IOSR Journal of Mechanical and Civil Engineering (IOSR-JMCE)
e-ISSN: 2278-1684,p-ISSN: 2320-334X, Volume 19, Issue 1 Ser. III (Jan. – Feb. 2022), PP 46-57
www.iosrjournals.org

Modeling and Control of X-Shape Quadcopter

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***Abstract**: A quadcopter is a form of unmanned aerial vehicle with several rotors. Surveillance, military operations, fire detection, agriculture, spyware, and a variety of other applications are all new to it. Because of their dependability, cost effectiveness, and multi-functionality, they are frequently used in many locations. The quadrotor project still facing some drawbacks and challenges since it is a highly nonlinear, underactuated system. It has six degrees of freedom but only four actuators which make it very difficult to be controlled. A detail derivation of the mathematical model of Quadcopter is presented. The quadcopter's model is divided into two subsystems; rotational subsystem (roll, pitch and yaw) and translational subsystem (altitude, x and y position). The rotational subsystem is fully actuated whereas the translational subsystem is underactuated. The model is controlled using a PID controller. The PID gains; k_p, k_i, and k_d. are obtained using auto tuning and fined using manual tuning. The responses for heading, altitude, x-position and y-position are acceptable.*

***Keywords**: Quadcopter – PID Control – UAV*

Date of Submission: 02-02-2022 Date of Acceptance: 15-02-2022

I. Introduction

A quadcopter, drone, or quad-rotor all refer to the same thing as a helicopter but have four motors. Its motors are oriented upwards and distributed in a square configuration at an equal distance from the center of mass. Quadcopters falls in the category of vertical take-off and landing (VTOL) UAVs [1].

The control of the quadcopter in hovering, maneuvers and take off is done by controlling the speed of four rotors. On the beginning of the UAVs, it is limited to the military applications with very large sizes then after the very fast development, the size of UAVs is gotten more small and compact size.
This paved the way for its entry into many other applications such as Aerial Photograph, Agriculture, 3-D Mapping, Shipping, and delivery [3]. The problem facing the control of the quadcopter is that the quadcopter is a highly nonlinear, multi-variable system and since it has a six Degrees of Freedom (DOF) but only four actuators so, it is an underactuated system [2].

Since the quadcopter has 6-DOFs, 6 variables will be used to express its position and orientation in space (x, y, z, φ, θ, ψ). The distances of the quadrotor's center of mass along the x,y and z axes respectively from a fixed inertial frame will be presented as x, y and z. The three Euler angles; φ, θ and ψ represent the orientation of the quadrotor. The roll and pitch movement (φ, θ) represents the attitude of the quadcopter, while ψ (yaw) referred to the heading of the quadcopter and z refered to the altitude [3].

There are many different methods to control the quadcopter, PID controller, back-stepping control, non-linear control, LQR controllers and nonlinear controllers with nested saturations The PID-controller is the most popular method [3].

In this article a mathematical model for an under actuated six-degrees of freedom (6 DoF) quadcopter is derived based on Newton-Euler method. In addition the actuation forces are considered by modelling the aerodynamic forces and coefficients. A detail derivation of the mathematical model of Quadcopter is presented. The quadcopter's model is divided into two subsystems; rotational subsystem (roll, pitch and yaw) and translational subsystem (altitude, x and y position). The rotational subsystem is fully actuated whereas the translational subsystem is underactuated. The model will be controlled using a PID controller. The PID gains; k_p, k_i, and k_d. are obtained using auto tuning and fined using manual tuning. The responses for heading, altitude, x-position and y-position show a good behavior and they are acceptable.

II. System Modeling

Before talking about the kinematic and dynamic model of the quadcopter, A brief introduction about how to generate a thrust, roll, pitch and yaw with only four motors will be presented. And also show how the spin direction allows us to decouple one motion from the other.

Spinning a propeller of a motor causes the thrust motion since it pushes air down causing the reaction force in the opposite direction that is up. If the motor is placed at the center of gravity of an object the applied force will cause the net object to move in pure translation as shown Fig. 1 .The object will hover in place in case the thrust force is truly equal and opposite to the gravity force as shown in Fig. 2.

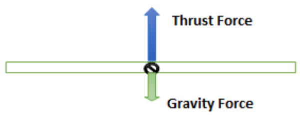

Fig.1: Lift force generated by motor

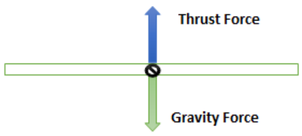

Fig.2: Hovering due to equality of lift and gravity forces

By applying a force at a certain distance from the center of gravity, both a torque as well as a translational motion will be produced about the center of gravity as shown in Fig. 3.

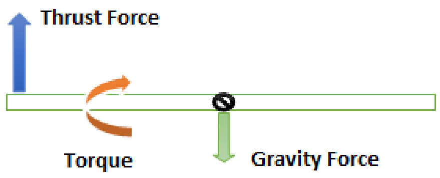

Fig. 3: Force produces torque

By applying a counter force in the right side of the center of the gravity and each force is half of the weight, the bar will remain stationary since forces and torques will cancel each other as shown in Fig. 4.

