{9}{2}F_5-\frac{9}{2}F_6-\frac{9}{2}F_7+ \frac{9}{2}F_8-\frac{45}{2}F_9+27F_{10}\right)+\xi^3\left(-\frac{9}{2}F_1+\frac{9}{2}F_2+\frac{27}{2}F_4-\frac{27}{2}F_5\right)+\eta^3\left(-\frac{9}{2}F_1+\frac{9}{2}F_3+\frac{27}{2}F_9-\frac{27}{2}F_8\right)+ \xi^2\eta\left(-\frac{27}{2}F_1+27F_4-\frac{27}{2}F_5+\frac{27}{2}F_6+\frac{27}{2}F_9-27F_{10}\right)+\xi\eta^2\left(-\frac{27}{2}F_1+\frac{27}{2}F_4+\frac{27}{2}F_7-\frac{27}{2}F_8+27F_9- 27F_{10}\right)\Big]d\xi\ d\eta \tag{2.4}$$

Using the law, $\int_0^{\eta}\int_0^{1-\xi}\xi^p\eta^q\ d\xi\ d\eta = \frac{p!q!}{(p+q+2)!}$ in Eq. (2.4) we get,

$$|J|\int_0^{\eta}\int_0^{1-\xi}F\ d\xi d\eta$$

$$= |J|\left[\frac{1}{2}F_1 + \frac{1}{6}\left(-\frac{11}{2}F_1 + F_2 + 9F_4 - \frac{9}{2}F_5\right) + \frac{1}{6}\left(-\frac{11}{2}F_1 + F_3 - \frac{9}{2}F_8 + 9F_9\right)\right.$$
$$+ \frac{1}{12}\left(9F_1 - \frac{9}{2}F_2 - \frac{45}{2}F_4 + 18F_5\right) + \frac{1}{12}\left(9F_1 - \frac{9}{2}F_3 + 18F_8 - \frac{45}{2}F_9\right)$$
$$+ \frac{1}{24}\left(18F_1 - \frac{45}{2}F_4 + \frac{9}{2}F_5 - \frac{9}{2}F_6 - \frac{9}{2}F_7 + \frac{9}{2}F_8 - \frac{45}{2}F_9 + 27F_{10}\right)$$
$$+ \frac{1}{20}\left(-\frac{9}{2}F_1 + \frac{9}{2}F_2 + \frac{27}{2}F_4 - \frac{27}{2}F_5\right) + \frac{1}{20}\left(-\frac{9}{2}F_1 + \frac{9}{2}F_3 + \frac{27}{2}F_9 - \frac{27}{2}F_8\right)$$
$$+ \frac{1}{60}\left(-\frac{27}{2}F_1 + 27F_4 - \frac{27}{2}F_5 + \frac{27}{2}F_6 + \frac{27}{2}F_9 - 27F_{10}\right)$$
$$\left. + \frac{1}{60}\left(-\frac{27}{2}F_1 + \frac{27}{2}F_4 + \frac{27}{2}F_7 - \frac{27}{2}F_8 + 27F_9 - 27F_{10}\right)\right]$$

$$|J|\int_0^{\eta}\int_0^{1-\xi} F\,d\xi\,d\eta$$
$$= |J|[F_1\left(\frac{1}{2} - \frac{11}{12} - \frac{11}{12} + \frac{9}{12} + \frac{9}{12} + \frac{18}{24} - \frac{9}{40} - \frac{9}{40} - \frac{27}{120} - \frac{27}{120}\right) + F_2\left(\frac{1}{6} - \frac{9}{24} + \frac{9}{40}\right)$$
$$+ F_3\left(\frac{1}{6} - \frac{9}{24} + \frac{9}{40}\right) + F_4\left(\frac{9}{6} - \frac{45}{24} - \frac{45}{48} + \frac{27}{40} + \frac{27}{120} + \frac{27}{60}\right)$$
$$+ F_5\left(-\frac{9}{12} + \frac{8}{12} + \frac{9}{48} - \frac{27}{40} - \frac{27}{120}\right) + F_6\left(-\frac{9}{48} + \frac{27}{120}\right) + F_7\left(-\frac{9}{48} + \frac{27}{120}\right)$$
$$+ F_8\left(-\frac{9}{12} + \frac{8}{12} + \frac{9}{48} - \frac{27}{40} - \frac{27}{120}\right) + F_9\left(\frac{9}{6} - \frac{45}{24} - \frac{45}{48} + \frac{27}{40} + \frac{27}{120} + \frac{27}{60}\right)$$
$$+ F_{10}\left(\frac{27}{24} - \frac{27}{60} - \frac{27}{60}\right)]$$
$$= |J|\left[\frac{1}{60}(F_1 + F_2 + F_3) + \frac{3}{80}(F_4 + F_5 + F_6 + F_7 + F_8 + F_9) + \frac{9}{40}F_{10}\right]$$
$$\therefore \iint_\Delta F(x,y)\,dxdy = |J|\left[\frac{1}{60}(F_1 + F_2 + F_3) + \frac{3}{80}(F_4 + F_5 + F_6 + F_7 + F_8 + F_9) + \frac{9}{40}F_{10}\right]$$

$$= \frac{|J|}{240}[\mathbf{4(F_1 + F_2 + F_3) + 9(F_4 + F_5 + F_6 + F_7 + F_8 + F_9) + 54\,F_{10}}] \qquad (2.5)$$

