*IOSR Journal of Mechanical and Civil Engineering (IOSR-JMCE)*
*e-ISSN: 2278-1684,p-ISSN: 2320-334X, Volume 17, Issue 1 Ser. IV (Jan - Feb 2020), PP 52-64*
*www.iosrjournals.org*

# Exact Finite Element Formulation for Flexural Vibration of Axially Pre-Loaded Euler-Bernoulli Beams

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***Abstract:*** *The present paper investigate the dynamic analysis for the flexural vibration of Euler-Bernoulli (EB) axially preloaded beams subjected to various harmonic forces. The governing flexural vibration equation and related boundary conditions for the beams are derived using Hamilton's variational principle. The exact closed form solution is consequently used to develop a family of exact shape functions which exactly satisfy the homogeneous solution of the governing field equation. A super-convergent two-noded finite beam element based on exact shape functions is then formulated. The proposed beam element developed involves no special discretization errors normally encountered in conventional finite element formulations and provide results in excellent agreement with a minimal number of degrees of freedom. The present finite element solution is shown to successfully capture the static and steady state dynamic responses of EB beams. It is also able to predict the natural flexural bending frequencies and the corresponding mode shapes as well as the critical buckling loads. The validity and the accuracy of the present finite beam element solution is achieved throughout the numerical examples presented and compared with well-established ABAQUS finite beam solution.*

***Key Words****: Static axial force, preloaded beam, steady state response, harmonic excitations, exact shape functions, super-convergent finite beam element.*

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Date of Submission: 05-02-2020 Date of Acceptance: 20-02-2020

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## I. Introductionand Objective

The problem of dynamic flexural vibration of axially preloaded beams are of considerable practical interest, and have wide applications in civil, mechanical and aircraft industry. The structural beam elements are in a state of preload due to the application of axial constant loads. For instance, the helicopter rotor blades and structural elements attached through semi-rigid connections, and the centrifugal forces acting on the blades are modelled as axial static forces. The study of the influence of a constant axial compressive loads on flexural vibrational characteristics (natural frequencies and mode shapes) of beams with different boundary conditions has been well established. Among them, Banerjee and Williams [1] formulated the exact analytical expressions for dynamic stiffness matrix to study the coupled flexural-torsional vibration of an axially loaded Timoshenko beam element. Hashemi and Richard [2] developed a dynamic finite element formulation for coupled bending-torsional vibration of axially loaded members with asymmetric cross-sections based on the Bernoulli-Euler and St. Venant beam theories. Banerjee [3] derived the exact expressions for the frequency equation and mode shapes for coupled bending and torsion vibrations of uniform Timoshenko beams with cantilever end conditions. The effect of axial force together with the effect of shear deformation and rotary inertia was taken into account in the formulation. Li et al. [4] investigated the dynamic flexure-torsion coupled vibrations of axially loaded mono-symmetric thin-walled beams. The influence of axial static force and warping deformation on the coupled bending-torsional frequencies and mode shapes are studied. Al-Raheimy [5] presented the theoretical investigation of the free transverse vibrations of simply-supported beams subjected to axial static forces. Using the principle of virtual displacements, Prokic and Lukic [6] using the dynamic finite element method to investigate the flexural-torsion coupled vibrations of axially loaded thin-walled beams. Based on the Euler-Bernoulli bending and St. Venant torsion beam theories, Kashani et al. [7, 8 and 9] developed the dynamic finite element formulation to study the flexural-torsional coupled vibration of beams subjected to static axial load and end moment.

A literature survey on the subject shows that the publications exist studied only the effect of axial static forces on the free vibration characteristics (i.e., natural frequencies and mode shapes) of beams with no research publication appears to have taken into account the axial static force on the dynamic response of beams subjected to harmonic bending excitations. In addition, most publications based on finite element formulation use approximate shape functions while in this study the finite element formulation based on the exact shape functions which exactly satisfied the exact solution of the dynamic field equation. The present work attempts to fill this gap. A particular objective of the present study is to develop a finite element formulation for investigating the dynamic analysis of axially preloaded EB beams under harmonic bending excitations.

## II. Formulation of the Problem

An axially preloaded beam of length $\ell$ subjected to transverse dynamic forces is considered as shown in Figure (1-a). A Cartesian reference system $(X,Y,Z)$ is employed in the present formulation, in which $X$ is the beam axis, $Y,Z$ are assumed to be the principal axes of the cross-section. In the present formulation, it is assumed that:

1. The beam is prismatic and made of an isotropic homogenous linear elastic material,
2. The cross-section of the beam is doubly symmetric,
3. The displacements and slopes are assumed sufficiently small,
4. The beam is based on Euler-Bernoulli hypothesis in which the cross-sections are plane and perpendicular to the longitudinal axis remain plane and perpendicular to the axis of bending. As a result, the effects of transverse shear deformation and rotatory inertia are not taken into account.
5. The damping of the beam is negligible.

The positive directions of the displacements, forces and moments are coincided with the positive coordinates as shown in Figure (1-a).

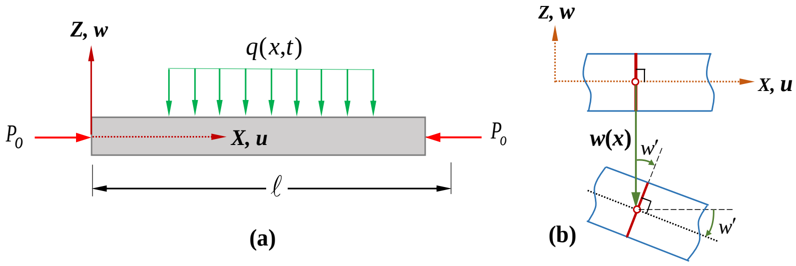


**Figure (1):** An axially loaded EB beam subjected to transverse dynamic force

### 2.1 Strain-Stress Functions

Based on the Euler-Bernoulli assumptions, the displacements shown in Figure (1-b) are given as:

$$\bar{u}(x,z,t) = -z\,w'(x,t)\ ,\quad \bar{v}(x,z,t) = 0 \ \text{and} \tag{1}$$

$$\bar{w}(x,z,t) = w(x,t)$$

(2)

where $\bar{u}(x,z,t)$ is the longitudinal displacement, $\bar{v}(x,z,t)$ and $\bar{w}(x,z,t)$ are the lateral and transverse displacements, respectively, and $w'(x,t)$ denotes the derivative of transverse displacement $w(x,t)$ with respect to coordinate $x$ .

