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Lagrangian Mechanics of a Double Atwood Machine

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Abstract
This experiment investigates the mechanics and conserved quantities of a Double Atwood Machine using Lagrangian mechanics and principles of classical dynamics. The experimental setup, comprising measured masses, precision pulleys, and ultrasonic sensors, was designed to model and verify theoretical predictions. Displacement data was collected using an Arduino Uno microcontroller with an HC-SR04 ultrasonic sensor, and the motion of the system was recorded across multiple trials for three distinct masses. Quadratic regression was applied to displacement-time data, yielding equations of motion that align with theoretical expectations. The conserved quantities of the system were calculated and verified by comparing experimental data with derived equations, highlighting minimal deviations indicative of system fidelity. This study demonstrates the utility of experimental setups for validating classical mechanics while addressing minor limitations in the apparatus. Visualizations through Desmos graphing tools and detailed mathematical analysis illustrate the system's behavior, emphasizing the practical application of conserved quantities in dynamic systems.[1]

I. Introduction

As previously discussed, this investigation will focus on Lagrangian mechanics and conserved quantities. Before the completion of this experiment, it is necessary to have a basic understanding of Lagrangian mechanics and conserved quantities. Beginning with the former, Lagrangian mechanics is an approach to classical mechanics that utilizes calculus of variations, and in particular, the idea of least action, or stationary integrals. This concept is that in any system, the path taken will be the one that minimizes the action.[2] This is shown in the equation below, with the path taken being from point x_1 to x_2, and the action integral being S. This is the general form for the action integral.[3]

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]dx$$

This is the fundamental idea of the calculus of variations and is very important regarding Lagrangian mechanics. In Lagrangian mechanics, the action is the integral of the difference between potential and kinetic energy, or more particularly:

$$S = \int_{x_1}^{x_2} (K(y(x), y'(x), x) - U((y(x), y'(x), x))dx$$

Here, the kinetic energy is represented by K, and the potential energy by U. Thus the path taken in minimizing the difference of the two. Here the integrand is known as the Lagrangian, which is given by $L = K - U$. From the original equation, a derivation can be taken that utilizes derivatives and the idea of finding minima. This provides the Euler-Lagrange equation, which is the relation $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$. This is the fundamental equation used to solve systems via Lagrangian mechanics, with $f = K - U$.

This report will investigate conserves quantities, or quantities in the system that remain constant throughout the course of motion. Systems often have conserved quantities due to certain symmetries, which is described in Noether's Theorem.[4] In particular, the conserved quantity is described by $C = \sum \frac{\partial L}{\partial \dot{s}_i}\frac{\partial s_i}{\partial \varsigma}$, where s there is a change of coordinates from the initial generalized coordinate q, described y above. That is, $s_i = q_i(\varsigma)$.

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[1] Desmos. *Desmos Graphing Calculator*. Desmos, Inc., 2024.
[2] Feynman, Richard. "The Feynman Lectures on Physics Vol. II Ch. 19: The Principle of Least Action." Feynman Lectures, www.feynmanlectures.caltech.edu/II_19.html. Accessed 1 Dec. 2024.
[3] Taylor, John R. Classical Mechanics. University Science Books, 2005.
[4] Baez, John. "Noether's Theorem in a Nutshell." UCR Math, math.ucr.edu/home/baez/noether.html. Accessed 1 Dec. 2024.

This experiment plans to compare these concepts described above to the real world. The diagram in **Figure 01** will be observed, with the conserved quantity of the system being measured. For this comparison to be made, the conserved quantity of the system must be calculated analytically.