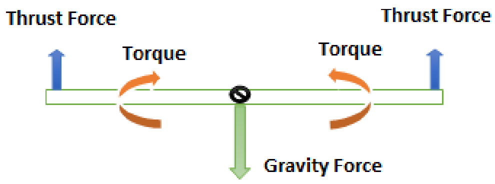

Fig. 4: Counter forces and torques cancel each others

The torque will be doubled and the bar will start to rotate in case the both motors are spinning in the same direction as shown in Fig. 5.

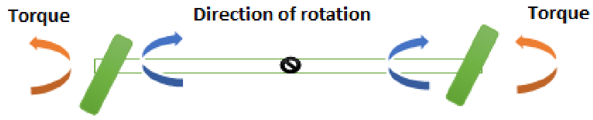

Fig. 5: Rotating bar due to doubled torques

Now, the two motors should be spin in the opposite directions to counter this torque. It does not matter where the counter rotating motors are placed as long as two of them spin in one direction and the other in the opposite direction. Designers of quadcopter choose to have opposite motors to spin in the same direction. It's due to the way yaw or the flat spinning motion interacts with roll and pitch.

We can command the yaw motion by slowing down two motors that are running in the same direction and speeding up the other two. As a result, we will be able to yaw without affecting thrust.

Similarly, roll and pitch can be considered. To roll, we decrease one of the front pairs and increase the other, resulting in a rolling torque. To pitch, we decrease one of the front pairs and increase the other, resulting in a pitching torque; both of these motions have no effect on yaw.

Newton-Euler formalism will be used to derive the kinematics and dynamics models of a quadrotor with assuming the following points:

- The structure is rigid and symmetrical.
- The center of gravity of the quadrotor coincides with the body fixed frame origin.
- The propellers are rigid.
- Thrust and drag are proportional to the square of propeller's speed.

A. *Kinematic model*

To define the coordinate frames shown in Fig.6, the fixed reference frame with X_E, Y_E and Z_E axes and the body frame with X_b, Y_b and Z_b axes. The fixed frame is an inertial frame on a specific location at ground level, where the axes point to the North, East and Downwards respectively. The body frame is located in the center of the quadrotor body, while the x-axis is located between propeller 1 and propeller 3, the y-axis is located between propeller 1 and propeller 4, and the z-axis pointing downward [3].

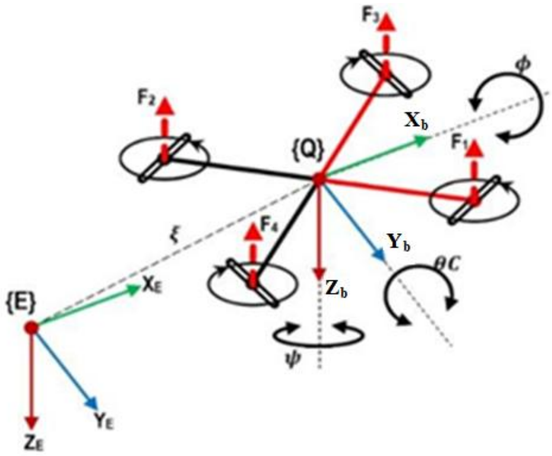

Fig.6: Quadcopter reference frame

The distance between the origin of the fixed frame and the origin of the body frame represents the absolute position of the quadrotor's center of mass $\xi = [X_E Y_E Z_E]^T$.The orientation of the quadrotor is described by the rotation R from the body frame to the inertial frame. The quadrotor's orientation is specified by roll, pitch, and yaw angles (φ; θ and ψ), which indicate rotations about the X_b, Y_b and Z_b-axes respectively. Assuming the sequence of rotation to be roll (φ), pitch (θ) then yaw (ψ) [3].

There are some ways to get the orientations of the vehicle such as trigonometric functions, Euler angles and quatrains. Use Euler angles to find the final orientation of the vehicle with respect to the body frame by using the transformation of an inertial frame to body frame [1].

The transformation matrix R shown in equation (1) which will be used for inertial to body frame transformation is calculated by multiplying all the three rotational matrices.

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = [R] \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (1)$$

Where;

$$R= \begin{bmatrix} \cos(\theta)\cos(\Psi) & \cos(\theta)\sin(\Psi) & -\sin(\theta) \\ -\cos(\emptyset)\sin(\Psi)+\cos(\Psi)\sin(\theta)\sin(\emptyset) & \cos(\Psi)\cos(\emptyset)+\sin(\theta)\sin(\emptyset)\sin(\Psi) & \cos(\theta)\sin(\emptyset) \\ \sin(\Psi)\sin(\emptyset)+\cos(\Psi)\cos(\emptyset)\sin(\theta) & -\sin(\emptyset)\cos(\Psi)+\cos(\emptyset)\sin(\theta)\sin(\Psi) & \cos(\theta)\cos(\emptyset) \end{bmatrix}$$

To get information on the angular velocity of the quadcopter, an on-board Inertial Measurement Unit (IMU) is generally utilized, which provides the velocity in the body coordinate frame. To relate the Euler rates $\dot{\eta} = [\dot{\emptyset}\ \dot{\theta}\ \dot{\Psi}]^T$ that are measured in the inertial frame and angular body rates ω=[p q r]^T, a transformation is obtained in equation (2).

$$\omega = R_r \dot{\eta} \qquad (2)$$

Where:

$$R_r = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\emptyset & \sin\emptyset\ \cos\theta \\ 0 & -\sin\emptyset & \cos\emptyset\cos\theta \end{bmatrix}$$

Around the hover position, small angle assumption is made where cos $\emptyset = 1$, cos $\theta = 1$ and sin $\emptyset$= sin θ = 0, thus R_r can be simplified to an identity matrix I [3].