### 2.2 Potential Energy Expression

The strain energy stored in the Euler-Bernoulli beam element due to bending is simply obtained as:

$$U_b = \frac{1}{2}\int_V \sigma_{xx}\varepsilon_{xx} dV = \frac{1}{2}\int_0^{\ell} EI\left(w''(x,t)\right)^2 dx$$

(3)

where $E$ is the modulus of elasticity, the axial strain is $\varepsilon_x = d\bar{u}/dx = -w''(x,t)$ and $I = \int_A z^2 dA$ is the moment of inertia of the beam cross-section.

The beam is under axial static compressive force $P_o$ applied at the centroid of the cross-section, and transverse dynamic forces and moments, distributed force $q(x,t) = \bar{q}(x)e^{i\Omega t}$ , concentrated forces $F_z(x,t) = \bar{F}_z(x)e^{i\Omega t}$ and bending moments $M_y(x,t) = \bar{M}_y(x)e^{i\Omega t}$ applied at the beam ends, i.e., $x = 0,\ell$ , as shown in Figure (2).

The potential energy of the axial constant compressive force $P_o$ is given by:

$$U_P = -\frac{1}{2}\int_0^{\ell} P_o\left(w'(x,t)\right)^2 dx \tag{4}$$

The potential energy of the external dynamic forces and moments is given by:

$$U_q = \int_0^{\ell} q(x,t)\, w(x,t)\, dx + \left[F_z(x,t) w(x,t)\right]_0^{\ell} + \left[M_y(x,t) w'(x,t)\right]_0^{\ell} \quad (5)$$

The total potential energy functional of the Euler-Bernoulli beam is:

$$\Pi = U_b + U_P - U_q$$

$$= \frac{1}{2}\int_0^{\ell}\left[EI\left(w''(x,t)\right)^2 - P_o\left(w'(x,t)\right)^2 - q(x,t)w\right]dx - \left[F_z(x,t)w(x,t)\right]_0^{\ell} - \left[M_y(x,t)w'(x,t)\right]_0^{\ell} \quad (6)$$

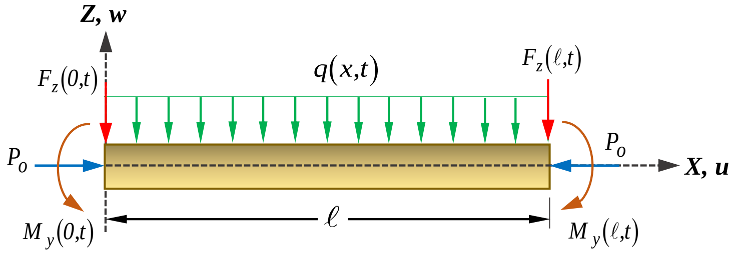


**Figure (2):** An axially loaded EB beam under various harmonic forces and moments

### 2.3 Kinetic Energy of Beam

The kinetic energy of the beam element is given by:

$$T = \frac{1}{2}\int_0^{\ell} \rho A\left[\dot{w}(x,t)\right]^2 dx \quad (7)$$

Based on the Euler-Bernoulli hypothesis, the rotary inertia of the beam is neglected since the beam is slender, $\rho$ is the mass density of the beam, $A$ is the cross-sectional area of the beam, and $\dot{w}(x,t)$ denotes the derivative of flexural displacement $w(x,t)$ with respect to time $t$.

### 2.4 Derivation Of Dynamic Bending Equation

The governing dynamic equation of flexural vibration and the associated boundary conditions for Euler-Bernoulli beam can be derived conveniently using the Hamilton's variational principle, which can be stated in the following form for an undamped free vibration analysis:

$$\int_{t_1}^{t_2}\left(\delta T - \delta \Pi\right)dt = 0, \qquad for\ \ \delta w(t_1) = \delta w(t_2) = 0 \quad (8)$$

Substituting equations (6) and (7) into equation (8), the governing dynamic flexural vibration of axially loaded Euler-Bernoulli beam subjected to dynamic force is obtained as:

$$\rho A \ddot{w}(x,t) + EI w^{iv}(x,t) - P_o w''(x,t) = q(x,t) \quad (9)$$

The essential and natural boundary conditions are given as:

$$\left[\delta w(x,t)\right]_0^{\ell} = 0 \ \text{, and } \left[\delta w'(x,t)\right]_0^{\ell} = 0 \quad (10)$$

$$\left[EI\, \delta w''(x,t) - M_y(x,t)\right]_0^{\ell} = 0 \text{, and } \left[EI\, \delta w'''(x,t) - P_o \delta w'(x,t) + F_z(x,t)\right]_0^{\ell} = 0$$

(11)

### 2.5 Harmonic Functions

For undamped system, the beam is assumed to be subjected to distributed harmonic forces $q(x,t)$ along the beam and end harmonic forces $F_z(x,t)$ and moments $M_y(x,t)$ applied at the beam ends $x = 0, \ell$, i.e.,

$$q(x,t),\ F_z(x,t), M_y(x,t) = \left[\bar{q}(x),\ \bar{F}_z(x)\Big|_0^{\ell}, \bar{M}_y(x)\Big|_0^{\ell}\right] e^{i\Omega t} \quad (12)$$

where $\Omega$ is the circular exciting frequency of the applied harmonic forces, and $i = \sqrt{-1}$ is the imaginary constant. Under the given applied harmonic forces and moments, the displacement corresponding to the steady state component of the flexural bending response is assumed to take the following form:

$$w(x,t)=W(x)\,e^{i\Omega t} \tag{13}$$

in which $W(x)$ is the beam displacement amplitude. Since the present formulation is proposed to capture only the steady state bending response of the Euler-Bernoulli beams, the transverse displacement field postulated in equation (13) disregard the transient bending response of the beam.

By substituting equations (12) and (13) into equation (9), one obtains:

$$EIW^{iv}(x)-P_oW''(x)-\rho A\Omega^2W(x)=\bar{q}(x) \tag{14}$$

Equation (14) governs the flexural bending vibration of axially pre-loaded EB beam subjected to transverse dynamic forces and moments.

### 2.6 Exact Solution of Bending Equation

The exact homogeneous solution of the governing field equation (14) is obtained by setting the right-hand side of the equation to zero, i.e. $\bar{q}(x)=0$. It is assumed that the homogeneous solution of the flexural displacement to take the following exponential form:

$$W(x)=C_i\,e^{\beta_i x} \tag{15}$$

Substituting equation (15) into equation (14), yielding a quartic equation have the following four distinct roots:

$$\beta_{1,2}=\pm\sqrt{\left(\frac{P_o}{2EI}\right)+\sqrt{\left(\frac{P_o}{2EI}\right)^2+\left(\frac{\rho A\Omega^2}{EI}\right)}}\text{ , and }\beta_{3,4}=\pm i\sqrt{-\left(\frac{P_o}{2EI}\right)+\sqrt{\left(\frac{P_o}{2EI}\right)^2+\left(\frac{\rho A\Omega^2}{EI}\right)}}$$

Therefore, the general solution for the flexural bending displacement $W(x)$ is then obtained as:

$$W(x)=C_1e^{\beta_1x}+C_2e^{\beta_2x}+C_3e^{\beta_3x}+C_4e^{\beta_4x}$$

(16)

where $C_i$ $(i=1,2,3,4)$ are unknown integration constants which can be obtained from the problem boundary conditions in order to determine the exact closed-form solution.