To solve this problem, the Lagrangian, which is kinetic energy minus potential energy, must first be calculated. Using the designed coordinate system described in the diagram, the kinetic energy, based on the formula $K = \frac{1}{2}mv^2$, will be:

$$K = \frac{1}{2}(0.2)\dot{x}^2 + \frac{1}{2}(0.04)(\dot{x} - \dot{y})^2 + \frac{1}{2}(0.1)(\dot{x} + \dot{y})^2$$

From here, potential energy is calculated using the formula $U = mg\Delta h$. This results in the potential energy below.

$$U = (0.2)g(L - x) + (0.1)g(x + y) + (0.04)g(x + (l - y))$$

Now, the Lagrangian is just given by the difference of these terms. This gives the result shown below.

$$L = \frac{1}{2}(0.2)\dot{x}^2 + \frac{1}{2}(0.04)(\dot{x} - \dot{y})^2 + \frac{1}{2}(0.1)(\dot{x} + \dot{y})^2 - (0.2)g(L - x) - (0.1)g(x + y) - (0.04)g\big(x + (l - y)\big)$$

Now, if a displacement ϵ is caused to y and x, it is desired to see how the system will change. Applying this translation to each variable, it is seen that $x \to x + \epsilon, y \to y + 2\epsilon$. Since y is dependent on x, increasing the x coordinate by ϵ increases y by ϵ. Thus $y \to y + 2\epsilon$.

From here, based on the formula for conserved quantities the derivative of both variables with respect to ϵ is taken, resulting in $\frac{\partial x}{\partial \epsilon} = 1$ and $\frac{\partial y}{\partial \epsilon} = 2$.

Thus, the conserved quantity of the system, given by $C = \sum \frac{\partial L}{\partial \dot{s}_i}\frac{\partial s_i}{\partial \varsigma}$, is $C = \frac{\partial L}{\partial \dot{x}}\frac{\partial x}{\partial \epsilon} + \frac{\partial L}{\partial \dot{y}}\frac{\partial y}{\partial \epsilon} = constant$. Using the values shown above, is the same as $C = \frac{\partial L}{\partial \dot{x}} + (2)\frac{\partial L}{\partial \dot{y}}$. Now, evaluating for the two unknown terms, the derivative of the Lagrangian is taken with respect to $\dot{x}$. This results in $\frac{\partial L}{\partial \dot{x}} = (0.2 + 0.04 + 0.1)\dot{x} + (0.1 - 0.04)\dot{y}$. This can be simplified to $\frac{\partial L}{\partial \dot{x}} = (0.34)\dot{x} + (0.06)\dot{y}$ Now doing the same with $\dot{y}$, $\frac{\partial L}{\partial \dot{y}} = (-0.04 + 0.1)\dot{x} + (0.1 + 0.04)\dot{y}$. This is the same as $\frac{\partial L}{\partial \dot{y}} = (0.06)\dot{x} + (0.14)\dot{y}$. Now, plugging these into the formula previously shown, $C = \frac{\partial L}{\partial \dot{x}} + (2)\frac{\partial L}{\partial \dot{y}}$, the final conserved quantity of $C = 0.46\dot{x} + 0.34\dot{y}$ is derived. This conserved quantity will be measured experimentally in this system in order to determine the accuracy of this calculation based on the theory discussed previously.

II. Methods

This experiment aims to verify the mechanics of the Double Atwood Machine, and more specifically, the conserved quantities of the system. A depiction of the system is displayed below in **Figure 01**.

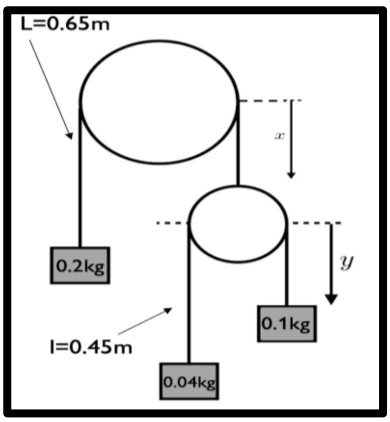

Figure 01: Double Atwood Machine system to be tested.

To model this system in the real world, the masses used were 200g, 100g, and 40g masses provided by the Georgia Institute of Technology School of Physics. The rope sections of length 0.65m and 0.45m were pieces of rope measured and cut from a 1/8-inch spool of rope (RELIABILT 0.125-in x 48-ft Braided Nylon Rope). The circular masses on which the taught rope rested was a 15mm wheel diameter pulley rated for a 35kg load (YAMASO, 304 Stainless Steel M15 Silver Single Pulley).

