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Closed form buckling analysis of thin rectangular plates

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Abstract: *This paper presents closed form buckling analysis of rectangular thin plates. It minimizes the total potential energy functional with respect to deflection function and obtained the Euler-Bernoulli equation of equilibrium of forces for the plate. Using split-deflection method, the equilibrium equation was uncoupled into two separate equations. The function satisfying each of the two equations was determined. Exact solution of Euler-Bernoulli governing equation for the plate was obtained as a product of the functions. Nine distinct deflection functions for plates were obtained after satisfying nine different boundary conditions. The paper went further to obtain the formula for calculating the critical buckling load of the plate by minimizing the total potential energy functional with respect coefficient of deflection. Numerical examples were carried out using two plates. One of the plates has two adjacent edges clamped and the other edges simply support (ccss). The other plate has one edge clamped and the other three edges simply supported (csss). The critical buckling loads obtained for the two plates were compared with the ones from an earlier study, which used polynomial deflection equation. For square ccss the values of the non dimensional critical buckling loads are 61.706 and 64.73 for the present and past studies respectively. For csss plate the values are 56.429 and 56.807 respectively for the present and past studies. The percentage difference between the values from the present and past studies are 4.67% for ccss and 0.67% for csss. It could be seen that the differences are not too significant.*

Keywords: *Buckling analysis, Potential energy, thin plates, Split deflection, critical buckling load*

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I. Introduction

There are basically two types of variational calculus for plate analysis as used herein. They are general variation and direct variation. General variation is the minimization of total potential energy with respect to displacement function whereas direct variation is the minimization of total potential energy with respect to coefficient of displacement. When the total potential energy functional is minimized with respect to coefficient of displacement, an equation typical of Ritz equation is obtained [1- 3]. On the other hand minimizing the total potential energy functional with respect to displacement function (instead of the coefficient of the displacement function) gives equation, which is typical of Euler-Bernoulli equation of equilibrium of forces. The integrand of the Euler-Bernoulli equation of equilibrium of forces is what is commonly referred to as "governing equation" [4-8]. Direct variation results to Raleigh's conservation of energy equation or Ritz (Rayleigh- Ritz) equation, from where the coefficient for the analysis is directly obtained. Thus, to arrive at exact solution for any plate analysis the two types of variational calculus must be employed. Only exact displacement functions can yield exact coefficient for the analysis. Approximate displacement functions can only give approximate coefficient for the analysis. The only way of getting exact displacement function is through direct integration of governing equation and satisfaction of the boundary condition. Most scholars assume displacement functions, which most times satisfies the boundary conditions but did not make attempt (through direct integration of the governing equation) to know if it satisfies the governing equation [9, 10]. The reason why scholar assume deflection function is to circumvent the integration of the governing equation. Another approach that circumvents the integration of the governing equation is numerical approach like finite element method. The results obtained from the numerical approach are termed approximate [11-13].

To this end, this paper tried to use split deflection approach (according to [14-16]) to uncouple and subsequently integrate the rectangular plate governing equation. After integration of the governing equation, the paper would go ahead to satisfy the boundary condition for various rectangular plates to obtain their distinct exact deflection equations in form of orthogonal trigonometric functions.

II. Theoretical approach

2.1 General Variational calculus and separation of governing equation of thin rectangular plate

The total potential energy equation for a thin rectangular plate with in-plane load along x-axis is given by [12]:

$$\Pi = \frac{bD}{2a^3}\int_0^1\int_0^1\left(\left[\frac{d^2w}{dR^2}\right]^2 + \frac{2}{\beta^2}\left[\frac{d^2w}{dRdQ}\right]^2 + \frac{1}{\beta^4}\left[\left(\frac{d^2w}{dQ^2}\right)^2\right]\right)dR\,dQ - \frac{bN_x}{2a}\int_0^1\int_0^1\left(\frac{dw}{dR}\right)^2 dR\,dQ \qquad (1)$$

Minimizing equation (1) with respect to deflection function and rearranging gave:

$$\int_0^1\int_0^1\left(\frac{d^4w}{dR^4} + \frac{2}{\beta^2}\frac{d^4w}{dR^2dQ^2} + \frac{1}{\beta^4}\frac{d^4w}{dQ^4} + \frac{N_xa^2}{D}\frac{d^2w}{dR^2}\right)dR\,dQ = 0 \qquad (2)$$

For equation (2) to be true, its integrand must be zero. That is:

$$\frac{d^4w}{dR^4} + \frac{2}{\beta^2}\frac{d^4w}{dR^2dQ^2} + \frac{1}{\beta^4}\frac{d^4w}{dQ^4} + \frac{N_xa^2}{D}\frac{d^2w}{dR^2} = 0 \qquad (3)$$