## III. Finite Element Formulation

In this study, a new two noded finite beam element is developed for forced vibration analysis of axially loaded Euler-Bernoulli beams subjected to various transverse harmonic forces and moments. Figure (3) illustrates the proposed two-noded finite EB beam element with four degrees of freedom per element. A family of shape functions which exactly satisfy the exact homogeneous solution of the governing filed equation presented in equation (16) is employed to derive the exact stiffness and mass matrices as well as the load potential vector for the beam element.

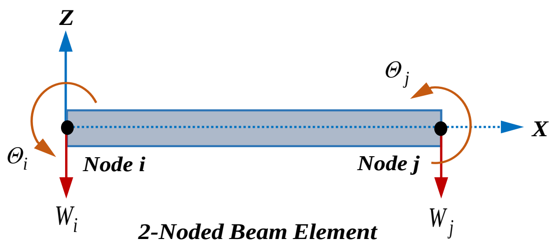


Figure (3): Two-noded Euler-Bernoulli beam element

### 3.1 Formulating Exact Shape Functions

The exact homogeneous solution related to steady state flexural bending response presented in equation (16) can be written in matrix form as:

$$W(x)=\langle E(x)\rangle_{1\times4}\{C\}_{4\times1} \tag{17}$$

where $\langle E(x)\rangle_{1\times4}=\langle e^{\beta_1x}\ e^{\beta_2x}\ e^{\beta_3x}\ e^{\beta_4x}\rangle_{1\times4}$ and $\langle C\rangle_{1\times4}=\langle C_1\ C_2\ C_3\ C_4\rangle_{1\times4}$. In order to relate the flexural displacement to nodal displacements, the vector of integration constants $\langle C\rangle_{1\times4}$ can be expressed in terms of the nodal transverse displacements and slopes $\langle W_n\rangle_{1\times4}=\langle W_1\ \Theta_1\ W_2\ \Theta_2\rangle_{1\times4}$ by enforcing the following conditions $W(0)=W_1$, $W'(0)=\Theta_1$, $W(\ell)=W_2$ and $W'(\ell)=\Theta_2$, yielding:

$$\{W_n\}_{4\times1}=\begin{Bmatrix} W_1 \\ \Theta_1 \\ W_2 \\ \Theta_2 \end{Bmatrix}_{4\times1}=\begin{Bmatrix} W(0) \\ W'(0) \\ W(\ell) \\ W'(\ell) \end{Bmatrix}_{4\times1}=\begin{bmatrix} \langle E(0)\rangle_{1\times4} \\ \langle E'(0)\rangle_{1\times4} \\ \langle E(\ell)\rangle_{1\times4} \\ \langle E'(\ell)\rangle_{1\times4} \end{bmatrix}_{4\times4}\{C\}_{4\times1}=[S]_{4\times4}\{C\}_{4\times1}$$

(18)

in which $[S]_{4\times4}=\begin{bmatrix} \langle E(0)\rangle_{1\times4} \\ \langle E'(0)\rangle_{1\times4} \\ \langle E(\ell)\rangle_{1\times4} \\ \langle E'(\ell)\rangle_{1\times4} \end{bmatrix}_{4\times4}=\begin{bmatrix} 1 & 1 & 1 & 1 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ e^{\beta_1\ell} & e^{\beta_2\ell} & e^{\beta_3\ell} & e^{\beta_4\ell} \\ \beta_1 e^{\beta_1\ell} & \beta_2 e^{\beta_2\ell} & \beta_3 e^{\beta_3\ell} & \beta_4 e^{\beta_4\ell} \end{bmatrix}_{4\times4}$.

From equation (18), by substituting into equation (16), one obtains:

$$W(x)=\langle E(x)\rangle_{1\times4}[S]^{-1}_{4\times4}\{W_n\}_{4\times1}=\langle H(x)\rangle_{1\times4}\{W_n\}_{4\times1} \qquad (19)$$

where $\langle H(x)\rangle_{1\times4}=\langle E(x)\rangle_{1\times4}[S]^{-1}_{4\times4}$ is the matrix of exact shape functions for the flexural bending response of Euler Bernoulli beams. It is noted that the shape functions presented in equation (19) exactly satisfy the homogeneous solution of the flexural bending equation (14).

### 3.2 Energy Expressions in Terms of Nodal Displacements and Rotations

The first variation of the total kinetic energy $\delta T$ for Euler-Bernoulli beam is given in terms of nodal degrees of freedom by substituting equation (19) into equation (7), yields:

$$\delta T=-\langle \delta W_n\rangle_{1\times4}\int_0^\ell\left[\rho A\Omega^2\{H(x)\}_{4\times1}\langle H(x)\rangle_{1\times4}\right]\{W_n\}_{4\times1}e^{i\Omega t}dx \quad (20)$$

By substituting equations (19) into equation (6), the first variation of the total potential energy $\delta\Pi$ for Euler-Bernoulli beam in terms of nodal degrees of freedom is given by:

$$\begin{aligned}\delta\Pi=\langle\delta W_n\rangle_{1\times4}\Bigg[&\left(\int_0^\ell EI\{H''(x)\}_{4\times1}\langle H''(x)\rangle_{1\times4}-P_o\{H'(x)\}_{4\times1}\langle H'(x)\rangle_{1\times4}\right)\{W_n\}_{4\times1}\\ &-\left(\int_0^\ell \bar{q}(x)\{H(x)\}_{4\times1}dx+\left[\bar{F}_z(x)\{H(x)\}_{4\times1}\right]_0^\ell+\left[\bar{M}_y(x)\{H'(x)\}_{4\times1}\right]_0^\ell\right)\Bigg]e^{i\Omega t}\end{aligned} \quad (21)$$

### 3.3 Finite Element Matrix Formulation

By substituting equations (20) and (21) into Hamilton's principle equation (8), one obtains:

$$\left([\boldsymbol{K}_e]_{4\times4}-P_o[\boldsymbol{K}_g]_{4\times4}-\Omega^2[\boldsymbol{M}_e]_{4\times4}\right)\{\boldsymbol{W}_n\}_{4\times1}=\{\boldsymbol{F}_e\}_{4\times1}$$