The 0.1kg and 0.04kg masses were tied via a hook on the top of the mass to each end of the 0.45m rope. The rope measured 0.45m in length from the approximate center of masses for each mass. This rope was then laid to rest on the grove of the lower pulley in Figure 01. The 0.2kg mass was then tied to one end of the 0.65m rope and the pulley, which had a metal loop at its top, was tied to the other end. Once again, the 0.65m measurement was from the approximate center of mass of the pulley to the center of mass of the 0.2kg mass. This section of rope was then laid on the groove of the upper pulley. This pulley was then fastened to a taught rope that spanned across the testing room. This system was directly over a surface that served as the vertical zero-point for the coordinate system. This setup is shown in **Figure 02**.

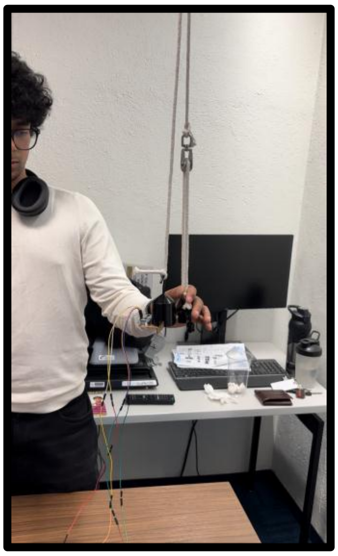

Figure 02: Actual setup used for testing.

In **Figure 02**, there is a noticeable module attached to the end of one of the masses. This is an HC-SR04 ultrasonic sensor used for measuring the distance from the table.[5] This was used in conjunction with the Arduino Uno R3, a microcontroller that is very user-friendly for projects similar to this one.[6] The Arduino and HC-SR04 are depicted in **Figure 03**, with all appropriate connections made. The Arduino was programmed to send out a sound wave that is reflected back and detected by its echo pin. The time difference between the wave being output and the reflected wave being received is multiplied by the speed of sound to provide a distance between the sensor and the nearest surface as a result of the equal $d = vt$. The code used for the module is included below in **Figure 04**.

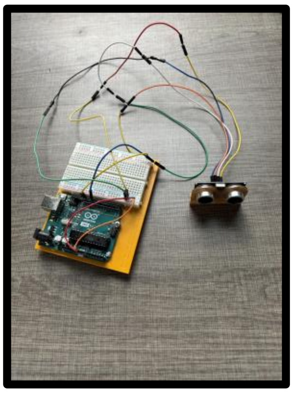

Figure 03: Arduino Uno R3 connected to HC-SR04 ultrasonic sensor.

[5] OSEPP Electronics LTD, *HC-SR04 Ultrasonic Sensor Datasheet.* [Online].
[6] Arduino, Arduino Uno R3 Datasheet. [Online].

Figure 04: Code used to program HC-SR04 to provide displacement measurements.

The code in **Figure 04** includes comments on what each line is doing. It is noticeable that it was initially set up to provide data from three modules, however some of that code was eventually commented out. This is due to the fact that when three modules were originally used for testing, they were placed on the surface below each respective mass and subsequently provided the distance to each mass. They were unable to distinguish between masses, however, so to remedy this testing was done with the sensor connected directly to the mass, facing the surface. Due to limitations in materials such as jumper wires to connect the sensors, only one sensor could be used for this setup. As a result, trials would be conducted for each of the masses, with distance from the surface only measured for one mass at a time. While not ideal, there should be little fluctuation in results from trial to trial. In regards to the added mass, the HC-SR04 sensor is only 0.0085kg, so the mass should be negligible in comparison to the expected results. That said, should these trials be conducted again, it would be wise to attach a sensor to each mass with only one sensor providing data, so that the 0.0085kg can be accounted for in calculations and the mass on each end is constant from trial to trial.