Equation (3) is coupled with the orthogonal deflection function, w. Thus, there is need to split the function into a product of two different functions as:

$$w = w_x \,.\, w_y \qquad (4)$$

Substituting equation (4) into equation (2) gave:

$$\int_0^1\int_0^1\left(w_y.\frac{d^4w_x}{dR^4} + \frac{2}{\beta^2}\frac{d^2w_x}{dR^2}.\frac{d^2w_y}{dQ^2} + \frac{1}{\beta^4}\frac{d^4w_y}{dQ^4}.w_x + \frac{N_xa^2}{D}.w_y.\frac{d^2w_x}{dR^2}\right)dR\,dQ = 0 \qquad (5)$$

Equation (5) could further be rearranged as shown in equations (6) and (7):

$$\int_0^1\int_0^1\left(\left[w_y.\frac{d^4w_x}{dR^4} + \frac{N_xa^2}{D}.w_y.\frac{d^2w_x}{dR^2}\right] + \left[\frac{1}{\beta^4}\frac{d^4w_y}{dQ^4}.w_x + \frac{2}{\beta^2}\frac{d^2w_x}{dR^2}.\frac{d^2w_y}{dQ^2}\right]\right)dR\,dQ = 0 \qquad (6)$$

$$\int_0^1\int_0^1\left(w_y.\frac{d^4w_x}{dR^4} + \frac{N_xa^2}{D}.w_y.\frac{d^2w_x}{dR^2}\right)dR\,dQ + \int_0^1\int_0^1\left(\frac{1}{\beta^4}\frac{d^4w_y}{dQ^4}.w_x + \frac{2}{\beta^2}\frac{d^2w_x}{dR^2}.\frac{d^2w_y}{dQ^2}\right)dR\,dQ = 0 \qquad (7)$$

Equation (7) could be zero if each of the two integrals is zero. That is:

$$\int_0^1\int_0^1\left(w_y.\frac{d^4w_x}{dR^4} + \frac{N_xa^2}{D}.w_y.\frac{d^2w_x}{dR^2}\right)dR\,dQ = 0 \qquad (8)$$

$$\int_0^1\int_0^1\left(\frac{1}{\beta^4}\frac{d^4w_y}{dQ^4}.w_x + \frac{2}{\beta^2}\frac{d^2w_x}{dR^2}.\frac{d^2w_y}{dQ^2}\right)dR\,dQ = 0 \qquad (9)$$

Carrying out the closed domain integrations of equations (8) and (9) with respect to Q and R respectively gave:

$$\int_0^1\left(w_2.\frac{d^4w_x}{dR^4} + \frac{N_xa^2}{D}.w_2.\frac{d^2w_x}{dR^2}\right)dR = 0 \qquad (10)$$

$$\int_0^1\left(\frac{1}{\beta^4}\frac{d^4w_y}{dQ^4}.w_1 + \frac{2}{\beta^2}w_3.\frac{d^2w_y}{dQ^2}\right)dQ = 0 \qquad (11)$$

Where: w_1, w_2 and w_3 are all constants defined as:

$$w_1 = \int_0^1 w_x\,dR\,; \qquad w_2 = \int_0^1 w_y\,dQ\,; \qquad w_3 = \int_0^1 \frac{d^2w_x}{dR^2}dR \qquad (12)$$

For equations (10) and (11) to be true, their integrands must be zero. Now, considering that integrand of equation (10) is zero, this gave:

$$\frac{d^4w_x}{dR^4} + \frac{N_xa^2}{D}.\frac{d^2w_x}{dR^2} = 0 \qquad (13)$$

Considering that integrand of equation (11) is zero and rearranging it gave:

$$\frac{d^4w_y}{dQ^4} + 2\beta^2\frac{w_3}{w_1}.\frac{d^2w_y}{dQ^2} = 0 \qquad (14)$$

The following constants were defined as:

$$g_1{}^2 = \frac{N_xa^2}{D} \quad ; \quad g_2{}^2 = 2\beta^2\frac{w_3}{w_1} \qquad (15)$$

With the definitions of equation (15), equations (13) and (14) became:

$$\frac{d^4w_x}{dR^4} + g_1{}^2 . \frac{d^2w_x}{dR^2} = 0 \qquad (16)$$

$$\frac{d^4w_y}{dQ^4} + g_2{}^2 . \frac{d^2w_y}{dQ^2} = 0 \qquad (17)$$

Equations (16) and (17) (or equations 13 and 14) are two different beam governing equations separated from the plate governing equation. The ready exact solutions of equation (16) and (17) are:

$$w_x = c_1 + c_2R + c_3e^{ig_1R} + c_4e^{-ig_1R} \qquad (18)$$

$$w_y = d_1 + d_2Q + d_3e^{ig_2Q} + d_4e^{-ig_2Q} \qquad (19)$$

The trigonometric transformation of equation (18) is:

$$w_x = a_0 + a_1R + a_2 \operatorname{Cos} g_1R + a_3 \operatorname{Sin} g_1R \qquad (20)$$