(22)

in which the elastic stiffness matrix for beam element $[\boldsymbol{K}_e]_{4\times4}$ is:

$$[\boldsymbol{K}_e]_{4\times4}=\int_0^\ell EI\{H''(x)\}_{4\times1}\langle H''(x)\rangle_{1\times4}dx\ \int_0^\ell$$

The geometric stiffness matrix for beam element $[\boldsymbol{K}_g]_{4\times4}$ is given as:

$$[\boldsymbol{K}_g]_{4\times4}=\int_0^\ell\{H'(x)\}_{4\times1}\langle H'(x)\rangle_{1\times4}dx$$

The beam element mass matrix $[\boldsymbol{M}_e]_{4\times4}$ is obtained as:

$$[\boldsymbol{M}_e]_{4\times4} = \int_0^{\ell} \rho A \{H(x)\}_{4\times1} \langle H(x)\rangle_{1\times4} dx$$

while, the load vector for beam element $\{\boldsymbol{F}_e\}_{4\times1}$ is given as:

$$\{\boldsymbol{F}_e\}_{4\times1} = \int_0^{\ell} \bar{q}(x)\{H(x)\}_{4\times1} dx + \left[\bar{F}_z(x)\{H(x)\}_{4\times1}\right]_0^{\ell} + \left[\bar{M}_y(x)\{H'(x)\}_{4\times1}\right]_0^{\ell}$$

The expressions for element elastic and geometric stiffness, mass matrices and load vector developed for one-dimensional two-noded Euler-Bernoulli axially loaded beam element with four degrees of freedom per node for flexural vibration response are evaluated by using the exact shape functions developed in this formulation.

## IV. Numerical Examples and Validation

In this section, several examples are presented in order to show the validity, accuracy and applicability of the new exact finite two-noded beam element developed in the present study. The new beam element can **(a)** capturing the steady state dynamic response of the beam under given harmonic forces, and **(b)** predicting the flexural natural frequencies and associated mode shapes of the Euler-Bernoulli beam from the steady state dynamic response. The present finite element formulation is based on the shape functions which exactly satisfy the exact solution of the bending field equation. This treatment eliminates mesh discretization errors arising in conventional interpolation schemes used in the finite element solutions and thus converge to the solution using a minimal number of degrees of freedom. As a result, it is observed that, the present results obtained based on a new finite beam element using a single two-noded beam element per span yielded the corresponding results which exactly matched with those based on the exact solutions. These results based on the present finite beam element (with two degrees of freedom per node) are compared with exact solutions available in the literature and Abaqus finite beam B13 element (with six degrees of freedom per node, i.e., three translation and three rotations). The effects of static axial compressive forces on the natural bending frequency and mode shape and steady state bending responses are also investigated.

### 4.1 Example (1) - Verification

A 3000mm cantilever beam of rectangular cross-section subjected to uniformly distributed transverse harmonic force $q(x,t) = 400\, e^{i\Omega t}\, N/m$ along the beam axis is shown in Figure (4). The beam made of steel material with the mechanical properties used for the analysis are: modulus of elasticity $E = 200GPa$, shear modulus $G = 80GPa$ and material density $\rho = 7800 kg/m^3$. The purpose of this example is to evaluate the validity and accuracy of the results obtained from the present finite element formulation.

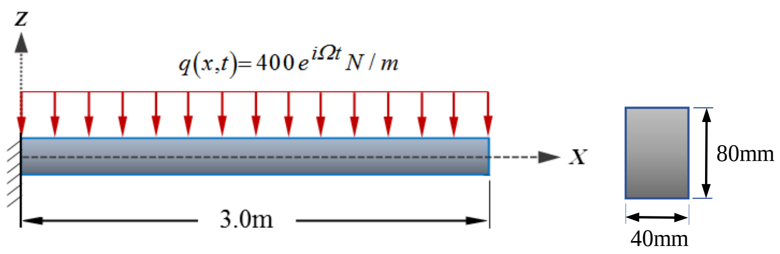


Figure (4): A cantilever beam under distributed transverse harmonic force

It is required to investigate the following:

(1) extracting the natural flexural frequencies and related mode shapes of the beam from steady state dynamic response analyses,

(2) conducting the quasi-static analysis of the cantilever beam under the given harmonic forces by adopting an exciting frequency $\Omega \approx 0.01\omega_1$, and

(3) computing the dynamic response of the cantilever beam at exciting frequency $\Omega = 1.50\omega_1$, where $\omega_1$ is the first natural flexural frequency of the cantilever beam.

The cantilever is modelled in Abaqus finite element by using 100 beam B31 element along the longitudinal axis of the cantilever beam. In other words, the model has 606 degrees of freedom in order to attain the required accuracy in this example. In constraint, the results obtained from the present finite beam element formulation use only one two-noded beam element with four degree of freedom to achieve the exact solution results.

**Exact Closed-Form Solution for Cantilever Beam**

The exact closed-form solution for cantilever under distributed harmonic force $q(x,t)=\bar{q}\, e^{i\Omega t}\, kN/m$ is obtained by substituting equation (14) into the boundary conditions: $W(0)=W'(0)=W''(\ell)=W'''(\ell)=0$ ,as:

$$W_c(x)=\langle E(x)\rangle_{1\times4}[G_c]^{-1}_{4\times4}\{Q_c\}_{4\times1}-\left\{\frac{\bar{q}}{\rho A\Omega^2}\right\}_{1\times1} \tag{23}$$

where $[G_c]_{4\times4}=\begin{bmatrix} 1 & 1 & 1 & 1 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ EI\beta_1^2 e^{\beta_1\ell} & EI\beta_2^2 e^{\beta_2\ell} & EI\beta_3^2 e^{\beta_3\ell} & EI\beta_4^2 e^{\beta_4\ell} \\ EI\beta_1^3 e^{\beta_1\ell} & EI\beta_2^3 e^{\beta_2\ell} & EI\beta_3^3 e^{\beta_3\ell} & EI\beta_4^3 e^{\beta_4\ell} \end{bmatrix}_{4\times4}$ , and $\langle Q_c\rangle_{1\times4}=\left\langle \frac{\bar{q}}{\rho A\Omega^2} \quad 0 \quad 0 \quad 0 \right\rangle_{1\times4}$

.