To start testing, the sensor is connected to one of the three masses being tested and fastened with adhesive to the bottom of the mass. For this experiment, the 0.2kg mass was tested first, so the HC-SR04 was attached to the bottom of the 0.2kg mass with adhesive. With the system now setup, trials can be completed. The Arduino Uno is turned on and connected via an extension cable to a computer that has access to the Arduino IDE software. The code provided in **Figure 04** provides the displacement measurement every 1 ms to the computer, allowing for the collection of data. With the sensor connected and ready to collect data, a researcher holds the center of the 0.04kg, 0.1kg, and 0.2kg massesat approximately level, however, separated so that when released, the masses do not collide. It is vital that the center of masses are at the same vertical positions, however not in contact when released from initial conditions. If the masses collide during the trial, the mechanics will be affected by the collision, and the data must be neglected. With the sensor sending data and the masses at proper initial conditions, the researcher releases the masses as another researcher collects video of the trial.

Once the masses reach steady state positions, the displacement results from the sensor are recorded for the mass being tested. Since data is collected every millisecond while the Arduino Uno is on, the first data points prior to the release of the masses are constant, and the data points after the steady state is also constant. For this reason, the data is collected from the last initial condition point to the first steady-state measurement. This led to 6 meaningful data points per trial, with the last initial condition measurement being the data point at $t = 0ms$, and the first steady state point being the measurement at $t = 5ms$. These data points are recorded in the table shown in **Figure 05**, and the trials are then run two more times for the 0.2kg mass being tested. After collecting three sets of data for the first mass, the sensor is moved to the second mass, and the same process is repeated. This is then repeated for the third mass. All the data collected is shown in **Figures 05**, **Figure 06**, and **Figure 07**.

0.2kg Mass					
Trial One		Trial Two		Trial Three	
Time	Distance	Time	Distance	Time	Distance
0	47.75	0	47.73	0	47.48
1	47.23	1	47.25	1	46.55
2	41.45	2	41.91	2	41.02
3	31.61	3	31.97	3	31.75
4	28.98	4	28.82	4	28.86
5	28.24	5	28.11	5	28.55

Figure 05: Displacement measurements for 0.2kg mass.

0.1kg Mass					
Trial One		Trial Two		Trial Three	
Time	Distance	Time	Distance	Time	Distance
0	46.55	0	46.54	0	46.02
1	45.22	1	44.91	1	45.69
2	43.32	2	43.89	2	45.47
3	43.01	3	42.72	3	44.79
4	42.62	4	42.83	4	44.35
5	42.38	5	42.51	5	4

Figure 06: Displacement measurements for 0.1kg mass.

0.04kg Mass					
Trial One		Trial Two		Trial Three	
Time	Distance	Time	Distance	Time	Distance
0	46.36	0	46.11	0	46.63
1	47.02	1	47.50	1	47.84
2	50.32	2	52.34	2	51.64
3	52.79	3	58.34	3	58.46
4	56.27	4	64.32	4	65.53
5	63.24	5	70.42	5	68.74

Figure 07: Displacement measurements for 0.04kg mass.

III. Analysis

After collecting all data, plots were created graphing position as a function of time for each of the trials. This was done using a Desmos graphing calculator.[7] A quadratic regression was also completed for all data sets. This resulted in the data displayed in **Figure 08** for the 0.2kg mass, with the first trial in green, the second in blue, and third in purple. The same process was completed for the 0.1kg mass and 0.04kg mass, with the plots displayed in **Figure 09** and **Figure 10** respectively. For **Figure 09**, trial one is in black, two in red, and three in orange. For **Figure 10**, trial one is in purple, two in blue, and three in green.

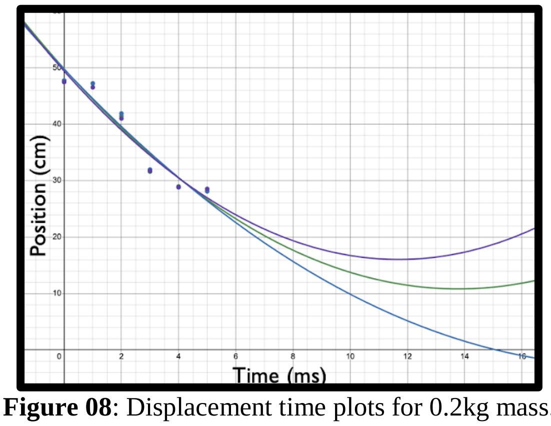

Figure 08: Displacement time plots for 0.2kg mass.