B. *Dynamic model:*

The quadcopter's motion can be divided into two subsystems; rotational subsystem (roll, pitch and yaw) and translational subsystem (altitude, x and y position). The rotational subsystem is fully actuated whereas the translational subsystem is underactuated.

III. Rotational equation of motion:

Newton-Euler method is used to derive the rotational equations of motion

$$J\dot{\omega} + \omega \times J\omega + M_G = M_B - M_a \quad (3)$$

Where:

J Quadrotor's diagonal inertia Matrix
$\dot{\omega}$ Angular body rates
M_G Gyroscopic moments due to rotors' inertia
M_B Moments acting on the quadrotor in the body frame
M_a Drag moment caused by aerodynamic effects.

The Gyroscopic moments are defined to be $\omega \times [\,0 \;\; 0 \;\; J_r\Omega_r]^T$

So the rotational equation of motion can be rewritten as,

$$J\dot{\omega} + \omega \times J\omega + \omega \times [0 \;\; 0 \;\; J_r\omega_r]^T = M_B - M_a \quad (4)$$

Where:

J_r rotors' inertia
Ω_r rotors' relative speed
$\Omega_r = -\Omega_1 + \Omega_2 - \Omega_3 + \Omega_4$

where $\Omega_1, \Omega_2, \Omega_3, and\ \Omega_4$ Rotors' angular speeds

To get a time-independent inertia matrix, we must calculate the rotational equations of motion in the body frame rather than the inertial frame.

Diagonal Inertia Matrix (J)

The inertia matrix for the quadcopter is a diagonal matrix, its diagonal elements are I_{xx} ,I_{yy}, I_{zz} which represents the mass moments of inertia about the principal axes in the body frame.

$$J = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \quad (5)$$

Gyroscopic Moment

The gyroscopic moment of rotor is a physical effect in which gyroscopic torques or moments attempt to align the spin axis of the rotor along with the inertial z-axis [3].

Moments Acting on the Quadcopter (M_B)

Each rotor in the quadcopter creates an upward thrust force F_i and generates a moment M_i with the direction opposite to the direction of rotation of the corresponding rotor i.

$$F_i = \frac{1}{2}\,\rho A C_T r^2 {\Omega_i}^2 \quad (6)$$

$$M_i = \frac{1}{2}\,\rho A C_D r^2 {\Omega_i}^2 \quad (7)$$

Where:

ρ Air density.
A Blade area.
C_T Thrust coefficient.
C_D Drag moment coefficient.
r Blade length.
Ω_i Angular velocity of rotor i.

Because the aerodynamic forces and moments are dependent on the propeller shape and air density, and the quadcopter's maximum altitude is generally limited, the air density may be assumed to be constant [3]. As a result the equations (6) and (7) can be rewritten as follows,

$$F_i = k_f {\Omega_i}^2 \quad (8)$$

$$M_i = k_M {\Omega_i}^2 \quad (9)$$

From the previous explanation of the quadcopter motion and how it generates its moments about x_b, y_b and z_b axes, now we will express the moment about each axis as follows:

The total moment about the x-axis,

$$M_x = -F_2 l + F_4 l$$
$$= -\left(k_f {\Omega_2}^2\right) l + \left(k_f {\Omega_4}^2\right) l$$
$$= l k_f (-{\Omega_2}^2 + {\Omega_4}^2) \quad (10)$$

The total moment about the y-axis,

$$M_y = F_1 l - F_3 l$$
$$= (k_f \Omega_1{}^2)l - (k_f \Omega_3{}^2)l$$
$$= lk_f(\Omega_1{}^2 - \Omega_3{}^2) \qquad (11)$$

The total moment about the z-axis, the moment is caused by the rotation of the rotors, thus the equation can be written as,

$$M_z = M_1 - M_2 + M_3 - M_4$$
$$= (k_M \Omega_1{}^2) - (k_M \Omega_2{}^2) + (k_M \Omega_3{}^2) - (k_M \Omega_4{}^2)$$
$$= k_M(\Omega_1{}^2 - \Omega_2{}^2 + \Omega_3{}^2 - \Omega_4{}^2) \qquad (12)$$

Combining equations (10), (11) and (12) in vector form, we get,

$$M_B = \begin{bmatrix} lk_f(-\Omega_2{}^2 + \Omega_4{}^2) \\ lk_f(\Omega_1{}^2 - \Omega_3{}^2) \\ k_M(\Omega_1{}^2 - \Omega_2{}^2 + \Omega_3{}^2 - \Omega_4{}^2) \end{bmatrix} \qquad (13)$$

Where:

k_f Aerodynamic force constant.

k_M Aerodynamic moment constant.

l The length of the quadcopter arm from the (C.G) to the propeller.

IV. Translational Equations of Motion

The quadcopter's translation equations of motion are derived in the inertial frame and are based on Newton's second law.

$$m\ddot{\xi} = \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} + RF_B - F_a \qquad (14)$$

Where:

ξ = $[x_E \ y_E \ z_E]^T$ Quadcopter’s distance from the inertial frame.

m Quadcopter’s mass.

g Gravitational acceleration g = 9.81m/s^2.