**Extracting Natural Frequencies and Mode Shapes**

For cantilever beam under uniformly distributed transverse harmonic force $q(x,t)=400\, e^{i\Omega t}\, kN/m$, the natural flexural frequencies are extracted from the steady state dynamic response analyses in which the exciting frequency $\Omega$ is varied from nearly zero to 500*Hz*. The peak transverse displacement at the cantilever tip as a function of the exciting frequency is shown in Figure (5). For the sake of comparison, two solutions based on the present finite beam element developed in this study and Abaqus finite element model are plotted on the same diagrams. Peaks on both diagrams indicate resonance and are thus indicators of the natural flexural frequencies of the given beam. Thus, the first five flexural natural frequencies extracted from the peaks related to the present finite beam element and Abaqus model solutions given in Table (1) are compared with exact closed-form solution presented in equation (23).

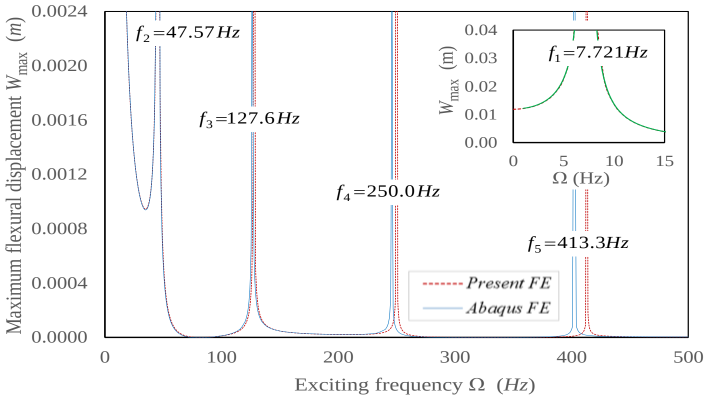


Figure (5): Steady state flexural response for cantilever beam under harmonic loading

The present finite element model based on a single beam element (2 dof) predicts the flexural natural frequencies in excellent agreement with those based on Abaqus beam model using 100 B31 elements (606 dof) and exact solution established by equation (23). This is a natural outcome that the finite element developed in this study based on the exact shape functions eliminates the mesh discretization errors.

**Table (1):** Natural flexural frequencies (*Hz*) for cantilever beam under harmonic transverse loading

| Frequency | Exact solution [1] | Present FE [2] | Abaqus FE [3] | %Difference =[1-2]/1 | %Difference =[1-3]/1 |
|---|---|---|---|---|---|
| 1 | 7.271 | 7.271 | 7.267 | 0.00% | 0.06% |
| 2 | 45.57 | 45.57 | 45.38 | 0.00% | 0.42% |
| 3 | 127.6 | 127.6 | 126.4 | 0.00% | 0.94% |
| 4 | 250.0 | 250.0 | 245.8 | 0.00% | 1.68% |

| | | | | | |
|---|---|---|---|---|---|
| 5 | 413.3 | 413.3 | 402.4 | 0.00% | 2.64% |

**Steady State Flexural Mode Shapes**

The first five normalized flexural mode shapes for the steady state transverse displacement and bending rotation responses of the cantilever beam under the given harmonic transverse loading using five different exciting frequencies at the peaks (i.e., $f_1 = 7.271Hz$, $f_2 = 45.57Hz$, $f_3 = 127.6Hz$, $f_4 = 250.0Hz$, $f_5 = 413.3Hz$) are shown in Figure (6). The normalized steady state flexural displacement and bending rotation mode shapes obtained from the present finite beam element (using 6 beam elements for the sake of comparison) and Abaqus beam model solution are plotted on the same diagrams exhibit excellent agreement.

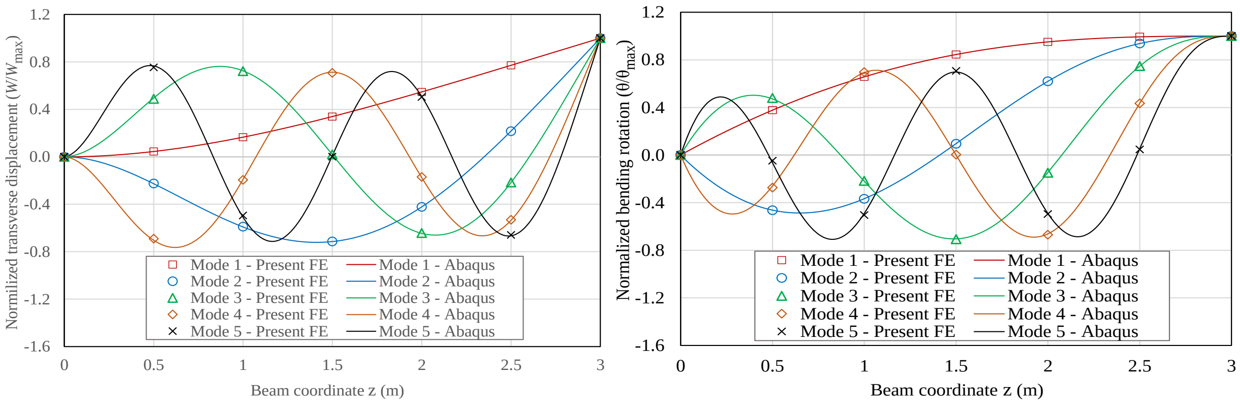


Figure (6): Steady state flexural displacement and bending rotation modes for cantilever beam

**4.2 Example (2) – Axial Static Force Effect**

A simply-supported steel beam ($E$=200GPa, $G$=77GPa, $\rho$=7850kg/m$^3$), having a length of 5000$mm$, width of 80$mm$ and depth of 80$mm$ subjected to distributed harmonic force $q(x,t) = 8.0\,e^{i\Omega t}\,kN/m$ in additional to axial static force $P_o$ acting through the centroid of the cross section is considered as shown in Figure (7).

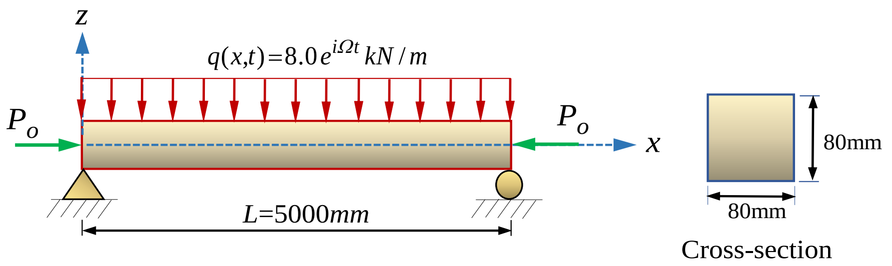


Figure (7): Simply supported axially preloaded beam under distributed harmonic force

It is required to (1) predicting the natural flexural frequencies and critical buckling loads for the beam, (2) investigating the axial static effect on natural flexural frequencies and steady state flexural dynamic response.