Where: $a_0 = c_0;\ a_1 = c_1;\ a_2 = c_2 + c_3;\ a_3 = jc_2 - jc_3$

The trigonometric transformation of equation (17) is:

$$w_y = b_0 + b_1Q + b_2 \operatorname{Cos} g_2Q + b_3 \operatorname{Sin} g_2Q \qquad (21)$$

Where: $b_0 = d_0;\ b_1 = d_1;\ b_2 = d_2 + d_3;\ b_3 = jd_2 - jd_3$

Equation (20) and (21) are the same for beams, which are oriented along x and y directions respectively.

2.2 Satisfaction of beam boundary conditions

The derivatives of equation (20) are:

$$\frac{dw_x}{dR} = a_1 - g_1a_2 \operatorname{Sin} g_1R + g_1a_3 \operatorname{Cos} g_1R \qquad (22)$$

$$\frac{d^2w_x}{dR^2} = -g_1{}^2a_2 \operatorname{Cos} g_1R - g_1{}^2a_3 \operatorname{Sin} g_1R \qquad (23)$$

$$\frac{d^3w_x}{dR^3} = g_1{}^3a_2 \operatorname{Sin} g_1R - g_1{}^3a_3 \operatorname{Cos} g_1R \qquad (24)$$

$$\frac{d^4w_x}{dR^4} = g_1{}^4a_2 \operatorname{Cos} g_1R + g_1{}^4a_3 \operatorname{Sin} g_1R \qquad (25)$$

2.2.1 Beam simply supported at both ends (S - S beam)

The boundary condition for S - S beam are:

$$w_x(R=0) = \frac{d^2w_x(R=0)}{dR^2} = w_x(R=1) = \frac{d^2w_x(R=1)}{dR^2} = 0 \qquad (26)$$

Substituting equation (20) into equation (26) (leveraging on equations 22 to 25) gave:

$$\begin{vmatrix} w_x(0) \\ \frac{d^2w_x(0)}{dR^2} \\ w_x(1) \\ \frac{d^2w_x(1)}{dR^2} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -g_1{}^2 & 0 \\ 1 & 1 & \operatorname{Cos} g_1 & \operatorname{Sin} g_1 \\ 0 & 0 & -g_1{}^2 \operatorname{Cos} g_1 & -g_1{}^2 \operatorname{Sin} g_1 \end{vmatrix} \begin{vmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{vmatrix} = 0 \qquad (27)$$

Solving for the determinant of equation (27) gave the non trivial solution as:

$$\operatorname{Sin} g_1 = 0 \qquad (28)$$

The value of g_1 that satisfies equation (28) is:

$$g_1 = m\pi \ \ [\text{where } m = 1, 2, 3 \ldots] \qquad (29)$$

Substituting equation (29) into equations (22) to (25) and satisfying the boundary conditions of equation (26) gave:

$$a_0 = a_1 = a_2 = 0 \qquad (30)$$

Substituting equations the constants of (29) and (30) into equation (20) gave:

$$w_x = a_3 \operatorname{Sin}(m\pi R) \qquad (31)$$

When similar thing was done on equation (21) for simply supported beam equation (32) was obtained:

$$w_y = b_3 \operatorname{Sin}(n\pi Q) \quad \{\text{where } n = 1, 2, 3 \ldots\} \qquad (32)$$

2.2.2 Beam clamped at both ends (C - C beam)
The boundary condition for C - C beam are:
$$w_x(R=0)=\frac{dw_x(R=0)}{dR}=w_x(R=1)=\frac{dw_x(R=1)}{dR}=0 \qquad (33)$$
Substituting equation (20) into equation (33) (leveraging on equations 22 to 25) gave:
$$\begin{vmatrix} w_x(0) \\ \frac{dw_x(0)}{dR} \\ w_x(1) \\ \frac{dw_x(1)}{dR} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & g_1 \\ 1 & 1 & \cos g_1 & \sin g_1 \\ 0 & 1 & -g_1 \sin g_1 & g_1 \cos g_1 \end{vmatrix} \begin{vmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{vmatrix} = 0 \qquad (34)$$