F_B Non-gravitational forces acting on the quadcopter in the bodyframe.

F_a The drag force caused by aerodynamic effects.

The nongravitational force is the thrust force acting on the quadcopter in the vertical direction (there is no rolling or pitching). This force can be expressed as follows,

$$F_B = \begin{bmatrix} 0 \\ 0 \\ -k_f(\Omega_1{}^2 + \Omega_2{}^2 + \Omega_3{}^2 + \Omega_4{}^2) \end{bmatrix} \qquad (15)$$

Because the thrust is upwards and the positive z-axis in the body frame is pointing downwards, the sign is negative.

F_B is multiplied by the rotation matrix R to transform the rotor thrust forces from the body frame to the inertial frame, allowing the equation to be used in any quadcopter configuration.

Aerodynamic Effects

1. Drag Force

The force generated due to the friction of the moving quadcopter body with the air, as the velocity of the quadcopter increases the drag force in turn increase. The drag force F_a can be approximated by,

$$F_a = k_t \dot{\xi} \qquad (16)$$

where $k_t$is a constant matrix called the aerodynamic translation coefficient matrixand $\dot{\xi}$ is the time derivative of the position vector ξ.

2. Drag Moment

Drag moment is the moment generated due to the friction of the moving quadcopter body with the air. The drag moment $M_a$can be approximated to be,

$$M_a = k_r \dot{\eta} \qquad (17)$$

Where; k_r is a constant matrix called the aerodynamics rotation coefficient matrix and $\dot{\eta}$ is the Euler rates.

V. State Space Model

In this section the mathematical equations of motion in a state space representation will be discussed, and this will help to make the control of the quadcopter easier to tackle.

A. *State Vector X*

Since the quadcopter has six degrees of freedom, the state vector that represents the position of the quadcopter in space and its angular and linear velocities will include twelve elements given in equations (18) and (19).

$$X = \begin{bmatrix} \emptyset & \dot{\emptyset} & \theta & \dot{\theta} & \Psi & \dot{\Psi} & x & \dot{x} & y & \dot{y} & z & \dot{z} \end{bmatrix}^T \tag{18}$$

$$X = [x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\ x_{11}\ x_{12}]^T \tag{19}$$

B. *Control Input Vector U*

A control input vector U, consisting of four inputs; U_1through $U_4$is defined as, $U = [U_1\ U_2\ U_3\ U_4\ ]^T$

Where:

$$U_1 = k_f({\Omega_1}^2 + {\Omega_2}^2 + {\Omega_3}^2 + {\Omega_4}^2) \tag{20}$$

$$U_2 = k_f(-{\Omega_2}^2 + {\Omega_4}^2) \tag{21}$$

$$U_3 = k_f({\Omega_1}^2 - {\Omega_3}^2) \tag{22}$$

$$U_4 = k_M({\Omega_1}^2 - {\Omega_2}^2 + {\Omega_3}^2 - {\Omega_4}^2) \tag{23}$$

Equations (20) to (23) it can be rewritten in matrix form as follows,

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} k_f & k_f & k_f & k_f \\ 0 & -k_f & 0 & k_f \\ k_f & 0 & -k_f & 0 \\ k_M & -k_M & k_M & -k_M \end{bmatrix} \begin{bmatrix} {\Omega_1}^2 \\ {\Omega_2}^2 \\ {\Omega_3}^2 \\ {\Omega_4}^2 \end{bmatrix} \tag{24}$$

The rotor's velocity can be calculated from the control input as follows,

$$\begin{bmatrix} {\Omega_1}^2 \\ {\Omega_2}^2 \\ {\Omega_3}^2 \\ {\Omega_4}^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4k_f} & 0 & \frac{1}{2k_f} & \frac{1}{4k_M} \\ \frac{1}{4k_f} & -\frac{1}{2k_f} & 0 & -\frac{1}{4k_M} \\ \frac{1}{4k_f} & 0 & -\frac{1}{2k_f} & \frac{1}{4k_M} \\ \frac{1}{4k_f} & \frac{1}{2k_f} & 0 & -\frac{1}{4k_M} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} \tag{25}$$

The rotors' velocities can be calculated from the control inputs by taking the square root of the previous equation as follows,

$$\Omega_1 = \sqrt{\frac{1}{4k_f}U_1 + \frac{1}{2k_f}U_3 + \frac{1}{4k_M}U_4} \tag{26}$$

$$\Omega_2 = \sqrt{\frac{1}{4k_f}U_1 - \frac{1}{2k_f}U_2 - \frac{1}{4k_M}U_4} \tag{27}$$

$$\Omega_3 = \sqrt{\frac{1}{4k_f}U_1 - \frac{1}{2k_f}U_3 + \frac{1}{4k_M}U_4} \tag{28}$$

$$\Omega_4 = \sqrt{\frac{1}{4k_f}U_1 + \frac{1}{2k_f}U_2 - \frac{1}{4k_M}U_4} \tag{29}$$