Test Examples: Consider the some integrals $\iint_\Delta F(x,y)dxdy$ to evaluate with vertices $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ by the above Formula (Eq. (2.5)). The followings listed in Table no 2 have been calculated by computer program using the formula:

Table no 2: Examples of triangular domain integrals evaluated by formula given by Eq. (2.5)

Integrand $\boldsymbol{F(x,y)}$	Vertices of the domain	Computed approximate value of the integral	Exact value of the integral
$\mathbf{2-3x+xy}$	(0,0), (1,0), (1,3)	1.1250	1.1250
$(\boldsymbol{x+y})^{\frac{1}{2}}$	(0,0), (1,0), (0,1)	0.396583	0.4
$\boldsymbol{e}^{\lvert x+y-1\rvert}$	(0,0), (1,0), (0,1)	0.7184018	0.71828183
$(\boldsymbol{x}^2+\boldsymbol{y}^2)^{-\frac{1}{2}}$	(0,0), (1,0), (1,1)	0.70874297	0.88137358
$(\boldsymbol{x+y})^{-\frac{1}{2}}$	(0,0), (1,0), (1,1)	0.62232725	0.6666667

V. Result

Result Discussion: Some important remarks can be drawn from the Table no 2 as in the following:

1. Numerical integration formula employing cubic triangular element is faster and gives exact result for the integrals of polynomial integrands.
2. Depending on the nature of the non-polynomial integrand the convergence is faster than other derived formula.
3. Formula can be applied for the integrals when the integrand has the singularity at any vertices of the triangle.

In the next section we wish to present the integration formula that will be more efficient and applicable for evaluating the integral of the integrand with or without singularity at vertices.

VI. Conclusion

For the first time, we have presented numerical integration two formulae for triangular domain integrals which are analogous in nature to trapezoidal, Simpsons and Weddles formulae which are applicable for one dimensional domain integrals. Integration formulae (given in Eq. (1.8) and Eq. (2.5)) so presented employing linear and cubic triangles are applicable to evaluate the triangular domain integrals of integrand with and without having singularity at vertices of the triangle. Through several test cases it is investigated that the desired accuracy of the domain integrals can be obtained. For the general purpose, we believe that the second integration formula will find better application in many areas of science and engineering.

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A.1. Appendix

We, for clarity and reference, are appending FORTRAN code for the first order integration formula. We hope that the code for the 3rd order formula can be developed on the same way.