Solving for the determinant of equation (34) and simplifying gave the characteristic equation as:
$$2\cos g_1 + g_1 \sin g_1 - 2 = 0 \qquad (35)$$
The value of g_1 that satisfies equation (35) is:
$$g_1 = 2m\pi \ \{\text{where } m = 1, 2, 3 \ldots\} \qquad (36)$$
Substituting equation (36) into equations (22) to (25) and satisfying the boundary conditions of equation (33) gave:
$$a_1 = a_3 = 0; \quad a_0 = -a_2 \qquad (37)$$
Substituting the constants of equations (36) and (37) into equation (20) gave:
$$w_x = a_2(\cos 2m\pi R - 1) \qquad (38a)$$
$$w_x = a_0(1 - \cos 2m\pi R) \qquad (38b)$$
When similar thing was done on equation (21) for beam clamped at both ends equation (39) was obtained:
$$w_y = b_2(\cos 2m\pi Q - 1) \qquad (39a)$$
$$w_y = b_0(1 - \cos 2m\pi Q) \qquad (39b)$$
2.2.3 Beam clamped at one end and simply supported at the other end (C - S beam)
The boundary condition for C - S beam are:
$$w_x(R=0)=\frac{dw_x(R=0)}{dR}=w_x(R=1)=\frac{d^2w_x(R=1)}{dR^2}=0 \qquad (40)$$
Substituting equation (20) into equation (40) (leveraging on equations 22 to 25) gave:
$$\begin{vmatrix} w_x(0) \\ \frac{dw_x(0)}{dR} \\ w_x(1) \\ \frac{d^2w_x(1)}{dR^2} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & g_1 \\ 1 & 1 & \cos g_1 & \sin g_1 \\ 0 & 0 & -{g_1}^2 \cos g_1 & -{g_1}^2 \sin g_1 \end{vmatrix} \begin{vmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{vmatrix} = 0 \qquad (41)$$

Solving for the determinant of equation (41) gave the characteristic equation as:
$$g_1 \cos g_1 - \sin g_1 = 0 \qquad (42)$$
The value of g_1 that satisfies equation (35) is:
$$g_1 = 4.49340946 \qquad (43)$$
Substituting equation (43) into equations (22) to (25) and satisfying the boundary conditions of equation (40) gave:
$$a_0 = g_1 a_3 \ ; \quad a_1 = -g_1 a_3; \ a_2 = -g_1 a_3 \qquad (44)$$
Substituting the constants of equations (43) and (44) into equation (20) gave:
$$w_x = a_3(g_1 - g_1 R - g_1 \cos g_1 R + \sin g_1 R) \qquad (45)$$
When similar thing is done on equation (21) for beam clamped at end and simply supported at the other end equation (46) was obtained:
$$w_y = b_3(g_1 - g_1 Q - g_1 \cos g_1 Q + \sin g_1 Q) \qquad (46)$$

2.2.4 Beam clamped at one end and free at the other end (C - F beam)
The boundary condition for C - F beam are:

$$w_x(R=0) = \frac{dw_x(R=0)}{dR} = M(R=1) = V(R=1) = 0 \qquad (47)$$

The shear force and bending moment equations of beam in buckling mode are given as:

$$\frac{V}{D} = \frac{d^3w}{dR^3} + \frac{N_x a^2}{D}.\frac{dw}{dR} \qquad (48)$$

$$\frac{M}{D} = -\frac{d^2w}{dR^2} \qquad (49)$$

Satisfying the first condition of equation (47) using equation (20) gave:

$$w_x(0) = 0 = a_0 + 0 + a_2 + 0 \implies a_0 = -a_2 \qquad (50)$$

Satisfying the second condition of equation (47) using equation (22) gave:

$$\frac{dw_x(0)}{dR} = 0 = a_1 + g_1 a_3 \implies a_1 = -g_1 a_3 \qquad (51)$$

Substituting equations (22), (24) and the fourth condition of equation (47) into equation (48) gave:

$$0 = g_1{}^3 a_2 \operatorname{Sin} g_1 - g_1{}^3 a_3 \operatorname{Cos} g_1 + g_1{}^2.(a_1 - g_1 a_2 \operatorname{Sin} g_1 + g_1 a_3 \operatorname{Cos} g_1) \implies a_1 = 0 \qquad (52)$$

Substituting equation (52) into equation (51) gave:

$$a_3 = 0 \qquad (53)$$

Substituting equation (23) and the third condition of equation (47) into equation (49) gave:

$$0 = -g_1{}^2 a_2 \operatorname{Cos} g_1 - g_1{}^2 a_3 \operatorname{Sin} g_1 \implies a_2 \operatorname{Cos} g_1 = 0 \qquad (54)$$