In order to validate the results obtained from the present finite beam element formulation, exact solution and Abaqus finite beam element are used for comparison. The present finite element formulation used two beam elements with 6 degrees of freedom while Abaqus finite element employed 100 B31 beam elements with 606 degrees of freedom to capture the accurate results.

**Critical Buckling Loads and Natural Frequencies**

For the given simply supported beam subjected to harmonic force $q(x,t) = 8.0\,e^{i\Omega t}\,kN/m$, the critical buckling loads and natural flexural frequencies related to first four buckling and vibration mode shapes are extracted from the steady state flexural dynamic analyses in which the exciting frequency $\Omega$ varying from nearly zero to 750rad/sec , as illustrated in Figure (8). It is observed that the natural flexural frequencies $(\omega_1, \omega_2, \omega_3, \omega_4)$ decrease with the increase of static compressive loads $P_o$, and the decrease become more quickly when the compressive forces close to critical buckling loads $(P_{cr1}, P_{cr2}, P_{cr3}, P_{cr4})$. This is attributed to the decreasing of beam stiffens. In other words, the critical buckling loads (i.e., $P_{cri} \approx 0$) are achieved when the natural frequencies ($\omega_i$ $rad/\sec$) are nearly equal zero (i.e., $\omega_i \approx 0$). This leads to conclude that the simply-supported beam losses stability when the compressive forces approach to critical buckling loads.

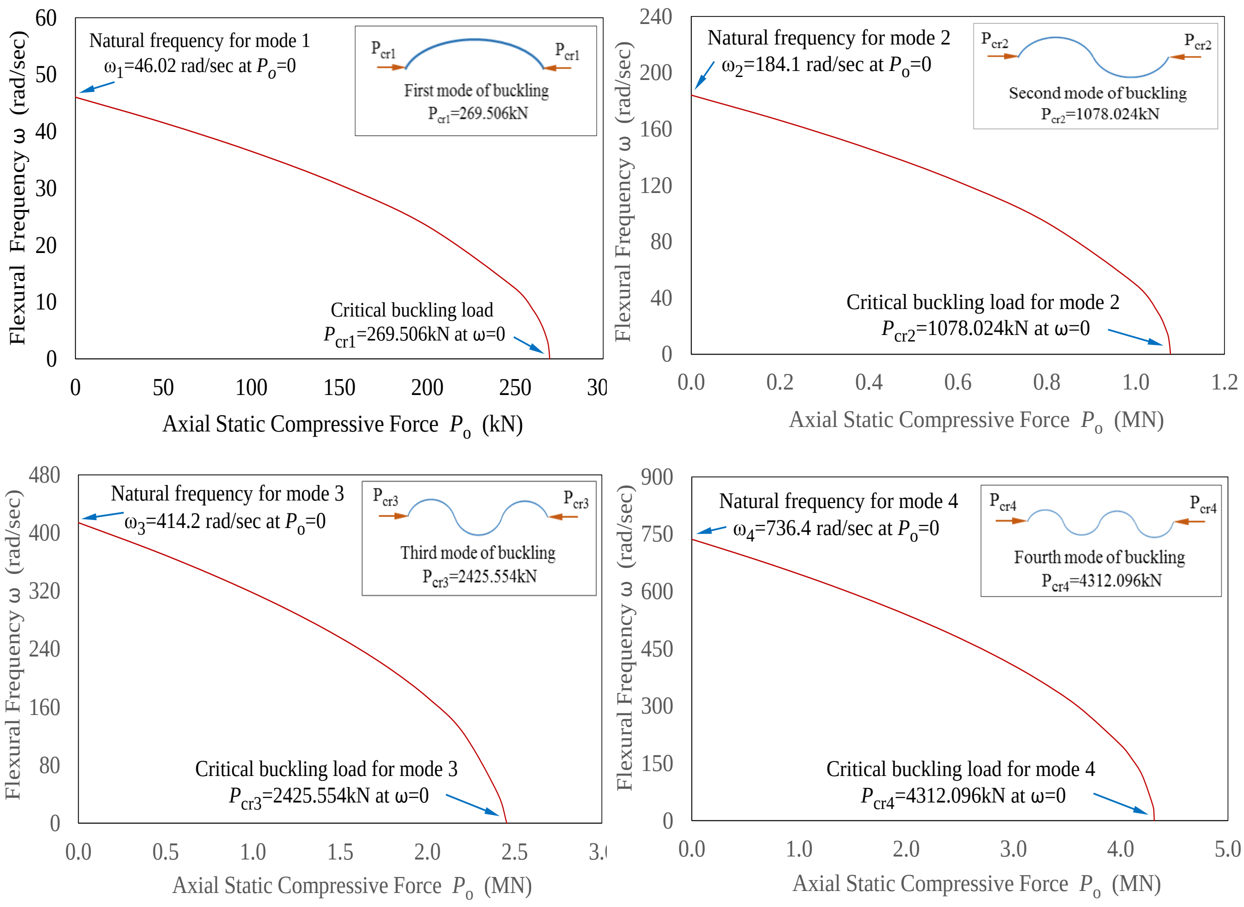


Figure (8): Natural flexural frequencies and critical buckling loads for simply-supported beam

Table (2) provides the first four natural flexural frequencies and critical buckling loads which extracted from the steady state flexural dynamic responses are based on three different solutions: the present finite element formulation, exact solution and ABAQUS finite element solution. As a general observation, excellent agreement is observed between present finite element solution and exact solution, while the corresponding results predicted by ABAQUS B31 beam model showed less agreement for the higher natural frequencies as well as the critical buckling loads. As can be noted form Table (2), ABAQUS beam model differ from those based on the exact solution by 0.02%-0.65% for natural flexural frequencies and by 0.04%-0.86% for the case of critical buckling loads. In constraint, the results computed from the present formulation exactly match with those obtained from the exact solution. As expected, the present finite beam element solution based on the exact shape functions which exactly satisfy the homogeneous form of the governing field equation, which in turn eliminates discretization errors induced in the conventional finite element formulations demonstrate excellent agreement with other solutions.