[7] Desmos. *Desmos Graphing Calculator*. Desmos, Inc., 2024.

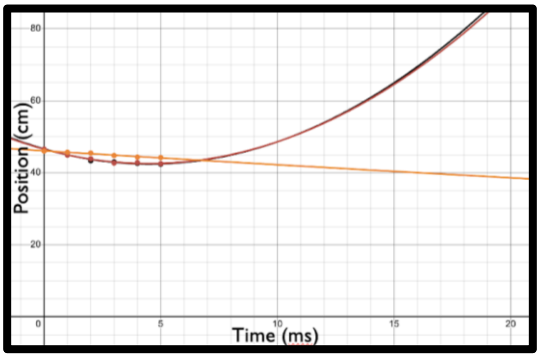

Figure 09: Displacement time plots for 0.1kg mass.

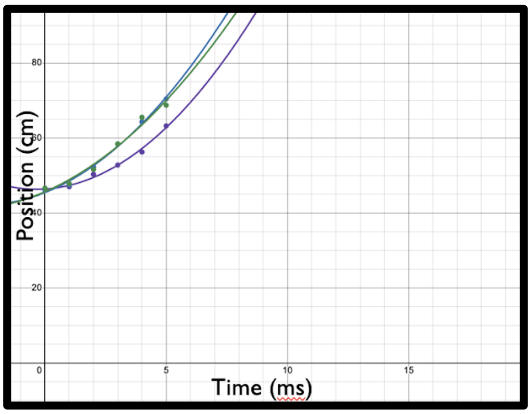

Figure 10: Displacement time plots for 0.04kg mass.

The regression equations are for the 0.2kg masses are as follows:
- $y(t) = 0.2054t^2 - 5.659t + 49.809, r^2 = 0.9235$
- $y(t) = 0.1359t^2 - 5.346t + 49.751, r^2 = 0.9241$
- $y(t) = 0.2439t^2 - 5.705t + 49.395, r^2 = 0.9285$

For the 0.1kg mass, these were:
- $y(t) = 0.2052t^2 - 1.853t + 46.602, r^2 = 0.9802$
- $y(t) = 0.1986t^2 - 1.779t + 46.529, r^2 = 0.9881$
- $y(t) = 0.00125t^2 - 0.3991t + 46.075, r^2 = 0.9711$

Finally, for the 0.04kg mass, these were:
- $y(t) = 0.5763t^2 + 0.3936t + 46.400, r^2 = 0.9907$
- $y(t) = 0.5020t^2 + 2.576t + 45.463, r^2 = 0.9947$
- $y(t) = 0.4121t^2 + 2.809t + 45.673, r^2 = 0.9771$

The coefficients of determination r^2, were also included for each dataset. For all these regressions, the lowest correlation coefficient of determination was $r^2 = 0.9235$, suggesting that the regression equations very accurately reflected the data collected. After finding these initial equations, the average was taken for each of the masses to get an equation of motion that most accurately reflects the system. This led to the following:
- $y_{0.2kg}(t) = 0.1951t^2 - 5.570t + 49.652$
- $y_{0.1kg}(t) = 0.1350t^2 - 1.344t + 46.402$
- $y_{0.04kg}(t) = 0.4968t^2 - 1.926t + 45.845$

These are plotted in **Figure 11**. They are only plotted from zero to five milliseconds, as that was the range of motion. For these equations, the displacement is in cm, and the time in milliseconds.