The value of g_1 that satisfies equation (54) is:

$$g_1 = \frac{m\pi}{2} \ \{\text{where: } m = 1, 3, 5, 7, . \ . \ .\} \qquad (55)$$

Substituting the constants of equations (50), (52), (53) and (55) into equation (20) gives:

$$w_x = a_2\left(\operatorname{Cos}\frac{m\pi R}{2} - 1\right) \qquad (56)$$

When similar thing was done on equation (21) for beam clamped at one end and free at the other end gave:

$$w_y = b_2\left(\operatorname{Cos}\frac{m\pi Q}{2} - 1\right) \qquad (57)$$

The deflection functions for the four boundary conditions were summarized on Table 1:

Table 1: Deflection function of beams of various boundary conditions

Boundary Conditions	w_x	w_y
S - S	$a_3 \operatorname{Sin}(m\pi R)$ $[m = 1, 2, 3 . . .]$	$b_3 \operatorname{Sin}(n\pi Q)$ $[n = 1, 2, 3 . . .]$
C - C	$a_2(\operatorname{Cos} 2m\pi R - 1)$ $[m = 1, 2, 3 . . .]$	$b_2(\operatorname{Cos} 2m\pi Q - 1)$ $[n = 1, 2, 3 . . .]$
C - S	$a_3(g_1 - g_1 R - g_1 \operatorname{Cos} g_1 R + \operatorname{Sin} g_1 R)$	$b_3(g_1 - g_1 Q - g_1 \operatorname{Cos} g_1 Q + \operatorname{Sin} g_1 Q)$
C - F	$a_2\left(\operatorname{Cos}\frac{m\pi R}{2} - 1\right)$ $[m = 1, 3, 5 . . .]$	$b_2\left(\operatorname{Cos}\frac{m\pi Q}{2} - 1\right)$ $[n = 1, 3, 5 . . .]$

Legend: $g_1 = 4.49340946$

2.3 Deflection function for rectangular plate under buckling

The configuration rectangular plate was defined on Figure 1. Strip x was defined by edges 2 and 4 where as strip y was defined by edges 1 and 3. The boundary conditions of the trips (strip x and strip y) and the edge numbering were used in defining the plate and its boundary conditions. The orthogonal deflection function of rectangular was defined in equation 4. Using this equation and table 1 (considering the strips as beams), the orthogonal deflection functions of rectangular plates of various boundary conditions were defined as presented on Table 2.

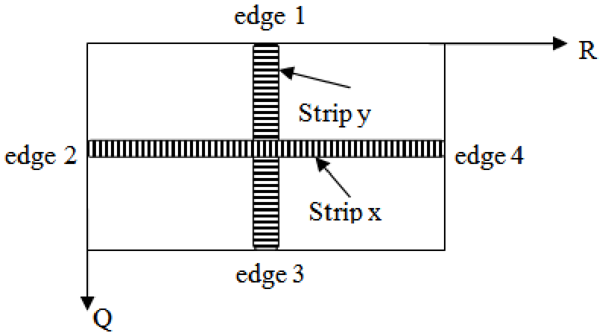

Figure 1: Configuration rectangular plate

2.4 Direct Variational calculus and Buckling load coefficient for thin rectangular plates

From the definitions of the orthogonal deflection functions as presented on Table 2, the deflection function could be defined as:

$$w = Ah \qquad (58)$$

Where "h" is the shape (profile) function of deflection and "A" is the deflection coefficient. Substituting equation (58) into equation (1) gave:

$$\Pi = \frac{A^2bD}{2a^3}\int_0^a\int_0^b\left(\left[\frac{d^2h}{dR^2}\right]^2 + \frac{2}{\beta^2}\left[\frac{d^2h}{dRdQ}\right]^2 + \frac{1}{\beta^4}\left[\frac{d^2h}{dQ^2}\right]^2\right)dR\,dQ - \frac{A^2bN_x}{2a}\int_0^a\int_0^b\left(\frac{dh}{dR}\right)^2 dR\,dQ \qquad (59)$$

Minimizing equation (59) with respect to the coefficient of deflection "A" gave:

$$\frac{\partial\Pi}{\partial A} = \frac{AbD}{a^3}\int_0^1\int_0^1\left(\left[\frac{d^2h}{dR^2}\right]^2 + \frac{2}{\beta^2}\left[\frac{d^2h}{dRdQ}\right]^2 + \frac{1}{\beta^4}\left[\frac{d^2h}{dQ^2}\right]^2\right)dR\,dQ - \frac{AbN_x}{a}\int_0^1\int_0^1\left(\frac{dh}{dR}\right)^2 dR\,dQ = 0.$$