C. *Rotational Equation of motion in state space form*

The equation of the total moments acting on the quadcopter body becomes,

$$M_B = \begin{bmatrix} lU_2 \\ lU_3 \\ U_4 \end{bmatrix} \tag{30}$$

By expanding the rotational equation of motion we get,

$$\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \ddot{\emptyset} \\ \ddot{\theta} \\ \ddot{\Psi} \end{bmatrix} + \begin{bmatrix} \dot{\emptyset} \\ \dot{\theta} \\ \dot{\Psi} \end{bmatrix} \times \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{\emptyset} \\ \dot{\theta} \\ \dot{\Psi} \end{bmatrix} + \begin{bmatrix} \dot{\emptyset} \\ \dot{\theta} \\ \dot{\Psi} \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ J_r\Omega_r \end{bmatrix} = \begin{bmatrix} lU_2 \\ lU_3 \\ U_4 \end{bmatrix} - M_a \tag{31}$$

Expanding equation (31), leads to,

$$\begin{bmatrix} I_{xx}\ddot{\emptyset} \\ I_{yy}\ddot{\theta} \\ I_{zz}\ddot{\Psi} \end{bmatrix} + \begin{bmatrix} \dot{\theta} I_{zz}\dot{\Psi} - \dot{\Psi} I_{yy}\dot{\theta} \\ \dot{\Psi} I_{xx}\dot{\emptyset} - \dot{\emptyset} I_{zz}\dot{\Psi} \\ \dot{\emptyset} I_{yy}\dot{\theta} - \dot{\theta} I_{xx}\dot{\emptyset} \end{bmatrix} + \begin{bmatrix} \dot{\theta} J_r \Omega_r \\ -\dot{\emptyset} J_r \Omega_r \\ 0 \end{bmatrix} = \begin{bmatrix} lU_2 \\ lU_3 \\ U_4 \end{bmatrix} - k_r \begin{bmatrix} \dot{\emptyset} \\ \dot{\theta} \\ \dot{\Psi} \end{bmatrix} \quad (32)$$

Using equation (32), the angular accelerations can be written in terms of the other variables as:

$$\ddot{\varphi} = \frac{l}{I_{xx}} U_2 - \frac{J_r}{I_{xx}} \dot{\theta}\Omega_r + \frac{I_{yy}}{I_{xx}} \dot{\psi}\dot{\theta} - \frac{I_{zz}}{I_{xx}} \dot{\theta}\dot{\psi} - k_r \dot{\emptyset} \quad (33)$$

$$\ddot{\theta} = \frac{l}{I_{yy}} U_3 - \frac{J_r}{I_{yy}} \dot{\varphi}\Omega_r + \frac{I_{zz}}{I_{yy}} \dot{\psi}\dot{\varphi} - \frac{I_{xx}}{I_{yy}} \dot{\varphi}\dot{\psi} - k_r \dot{\theta} \quad (34)$$

$$\ddot{\psi} = \frac{1}{I_{xx}} U_4 + \frac{I_{xx}}{I_{zz}} \dot{\varphi}\dot{\theta} - \frac{I_{yy}}{I_{zz}} \dot{\theta}\dot{\varphi} - \quad k_r \dot{\Psi} \quad (35)$$

To simplify equations (33) to (35), the inertia terms are written as:

$$a_1 = \frac{I_{yy} - I_{zz}}{I_{xx}}, \qquad a_2 = \frac{J_r}{I_{xx}}, \qquad a_3 = \frac{I_{zz} - I_{xx}}{I_{yy}}$$

$$a_4 = \frac{J_r}{I_{yy}}, \qquad a_5 = \frac{I_{xx} - I_{yy}}{I_{zz}}, \qquad b_1 = \frac{l}{I_{xx}}$$

$$b_2 = \frac{l}{I_{yy}}, \qquad b_3 = \frac{1}{I_{zz}}$$

Equations (33) through (35) can be rewritten in a simple form in terms of the system states, using the above definition of a1 to a5 and b1 to b3

$$\ddot{\varphi} = b_1 U_2 - a_2 x_4 \Omega_r + a_1 x_4 x_6 - k_r x_2 \quad (36)$$

$$\ddot{\theta} = b_2 U_3 - a_4 x_2 \Omega_r + a_3 x_2 x_6 - k_r x_4 \quad (37)$$

$$\ddot{\psi} = b_3 U_4 + a_5 x_2 x_4 - k_r x_6 \quad (38)$$

D. Translational Equations of motion in state space form

The equation of the upward force acting on the Quadcopter becomes,

$$F_B = \begin{bmatrix} 0 \\ 0 \\ -U_1 \end{bmatrix} \quad (39)$$

By expanding the translational equation of motion (14) and using equation (16) we get,

$$m \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} + \begin{bmatrix} c\theta c\Psi & c\Psi s\emptyset s\theta & s\emptyset s\Psi + c\emptyset c\Psi s\theta \\ c\theta s\Psi & c\theta c\Psi + s\emptyset s\Psi s\theta & c\emptyset s\Psi s\theta - c\Psi s\theta \\ -s\theta & c\theta s\emptyset & c\emptyset c\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -U_1 \end{bmatrix} - k_t \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \quad (40)$$