<u>Program for 1st order formula:</u>

```
C       PROGRAM AIOTS13.FOR
C ALGORITHMIC INTEGRATION OVER TRIANGULAR SURFACES SPLITING THE DOMAIN
C INTO M*M SUB-TRIAGLES AND AT NODE (X1, Y1) FUNCTION IS UNDEFINED.
C       ============================================================
      PARAMETER (N=500)
      DOUBLE PRECISION X(3),Y(3),SUM,SUM1,SUM2,XA(N),XB(N),XC(N),
     1 YA(N),YB(N),YC(N),AX,BX,CX,AY,BY,CY,XT(N,N),
     2  YT(N,N),TX,TY,INTV,U,V,XT1,YT1,XT2,YT2,XT3,YT3,EXACT,ERR
      Real F, G
C      HERE THE INTEGRAND SHOULD BE GIVEN AS STATEMENT FUNCTION
      G(U,V)=SQRT(U+V)
      F(U,V)=1.D0/G(U,V)

       OPEN( UNIT =2, FILE= 'a1.dat')
       OPEN( UNIT =3, FILE= 'a2.dat')
       EXACT=0.666667D0
C      ============================================================
C
C THIS PART WILL COMPUTE MID NODES ON THE SIDES AND FUNCTION VALUES AT
C THESE NODES AND MULTIPLIES BY 3 AND SUM THE PRODUCTS. TO DO SO ONLY
C CO-ORDINATES OF CORNER NODES ARE REQUIRED AS INPUTS.
C
      DO 2 JJ=1,3
       WRITE(*,3) JJ
 3      FORMAT('TPYE X AND Y COORDINATES OF NODE:-',I2)
       READ*,X(JJ),Y(JJ)
 2     CONTINUE
      AREA=(X(2)-X(1))*(Y(3)-Y(1))-(X(3)-X(1))*(Y(2)-Y(1))
C
C      THIS SECTION WILL TEST WHETHER THE FUNCTION IS DEFINED AT EACH
C      CORNER NODES OR NOT AND ON THAT BASIS NEW CORNER NODES WILL BE
C      CALCULATED USING THE DERIVED FORMULA
        PRINT*,'TYPE THE NUMBER OF TRIALS TO EVALUATE THE INTEGRAL MT'
           READ*,MT
            M=1
          XT1=X(1)
       YT1=Y(1)
          XT2=X(2)
       YT2=Y(2)
          XT3=X(3)
       YT3=Y(3)
      DO 5 MT1=1,MT
           WRITE(3,*)
C
          IF(G(XT1,YT1).EQ.0.D0) THEN
           XT1=((2.*M-1.)*X(1)+X(2))/DBLE(2.*M)
           YT1=((2.*M-1.)*Y(1)+Y(3))/DBLE(2.*M)
```

```
            ELSE IF(G(XT2,YT2).EQ.0.D0) THEN
             XT2=((2.*M-1.)*X(2)+X(1))/DBLE(2.*M)
             YT2=((2.*M-1.)*Y(2)+Y(3))/DBLE(2.*M)
            ELSE IF(G(XT3,YT3).EQ.0.D0) THEN
             XT3=((2.*M-1.)*X(3)+X(2))/DBLE(2.*M)
             YT3=((2.*M-1.)*Y(3)+Y(1))/DBLE(2.*M)
      END IF
           WRITE(3,*)XT1,YT1
           WRITE(3,*)XT2,YT2
           WRITE(3,*)XT3,YT3

            SUM=F(XT1,YT1)+F(XT2,YT2)+F(XT3,YT3)
c          PRINT*,'SUM=',SUM
c          PRINT*,'CORNER NODE:'
c          PRINT*,'XT1=',XT1, 'XT2=', XT2, 'XT3=',XT3
c          PRINT*,'YT1=',YT1, 'YT2=', YT2, 'YT3=',YT3
C           CALCULATION OF SIDE MID NODES
             SUM1=0.D0
             DO 7 I=1,M-1
               XA(I)=((M-I)*X(1)+I*X(2))/DBLE(M)
               YA(I)=((M-I)*Y(1)+I*Y(2))/DBLE(M)
               XB(I)=((M-I)*X(2)+I*X(3))/DBLE(M)
               YB(I)=((M-I)*Y(2)+I*Y(3))/DBLE(M)
          XC(I)=((M-I)*X(1)+I*X(3))/DBLE(M)
               YC(I)=((M-I)*Y(1)+I*Y(3))/DBLE(M)
           WRITE(3,*)XA(I),YA(I)
           WRITE(3,*)XA(I),YB(I)
           WRITE(3,*)XC(I),YC(I)

c            WRITE(*,*)I,'XA=',XA(I),I,'YA=',YA(I),I,'XB=',XB(I),I,'YB=',
c          1   YB(I),I,'XC=',XC(I),I,'YC=',YC(I)
C            WRITE(2,8)I,XA(I),I,YA(I),I,XB(I),I,YB(I),I,XC(I),I,YC(I)
C 8      FORMAT(2X,'XA(',I2,')=',D18.10,2X,'YA(',I2,')=',D18.10,/,
C    1   2X,'XB(',I2,')=',D18.10,2X,'YB(',I2,')=',D18.10,/,
C    2   2X,'XC(',I2,')=',D18.10,2X,'YC(',I2,')=',D18.10)
               AX=XA(I)
               AY=YA(I)
               BX=XB(I)
               BY=YB(I)
               CX=XC(I)
               CY=YC(I)
               SUM1=SUM1+3.D0*(F(AX,AY)+F(BX,BY)+F(CX,CY))
 7           CONTINUE
           IF(M.GT.2) THEN
C            WRITE(2,9)
C 9      FORMAT(/,'CO-ORDINATES OF NODES INSIDE THE TRIANGLE ARE AS:',/)
           SUM2=0.D0
           DO 10 L=2,M-1
           DO 11 J=1,L-1
             XT(L,J)= ((L-J)*XA(L)+J*XC(L))/DBLE(L)
             YT(L,J)= ((L-J)*YA(L)+J*YC(L))/DBLE(L)
       TX=XT(L,J)
             TY=YT(L,J)
           WRITE(3,*)TX,TY
C          WRITE(2,12) L,L,J,TX,L,J,TY
C12        FORMAT(2X,'ON THE LINE- ',I2,2X,'XT(',I2,I2,')=',D18.10,2X,
C    1   'YT(',I2,I2,')=',D18.10)
            SUM2=SUM2+6.D0*F(TX,TY)
 11        CONTINUE
```

```
 10        CONTINUE
           ELSE SUM2=0.D0
           END IF
           INTV=AREA*(SUM+SUM1+SUM2)/(6.D0*DBLE(M**2))
           ERR=DABS(INTV-EXACT)
     WRITE(2,20) MT1,M,INTV,ERR
 20   FORMAT(3X,'FOR TRIAL :',I3,2X,'M=',I3,2X,'INTV =',D18.10,2X,
    1'ERR =',D18.10)
      M=M+1
 5         CONTINUE
      PRINT*, 'YOUR DATA FILE NAME ', OUTFILE
      STOP
      END
C===============================****===============================
```