**Table (2):** The critical buckling loads and natural flexural frequencies of simply supported beam

| Mode Number | Critical Buckling Loads $P_{cr}$ (kN) | | | % Difference =[1-2]/1 | %Difference =[1-3]/3 |
|---|---|---|---|---|---|
| | Exact Solution | Present FE (4 dof) | Abaqus FE (606 dof) | | |
| 1 | 269.5 | 269.5 | 269.4 | 0.00% | 0.04% |
| 2 | 1078 | 1078 | 1076 | 0.00% | 0.19% |
| 3 | 2426 | 2426 | 2414 | 0.00% | 0.49% |

| | | | | | |
|---|---|---|---|---|---|
| 4 | 4312 | 4312 | 4275 | 0.00% | 0.86% |

| Mode Number | Natural Flexural Frequencies $\omega$ (rad/sec) | | | % Difference =[1-2]/1 | %Difference =[1-3]/3 |
|---|---|---|---|---|---|
| | Exact Solution | Present FE (4 dof) | Abaqus FE (606 dof) | | |
| 1 | 46.02 | 46.02 | 46.01 | 0.00% | 0.02% |
| 2 | 184.1 | 184.1 | 183.8 | 0.00% | 0.16% |
| 3 | 414.2 | 414.2 | 412.6 | 0.00% | 0.39% |
| 4 | 736.3 | 736.4 | 731.5 | -0.01% | 0.65% |

**Axial Static Force Effect on Natural Flexural Frequencies**

The first three natural flexural frequencies $(f_1, f_2, f_3)$ extracted from the steady state bending dynamic response analysis of simply-supported beam are shown in Figure (9) for different values of axial static forces varying from compression -200.0*kN* to tension 200*kN*. It is noted that the natural flexural bending frequencies decrease with the increase of axial static compressive forces, while an increase of axial tensile static force leads to increase the natural flexural frequencies. As expected, the axial tensile forces increase the natural flexural frequencies of the beam indicating an increase in the beam stiffness while the compressive force decrease the natural frequencies indicating reduction in the stiffness of beam.

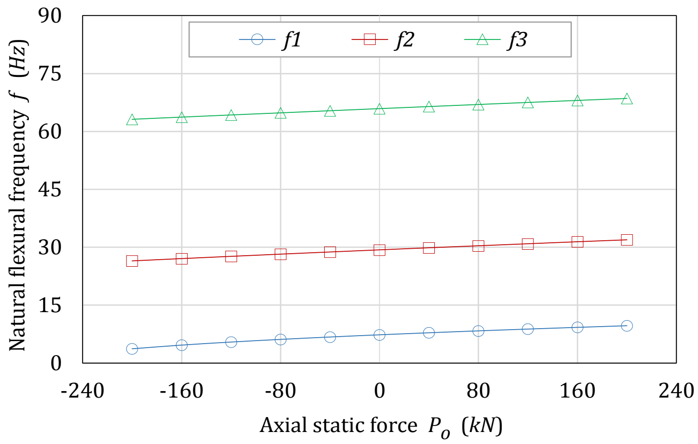


**Figure (9):** Axial static force effect on the natural flexural frequencies of simply-supported beam

**Axial Static Force Effect on Dynamic Response**

Figure (10) presents the steady state dynamic response for flexural displacement $(W)$ and bending rotation $(\Theta)$ versus the beam coordinate axis at exciting frequencies $\Omega = 1.48\omega_1 = 68.11 rad/\sec$ and for different values of axial static forces. It is obvious that the amplitudes of the flexural displacement and bending rotation decrease as the axial static force changes from tension $P_o = 1.5MN$ to compression $P_o = -1.5MN$. In other words, the axial tensile static force is increased, the natural flexural frequencies increase indicating an increase in beam stiffness while the axial compressive force has the opposite effect to that of axial tensile force, on the natural frequencies and the stiffness of a beam. Therefore, the results showed that the axial static compressive forces soften the beam whereas the tensile forces stiffen the beam.

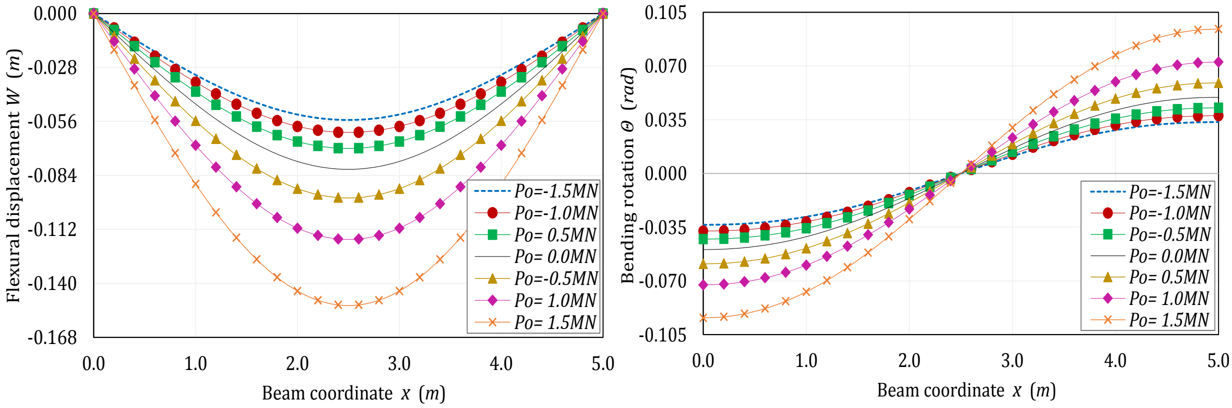


**Figure (10):** Flexural dynamic response of simply-supported beam under axial compressive force

### 4.3 Example (3) – Finite Element Formulation

This example is presented in order to show the ability and accuracy of the finite two-noded beam element developed in this study by comparing the obtained results for quasi-static and dynamic responses with those based on the Abaqus beam model results. A clamped-clamped EB beam under three harmonic bending forces; distributed force $q(x,t)=4.0\,e^{i\Omega t}\,kN/m$, concentrated transverse force $F(8m,t)=3.0\,e^{i\Omega t}\,kN$, and concentrated bending moment $M(6m,t)=2.0\,e^{i\Omega t}\,kNm$ is considered as illustrated in Figure (11). The beam has a circular cross-section of diameter 120mm, while the mechanical properties of the steel beam are: $E=200GPa$, $G=80GPa$ and $\rho=8000kg/m^3$.