Rewriting Equation (40) to have the accelerations in terms of the other variables, we get,

$$\ddot{x} = \frac{-U_1}{m} (\sin\varphi \sin\psi + \cos\varphi \cos\psi \sin\theta) - k_t \dot{x} \quad (41)$$

$$\ddot{y} = \frac{-U_1}{m} (\cos\varphi \sin\theta \sin\psi - \cos\psi \sin\varphi) - k_t \dot{y} \quad (42)$$

$$\ddot{z} = g - \frac{U_1}{m} (\cos\varphi \cos\theta) - k_t \dot{z} \quad (43)$$

Rewriting in terms of the state variable X,

$$\ddot{x} = \frac{-U_1}{m} (\sin x_1 \sin x_5 + \cos x_1 \cos x_5 \sin x_3) - k_t x_8 \quad (44)$$

$$\ddot{y} = \frac{-U_1}{m} (\cos x_1 \sin x_5 \sin x_3 - \cos x_5 \sin x_1) - k_t x_{10} \quad (45)$$

$$\ddot{z} = g - \frac{U_1}{m} (\cos x_1 \cos x_3) - k_t x_{12} \quad (46)$$

VI. Quadcopter Control

The quadcopter model will be controlled using a PID controller. PID controller block diagram is shown in Fig. 7. Knowing that the quadcopter has six degrees of freedom and only four controllers as the position in x and y depends on the roll and pitch orientation, so it is considered to be an underactuated system. This leads the attitude controller subsystem to be cascaded with position controller subsystem. The other two controller subsystems are the altitude controller which represents the control on the z-position and the heading controller which represents the orientation about the z-axis.

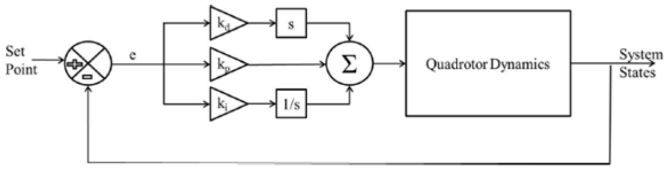

Fig.7: Block diagram for PID controller

A. Altitude Controller

The feedback signal from the quadcopter dynamics subsystem will be compared with the desired z-position and this error signal will be input to the altitude controller which modifies it using PID controller and produces the control signal U_1 shown in Fig. 8.

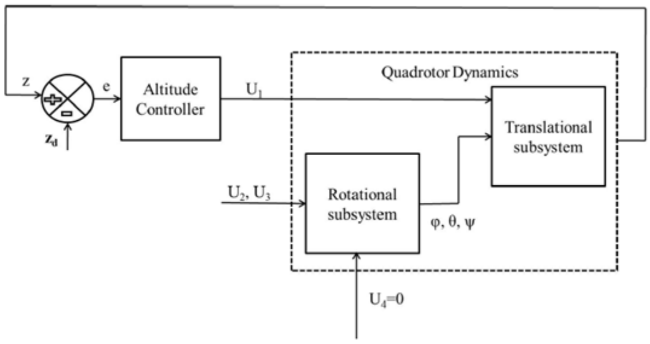

Fig. 8: Block diagram for altitude controller subsystem

B. Heading Controller

The feedback signal from the quadcopter dynamics subsystem is compared with the desired yaw angle and the error signal is input to the heading controller, then the controller modify the error signal using PID controller to produce a control signal U_4 that will be input to the rotational subsystem shown in Fig. 9.

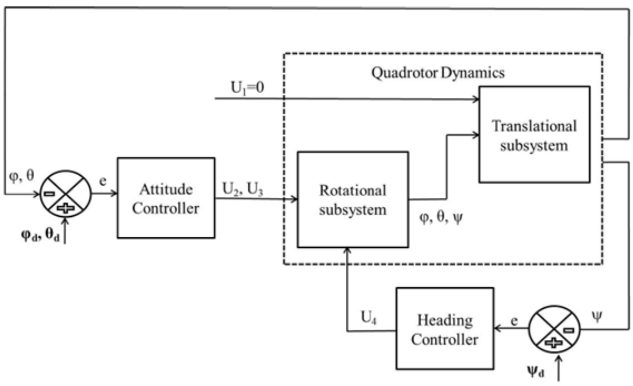

Fig.9: Block diagram for attitude and heading controller

C. Attitude Controller

The roll and pitch angles are feedback from the quadcopter dynamics subsystem and compared with the desired roll and pitch come from the position controller, the error signal is input to the attitude controller, then the controller modify the error signal using PD controller to produce a control signals U_2, U_3 that will be input to the rotational subsystem shown in Fig. 9.

D. Position Controller

The x and y positions are feedback from the quadcopter subsystem which compared with the desired x and y positions, the error signal is feed to the position controller to modify this error. The position controller also has a conversion from the inertial frame to body frame to make sure that the quadcopter is heading to the right position. The position controller block is shown in Fig.10.