That is, upon rearrangement:

$$\frac{N_xa^2}{D}k_N = \frac{k_x + \frac{2}{\beta^2}k_{xy} + \frac{1}{\beta^4}k_y}{k_N} \qquad (59)$$

Where the stiffness coefficients were defined as:

$$k_N = \int_0^a\int_0^b\left(\frac{dh}{dR}\right)^2 dR\,dQ\,; \quad k_x = \int_0^a\int_0^b\left(\frac{d^2h}{dR^2}\right)^2 dR\,dQ\,; \quad k_{xy} = \int_0^a\int_0^b\left(\frac{d^2h}{dRdQ}\right)^2 dR\,dQ$$

$$k_y = \int_0^a\int_0^b\left(\frac{d^2h}{dQ^2}\right)^2 dR\,dQ$$

Table 2: Deflection function of beams of various boundary conditions

Boundary Conditions	$\mathbf{w = w_x\,.\,w_y}$
$SSSS = (S-S)_{strip\ x}\,(S-S)_{strip\ y}$	$A\,Sin(m\pi R)\,Sin(n\pi Q)$ $[m = 1, 2, 3\ etc\,;\quad n = 1, 2, 3\ etc]$
$CCCC = (C-C)_{strip\ x}\,(C-C)_{strip\ y}$	$A(Cos\,2m\pi R - 1)\,(Cos\,2n\pi Q - 1)$ $[m = 1, 2, 3\ etc\,;\quad n = 1, 2, 3\ etc]$
$CSCS = (S-S)_{strip\ x}\,(C-C)_{strip\ y}$	$A\,Sin(m\pi R)(Cos\,2n\pi Q - 1)$ $[m = 1, 2, 3\ etc\,;\quad n = 1, 2, 3\ etc]$
$CCSS = (C-S)_{strip\ x}\,(C-S)_{strip\ y}$	$A(g_1 - g_1R - g_1\,Cos\,g_1R + Sin\,g_1R)$ $(g_1 - g_1Q - g_1\,Cos\,g_1Q + Sin\,g_1Q)$ $g_1 = 4.49340946$
$CSSS = (S-S)_{strip\ x}\,(C-S)_{strip\ y}$	$A\,Sin(m\pi R)\ (g_1 - g_1Q - g_1\,Cos\,g_1Q + Sin\,g_1Q)$ $[m = 1, 2, 3\ etc\,;\quad g_1 = 4.49340946]$
$CCCS = (C-S)_{strip\ x}\,(C-C)_{strip\ y}$	$A(g_1 - g_1R - g_1\,Cos\,g_1R + Sin\,g_1R)$ $(Cos\,2n\pi Q - 1)$ $n = 1, 2, 3\ etc;\quad g_1 = 4.49340946$
$CCFC = (C-C)_{strip\ x}\,(C-F)_{strip\ y}$	$A(Cos\,2m\pi R - 1)\left(Cos\frac{n\pi Q}{2} - 1\right)$ $[m = 1, 2, 3\ etc\,;\quad n = 1, 2, 3\ etc]$
$CSFS = (S-S)_{strip\ x}\,(C-F)_{strip\ y}$	$A\,Sin(m\pi R)\left(Cos\frac{n\pi Q}{2} - 1\right)$ $[m = 1, 2, 3\ etc\,;\quad n = 1, 2, 3\ etc]$
$CCFS = (C-S)_{strip\ x}\,(C-F)_{strip\ y}$	$A(g_1 - g_1R - g_1\,Cos\,g_1R + Sin\,g_1R)\left(Cos\frac{n\pi Q}{2} - 1\right)$ $[m = 1, 2, 3\ etc\,;\quad g_1 = 4.49340946]$

2.5 Numerical problem

Determine the buckling load applied along x-axis of rectangular plates: (i) with two adjacent edges clamped and the other two adjacent edges simply supported (CCSS) and (ii) with first edge clamped and the other three edges simply supported (CSSS). The shape functions, taken from Table 2 for the two plates, respectively are:

$$h = (g_1 - g_1R - g_1\,Cos\,g_1R + Sin\,g_1R)\ (g_1 - g_1Q - g_1\,Cos\,g_1Q + Sin\,g_1Q)$$

$$h = Sin(m\pi R)\ (g_1 - g_1Q - g_1\,Cos\,g_1Q + Sin\,g_1Q)$$

Where: $g_1 = 4.49340946$

With these shape functions, the stiffness coefficients for the two plates in the first mode of failure are:

(CCSS): $k_x = 85000.7326;\ k_{xy} = 53121.03833;\ k_y = 85000.7326;\ k_N = 4476.796382$