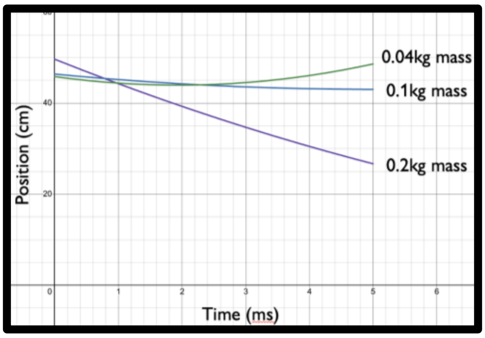

Figure 11: Average regression equations of motion for the three masses.

With these equations of motion found, the results can be compared to the theoretical conserved quantity. As previously discussed, this is $0.46\dot{x} + 0.34\dot{y} = constant$. Regarding the data collected and the initial system, $\dot{x}$ is clearly the negative of the velocity of the 0.2kg mass, as it points directly opposite the 0.2kg displacement as measured by the sensor, and thus directly opposite the velocity as well. Thus $\dot{x} = -y'_{0.2kg}(t)$. The motion of the 0.1kg mass is the motion of the pulley, or x, plus the motion of the 0.1kg mass relative to the pulley, y. Therefore, $\dot{x} + \dot{y} = y_{0.1kg}'(t)$, so $\dot{y} = y'_{0.1kg}(t) - \dot{x} = y'_{0.1kg}(t) + y'_{0.2kg}(t)$. Thus, to solve for the conserved quantity, the equation $C = -0.46y'_{0.2kg}(t) + 0.34(y'_{0.1kg}(t) + y'_{0.2kg}(t))$. Once again using the Desmos graphing calculator, the derivatives of the equations of motion are taken, and then added according to the desired equation. This leads to the plot in **Figure 12**, which is modeled by the equation $C(t) = 0.44976t + 2.1144$. These numbers were adjusted so that the conserved quantity is in kgm/s, and time is in s.

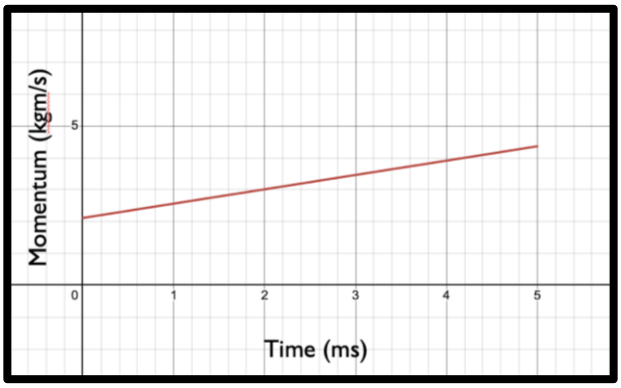

Figure 12: Conserved quantity as a function of time.

This equation means that the conserved quantity increases $0.44976kgm/s$ per second over the 0-5ms interval recorded. Multiplying this constant by the time of $5ms$, this shows an increase of $2.2488 \times 10^{-3}kgm/s$, which is just an 0.106% increase in the conserved quantity over this $5ms$ window of testing.

IV. Conclusion

This study effectively modeled and analyzed the dynamics of a Double Atwood Machine, with a particular focus on verifying the system's conserved quantities. Using Lagrangian mechanics and data collected via an Arduino-based ultrasonic sensor setup, the experimental results aligned closely with the theoretical predictions. The derived equations of motion exhibited strong consistency with expected behavior, and the calculated conserved showed minimal deviations, highlighting the reliability of the methodology.

The results confirm the practical applicability of classical mechanics in understanding and predicting the behavior of real-world dynamic systems. The experiment also illustrates the importance of integrating theoretical analysis with precise experimental design and computation tools. While minor limitations, such as slight variations due to sensor attachments and the mass of the added sensor, were present, these did not significantly impact the overall findings. Enhancing future setups with additional sensors could help improve accuracy and further mitigate these small discrepancies.

In conclusion, this research demonstrates how combining hands-on experimentation with theoretical mechanics can provide a deeper and more nuanced understanding of physical systems. It serves as a reminder of the importance of rigorous experimental design in validating fundamental principles of classical mechanics.

Arduino, *Arduino Uno R3 Datasheet.* [Online].

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