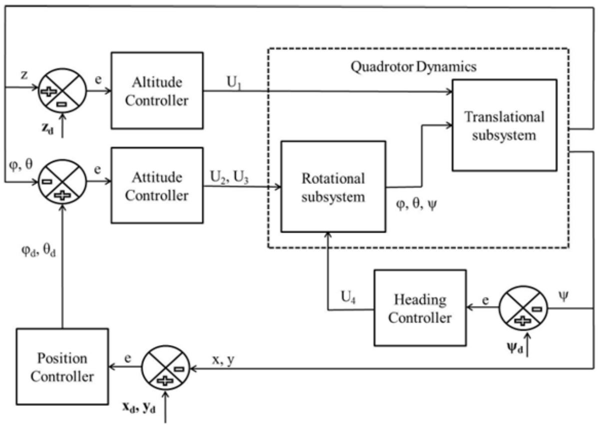

Fig.10: Block diagram for position controller

VII. Matlab Simulink

After these controllers are built and each controller produces one of the control input signal from U1 through U4, there is a need for subsystem to produce the input Ω_r(rotors' relative speed) to the quadcopter subsystem, thus a motor speed subsystem is built to calculate the speed of each rotor according to equations from (26) to (29) then create a relative speed subsystem to calculate the relative speed of rotors. Table.1 shows the input parameters of the quadcopter model.

The Simulink model is shown in Fig. 11. The quadcopter subsystem block contains two subsystems rotational subsystem which applied the three rotational equations (36) through (38) and the translational subsystem which applied the three translational equations (44) through (46). The two subsystems are shown in Fig. 12.

TABLE 1: QUADCOPTER INPUT PARAMETERS

Parameters	Description	Value	Units
$\boldsymbol{I_{xx}}$	Mass moment inertia about body frame's x-axis	0.01	$kg.m^2$
$\boldsymbol{I_{yy}}$	Mass moment inertia about body frame's y-axis	0.01	$kg.m^2$
$\boldsymbol{I_{zz}}$	Mass moment inertia about body frame's z-axis	0.01	$kg.m^2$
m	Quadrotor's mass	0.62	kg
$\boldsymbol{l}$	Moment arm	0.187	m
$\boldsymbol{J_r}$	Rotor inertia	6e-5	$kg.m^2$
$\boldsymbol{k_f}$	Aerodynamic force constant	3.13e-5	$N\ s^2$
$\boldsymbol{k_m}$	Aerodynamic moment constant	7.5e-7	$N\ m\ s^2$
$\boldsymbol{R_{mot}}$	Motor circuit resistance	0.6	
$\boldsymbol{K_{mot}}$	Motor torque constant	5.2	
$\boldsymbol{K_t}$	Aerodynamic translation coefficient	$diag(0.1,0.1,0.15)$	$N\ s/m$
$\boldsymbol{K_r}$	Aerodynamic rotation coefficient	$diag(0.1,0,0.15)$	$N\ m\ s$