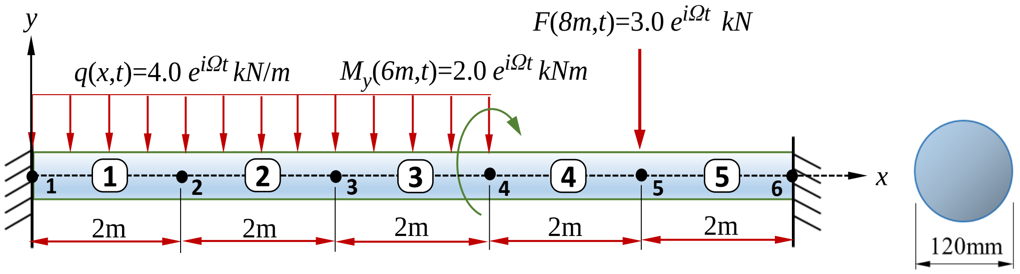


**Figure (11):** A clamped-clamped beam under various harmonic bending forces

In order to demonstrate the accuracy and capability of the present finite element based Euler-Bernoulli beam theory, the nodal degrees of freedom results for quasi-static flexural response and steady state flexural dynamic response are obtained and compared against the results based on established Abaqus finite beam element model. Under the present finite element solution, only five two-noded beam elements with twelve degrees of freedom are used to achieve the convergence while in Abaqus beam model solution, the model is consisted of two-hundred beam B31 elements with 1206 degrees of freedom along the beam axis to eliminate the discretization errors and attain the required accuracy of the solution.

**Quasi-Static and Dynamic Flexural Solutions**

The quasi-static flexural response of the clamped-clamped beam under the given harmonic bending forces and moments shown in Figure (11) approached by taking the value of the exciting frequency $\Omega$ significantly lower than the first natural flexural frequency of the given beam, i.e., $\Omega=0.01\omega_1\approx 0.3354\,rad/\sec$ (where the first natural flexural frequency of the clamped-clamped beam $\omega_1=33.54\,rad/\sec$) is plotted in Figure (12), while the steady state flexural dynamic response illustrated in Figure (13) is established at exciting frequency $\Omega=50\,rad/sec$. The results of quasi-static and steady state dynamic responses results obtained from the present finite element formulation based on five beam elements with 12 dof and Abaqus finite beam model using 200 beam B31 elements with 1206 dof are plotted on the same diagrams for comparison. It is observed that the nodal flexural displacements and rotations results obtained from both finite element solutions exhibit excellent agreement. Again, this is a natural outcome of the fact that, the present finite beam element formulation based on the exact shape functions which exactly satisfy the homogeneous solution of the governing flexural vibration equation. This treatment eliminates the discretization errors occurred in the conventional finite element formulations which based on approximate interpolation shape functions.

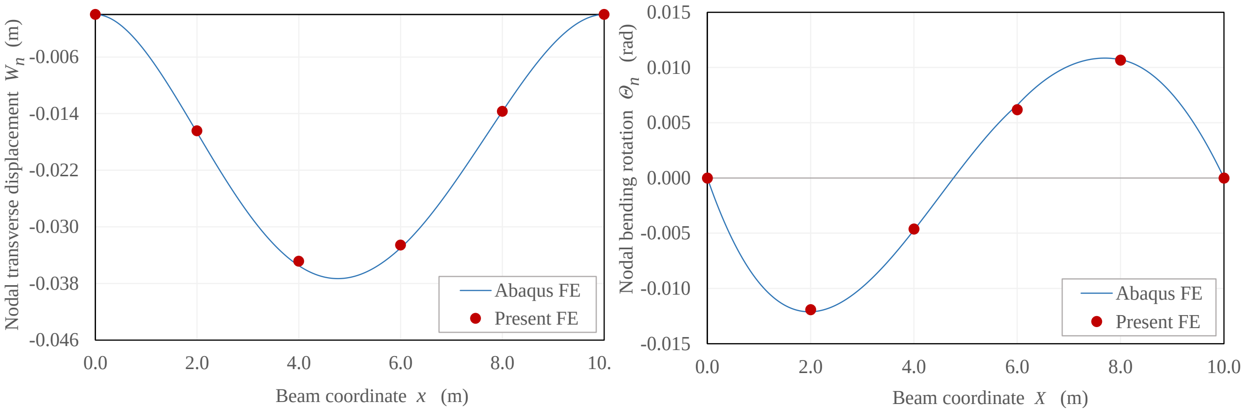


Figure (12): Quasi-static analysis for clamped-clamped beam

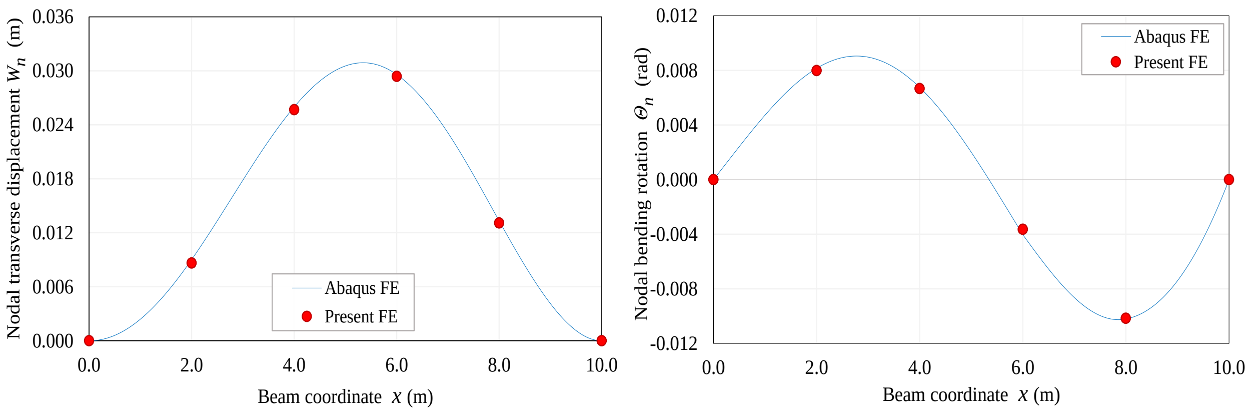


Figure (13): Dynamic analysis for clamped-clamped beam

## V. Summary and Conclusion

From the numerical results conducted throughout this study, the following concluding remarks are made:

- The governing field equation for the flexural bending vibration and related boundary conditions for axially loaded beams under harmonic bending excitations is derived through Hamilton variational principle.
- Exact closed-form solution of steady state flexural bending response of beams derived in this study is successfully used to formulate a family of exact shape functions which based on the homogeneous solution of the governing bending field equation.
- The exact shape functions are used to formulate a super-convergent finite beam element for the beams. The proposed beam element has a two nodes and four degrees of freedom.
- The new beam element involves no discretization errors and generally provides excellent results compared with Abaqus finite element solution while keeping the number of degrees of freedom a minimum.
- The present finite element formulation developed in this study are able to efficiently capture the quasi-static and steady state dynamic response of beams under harmonic bending loading. It is also capable of extracting the eigen-frequencies and eigen-modes as well as critical buckling loads.
- Results exhibit the axial static compressive load reduces the beam stiffness and result in lower predictions for natural flexural frequencies while the axial tensile load increases the natural flexural frequencies of the beam, indicating an increase in the stiffness of the beam.

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