(CSSS): $k_x = 946.0270604;\ k_{xy} = 1137.373296;\ k_y = 2188.056887;\ k_N = 95.85258152$

Substituting the stiffness coefficients for CCSS plate into equation (59) gave:

$$\frac{N_x a^2}{D} = \frac{85000.7326 + \frac{2}{\beta^2} \times 53121.03833 + \frac{1}{\beta^4} \times 85000.7326}{4476.796382} \quad . \text{ That is:}$$

$$\frac{N_x a^2}{D} = 18.987 + \frac{23.732}{\beta^2} + \frac{18.987}{\beta^4} \qquad (60)$$

Substituting the stiffness coefficients for CSSS plate into equation 59 gave:

$$\frac{N_x a^2}{D} = \frac{946.0270604 + \frac{2}{\beta^2} \times 1137.373296 + \frac{1}{\beta^4} \times 2188.056887}{95.85258152} \quad . \text{ That is:}$$

$$\frac{N_x a^2}{D} = 9.87 + \frac{23.732}{\beta^2} + \frac{22.827}{\beta^4} \qquad (61)$$

The results from the present study with those from earlier studies were compared using simple percentage difference, which was defined as:

$$\%\text{Diff} = \frac{|\text{Present} - \text{Past}|}{\text{Past}} \times 100 \qquad (62)$$

III. Results and discussions

From work of Ibearugbulem (2012), it was gathered that buckling loads for CCSS plate and CSSS plate were respectively given as:

$$\frac{N_x a^2}{D} = 21 + \frac{22.73}{\beta^2} + \frac{21}{\beta^4} \qquad (63)$$

$$\frac{N_x a^2}{D} = 9.882 + \frac{22.74}{\beta^2} + \frac{24.185}{\beta^4} \qquad (64)$$

The values of the non dimensional buckling load parameters for different aspect ratios were calculated using equations (60) and (61) presented on Table 3.

Table 3: Non dimensional critical buckling load parameter

	CCSS plate					CSSS plate		
β = b/a	**Present**	**Past**	**%Diff**		**β = b/a**	**Present**	**Past**	**%Diff**
1	61.706	64.73	4.67		1	56.429	56.807	0.67
1.1	51.5686	54.12841	4.73		1.1	45.07437	45.19407	0.26
1.2	44.62409	46.91204	4.88		1.2	37.35895	37.33696	0.06
1.3	39.67748	41.80239	5.08		1.3	31.90497	31.80546	0.31
1.4	36.03764	38.06341	5.32		1.4	27.92022	27.77959	0.51
1.5	33.28507	35.25037	5.58		1.5	24.92659	24.76595	0.65
1.6	31.1545	33.08325	5.83		1.6	22.62344	22.45515	0.75
1.7	29.47209	31.37939	6.08		1.7	20.81485	20.64619	0.82
1.8	28.12039	30.01589	6.31		1.8	19.36919	19.20438	0.86
1.9	27.0179	28.9078	6.54		1.9	18.19556	18.03697	0.88
2	26.10669	27.995	6.75		2	17.22969	17.07856	0.88

In addition, equations (63) and (64) were used to calculate the values of the non dimensional buckling load parameters as obtained by Ibearugbulem (2012) for different aspect ratios. The percentage differences between the values from the present study and those from Ibearugbulem (2012) were calculated using equation (62) and presented on the same Table 3 for both CCSS and CSSS plates. For CCSS plate, the average percentage difference between values from the present work and past work as recorded on Table 3 is 5.62% This means that the difference is not significant. However, it is of interest to note that Ibearugbulem (2012) used polynomial deflection function (assumed approximate function) in his work whereas; trigonometric function (product direct integration of governing equation) was used herein for the present work. Thus, the difference between the result from using polynomial deflection function as used by Ibearugbulem (2012) and the result from using trigonometric deflection function herein in mimicking the true shape of the deformed plated in buckling is not significant. Furthermore, it is observed from the same Table 3 that the average percentage difference between the values of non dimensional critical buckling obtained in this present work and the work of Ibearugbulem (2012) for csss plate is 0.6%. This small percentage difference mean that both the trigonometric deflection function used herein and the polynomial deflection used by Ibearugbulem (2012) were almost the same in mimicking the buckled deflection shape of CSSS plate.