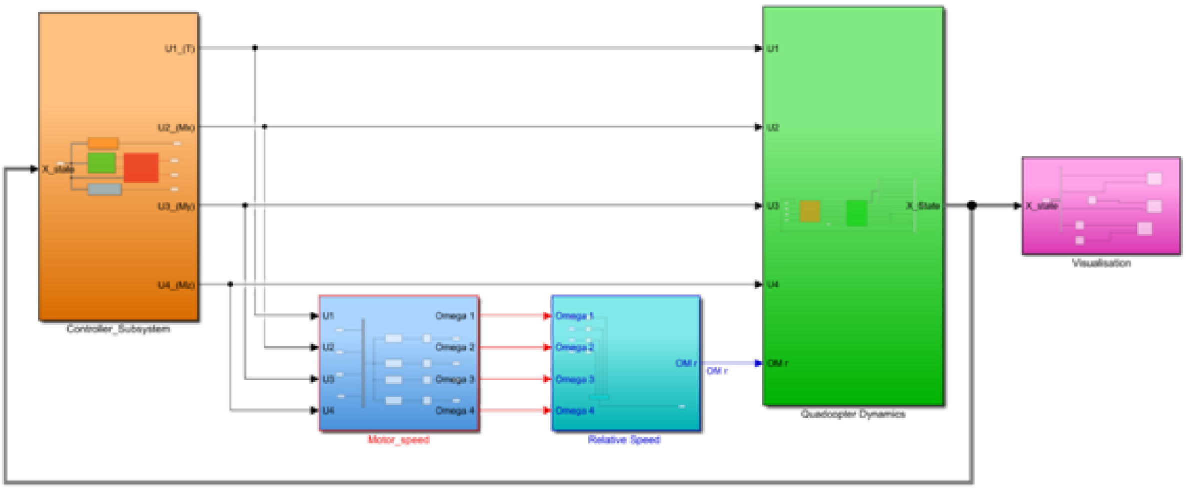

Fig. 11: Quadcopter MATLAB Simulink Model

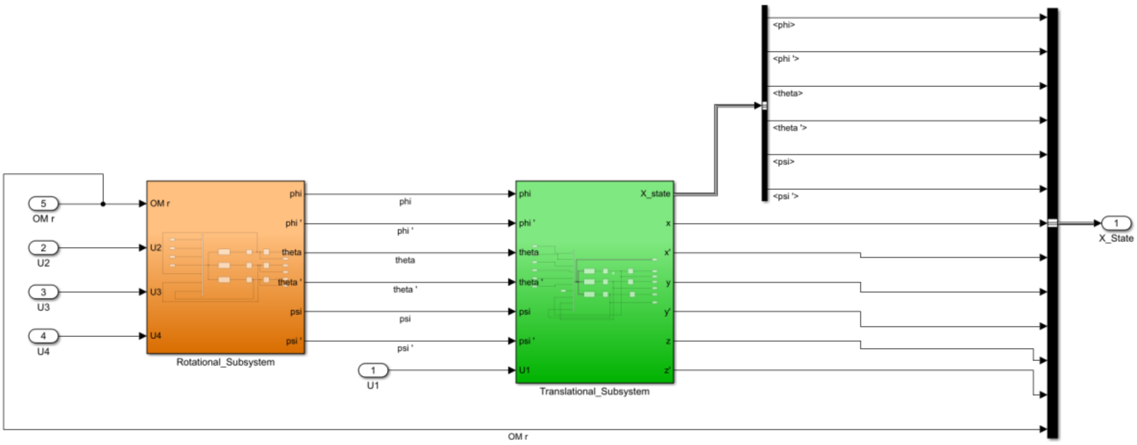

Fig. 12: Translational and Rotational subsystems

VIII. Simulink Results

The results shown in TABLE 2 are obtained using the PID tuner toolbox in MATLAB 2018a. The auto tuning method is used to determine the PID gains, since the result firstly not acceptable, then the values is fine-tuned manually.

TABLE 2: THE VALUE OF PID TUNED GAINS

	Desired value	k_p	k_i	k_d	Settling Time [sec]	Over-shooting	Rising Time [sec]
Altitude	1m	-0.11817	-7.4193e^{-3}	-4.01806e^{-1}	19.00	12.3%	1.98
X-Position	1m	2. 7236e^{-09}	0	1.7165e^{-05}	90.00	5.0%	24.48
Y-Position	1m	-1.0440e^{-08}	0	-2.3970e^{-05}	97.00	8.0%	27.72
Heading	45°	2.9623e^{-3}	0	9.24190e^{-2}	9.22	10.0%	1.55

For the PID gains of altitude, position and heading controllers it is specified as acceptable or not acceptable referring to settling time, rising time and maximum overshoot. The response of each controller; altitude response, heading response, x-position response and y-position response for the results shown in TABLE 2 is shown in Fig. 13 to Fig. 16.

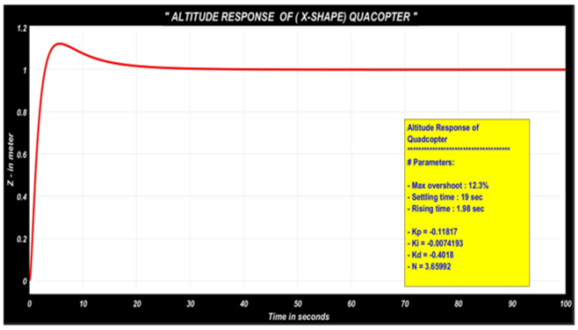

Fig.13: Altitude Response

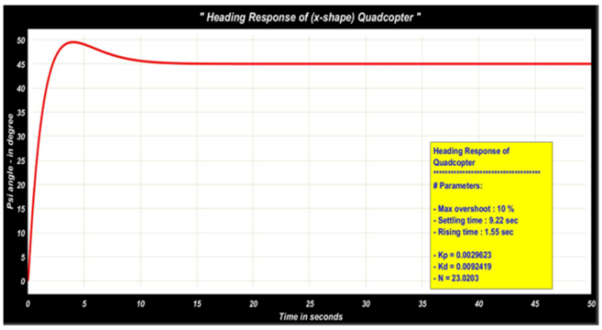

Fig.14: Heading Response

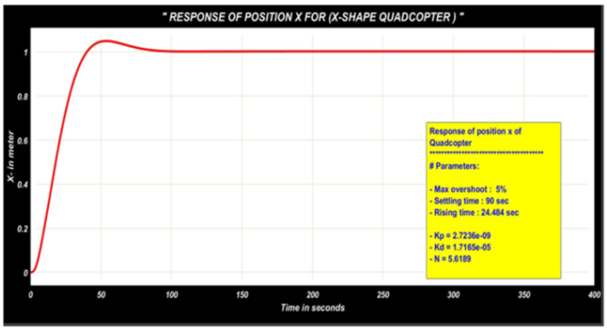

Fig.15: X-position Response

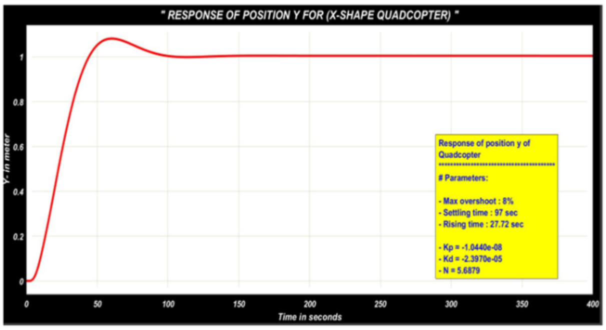

Fig.16: Y-position Response

IX. Conclusions

The quadrotor is an underactuated system. The mathematical model for an under actuated six-degrees of freedom (6 DoF) quadcopter is derived based on Newton-Euler method. The model is divided into two subsystems; rotational subsystem (roll, pitch and yaw) and translational subsystem (altitude, x and y position). The rotational subsystem is fully actuated whereas the translational subsystem is underactuated. The model is controlled using PID controller. The PID gains; k_p, k_i, and k_d. are obtained using auto tuning and fined using manual tuning. The responses for heading, altitude, x-position and y-position show a good behavior and they are acceptable.

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