IV. Conclusions

It becomes vivid here that the use of split deflection helped in uncoupling the governing differential equation of rectangular plate. This enabled the paper to separate the equation into two different and independent differential equations, whose solutions were quite easy to obtain. Hence, a critical observation reveals that exact solution of the governing equation in the form of orthogonal trigonometric function had been obtained for rectangular plate. Furthermore, by satisfying the boundary conditions of the nine different plates treated herein, nine distinct orthogonal trigonometric deflection functions were obtained for them. One interesting thing from this present work is the determination of distinct orthogonal trigonometric deflection functions for CCSS and CSSS plates analysis. It had been difficult, based on the literature available to the authors, to see scholars that analyzed CCSS and CSSS plates with trigonometric deflection function. Most of the earlier scholars, instead, usually resorted to numerical analysis for CCSS and CSSS plate.

References

[1]. Akhileshwar Singh (2013). Analysis of Stiffened Rectangular Plate. A thesis submitted to National Institute of Technology, Rourkela, Odisha, India in partial fulfillment for the degree of Master of Technology in Mechanical Engineering (Machine Design and Analysis).: Unpublished.

[2]. Ji-Huan He (2003). A Lagrangian for von Karman equations of large deflection problem of thin circular plate. Applied Mathematics and Computation vol. 143 (Elsevier Science Inc), pp. 543–549

[3]. R. N. Reddy (2007). Theory and Analysis of Elastic Plates and Shells. 2nd Ed. New York: CRC Press.

[4]. Bill Goodwine (2011). Engineering Differential Equations"Theory and Applications". London: Springer Science+Business Media. ISBN: 978-1-4419-7918-6

[5]. Vinesh V. Nishawala (2011). A study of large deflection of beams and Plates. A thesis submitted to the Graduate School—New Brunswick Rutgers, The State University of New Jersey in partial fulfillment of the requirements for the degree of Master of Science Graduate Program in Mechanical and Aerospace Engineering: Unpublished

[6]. A. A. Khdei & J. N. Reddy (1997). An exact solution for the bending of thin and thick cross-ply laminated beams. Composite Stuctures vol. 37 (Elsevier Science Inc), pp. 195-203

[7]. S. M. Ibrahim, E. Carrera, M. Petrolo and E. Zappino (2012). Buckling of composite thin walled beams by refined theory. Composite Structures vol. 94 (Elsevier Science Inc Elsevier Science Inc), pp. 563–570

[8]. D. J. Wagg (2004). A note on using the collocation method for modelling the dynamics of a flexible continuous beam subject to impacts. Journal of Sound and Vibration vol. 276, pp. 1128–1134

[9]. C. M. Wang, C. Y. Wang and J. N. Reddy (2005). Exact solutions for buckling of structural members. Florida: CRC Press LLC

[10]. Chai H. Yoo and Sung C. Lee (2011). Stability of structures "principles and applications". London: Elsevier Inc. ISBN 978-0-12-385122-2

[11]. M. Y. Osman, O. M. E. Suleiman (2017). Buckling Analysis of Thin Laminated Composite Plates using Finite Element Method. International Journal of Engineering Research And Advanced Technology (IJERAT), vol. 3, iss. 3, pp. 1 - 17

[12]. S. Han, S.Lee and G. Rus (2006). Postbuckling analysis of laminated composite plates subjected to the combination of in-plane shear, compression and lateral loading. International Journal of Solids and Structures vol. 43 (Elsevier Science Inc), pp. 5713–5735

[13]. S. K. Singh. and A. Chakrabarti (2012). Buckling analysis of laminated composite plates using an efficient C0 FE model. Latin American Journal of Solids and Structures vol. 1, pp. 1 – 13

[14]. Ibearugbulem, O. M., Ibearugbulem, C. N., Habib Momoh and Asomugha, U. C. (2016). Split-Deflection Method of Classical Rectangular Plate Analysis. International Journal of Scientific and Research Publications, Vol. 6, Iss. 5, pp. 147- 150

[15]. Ibearugbulem, O. M., Ibearugbulem, C. N., Habib Momoh and Asomugha, U. C. (2016). Buckling analysis of rectangular plate by Split-deflection method. International Journal of Emerging Technologies and Engineering (IJETE) Vol. 3, Iss. 5, pp. 27- 30

[16]. Ibearugbulem, O. M., Ibearugbulem, C. N., Habib Momoh and Asomugha, U. C. (2016). Free vibration analysis of rectangular plate by split-deflection method. International Journal of Engineering, Science and Mathematics, Vol. 5, Iss. 5, pp. 1- 11

[17]. Ibearugbulem, O. M. (2012). Application of a direct variational principle in Elastic Stability of Rectangular Flat Thin Plates. PhD. Thesis submitted to Postgraduate School, Federal University of Technology, Owerri, Nigeria

O. M. Ibearugbulem. "Closed form buckling analysis of thin rectangular plates." IOSR Journal of Mechanical and Civil Engineering (IOSR-JMCE) , vol. 16, no. 1, 2019, pp. 83-90