IOSR Journal of Mechanical and Civil Engineering (IOSR-JMCE)
e-ISSN: 2278-1684,p-ISSN: 2320-334X, Volume 15, Issue 1 Ver. III (Jan. - Feb. 2018), PP 31-39
www.iosrjournals.org

Bending Analysis of Isotropic Beam using "Hyperbolic Shear Deformation Theory"

Shaikh Nagma Saba[1], Dr, M.N. Mangulkar
[1](ME (Structures) Students, Department of CIVIL, Marathwada Institute of Technology Aurangabad, India)
[2](Associate Professor Department of Civil, Marathwada Institute of Technolog Aurangabad, India)

Abstract: *A new hyperbolic shear deformation theory for bending of isotropic beams, in which number of variables is same as that in the hyperbolic shear deformation theory, is developed. The theory takes into account transverse shear deformation effects; the noteworthy feature of theory is that the transverse shear stresses can be obtained directly from the use of constitutive relations with efficacy, satisfying the shear stress free condition on the top and bottom surfaces of the beam. Hence, the theory obviates the need of shear correction factor. The cantilever isotropic beam subjected to parabolic load is examined using the present theory. The scope of the present study is restricted to the linear analyses of beams with different aspect ratios. The beams can have cantilever as well as simple support boundary conditions. Results obtained are discussed critically with those of other theories.*

Keywords: *isotropic beam, deformation, principle of virtual work, equilibrium equations, displacement*

Date of Submission: 23-01-2018 Date of acceptance: 09-02-2018

I. Introduction

The shear deformation effects are more pronounced in the thick beams when subjected to transverse loads than in the thin beams under similar loading. The shear deformation effects are more significant in the thick beams. These effects are neglected in Elementary Theory of Beam (ETB). In order to describe the correct bending behavior of thick beams including shear deformation effects and the associated cross sectional warping, shear deformation theories are require. The wide spread use of shear flexible materials in aircraft, automotive, shipbuilding and other industries has stimulated interest in the accurate prediction of structural behavior of beams. The flexural analysis of thick beams led to the development of refined theories in order to address the correct structural behavior. The objective of this paper is to present a comprehensive review of refined shear deformation theories for shear deformable homogeneous, isotropic beams. More emphasis is placed on the recent advances in the modeling and analysis of isotropic beams.

Several researchers have presented their studies as, Rayleigh [9] included the rotatory inertia effect while later the effect of shear stiffness was added by Timoshenko [10]. Timoshenko showed that the effect of shear is much greater than that of rotatory inertia for transverse vibration of prismatic beams. The first correct boundary conditions for the Timoshenko beam were derived by Kruszewski [11] . Stephen and Levinson [22] have introduced a refined theory incorporating shear curvature, transverse direct stress and rotatory inertia effects. The governing differential equation is similar in form to the Timoshenko beam equation. However, the theory requires two coefficients, one for cross sectional warping and the second dependent on the transverse direct stresses. These coefficients for various cross sections are evaluated. Rychter [23] studied a rectangular beam bending theory, which incorporates the usual mean deflection and a rotation represented in terms of the relative axial displacement of the upper and lower surfaces of the beam. Soler [28] developed the higher order theory for thick isotropic rectangular elastic beams using Legendre polynomials Levinson [29] obtained the higher order beam theory providing the fourth order system of differential equations, satisfying two boundary conditions at each end of the beam. Ghugal [31, 32] has developed a beam theory including transverse shear deformation effect. Shi and Voyiadjis [37] have developed a new beam theory with the sixth order differential equations for the analysis of shear deformable beams with variational consistent boundary conditions and discussed the role of boundary conditions commensurate with the govering differential equations according to refined beam theories. Donnell [39] provided the series solution in the loading function, for the deflections and stresses in continuously loaded beams of rectangular cross section in terms of the top and bottom loading.

II. System of Development

The beam under consideration as shown in Fig. 3.1 occupies in $0-x-y-z$ Cartesian coordinate system the region:

$$0 \le x \le L \; ; \quad 0 \le y \le b \; ; \quad -\frac{h}{2} \le z \le \frac{h}{2} \tag{3.1}$$

where *x, y, z* are Cartesian coordinates, *L* and *b* are the length and width of beam in the *x* and *y* directions respectively, and *h* is the thickness of the beam in the *z*-direction. The beam is made up of homogeneous, linearly elastic isotropic material.

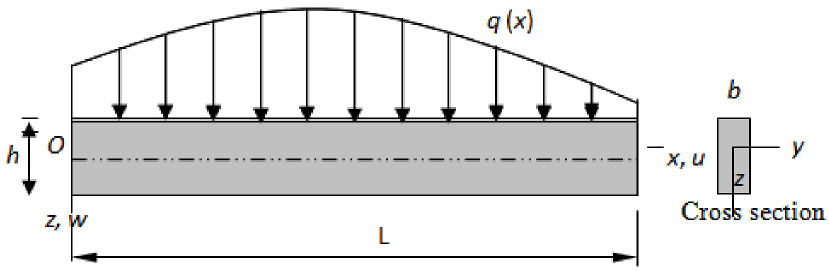

Fig. 1 Beam under bending in *x*-*z* plane

2.1 The displacement field

Based on the before mentioned assumptions, the displacement field of the present refined beam theory can be expressed as follows:

Axial and Transverse Displacement:

$$u(x,z) = -z\frac{dw}{dx} + \left[z\cosh\left(\frac{1}{2}\right) - \mathrm{hsinh}\left(\frac{z}{h}\right)\right]\varphi \tag{2.1}$$

$$w(x,z) = w(x)$$

Here, *u* is the axial displacement component in the *x* direction, and *w* is the transverse displacement in the *z* direction.

2.1 Strain-displacement relationships

Normal and shear strains are obtained within the framework of linear theory of elasticity using the displacement field given by Eqn. (2.1). These relationships are given as follows

Normal Strain:

$$\varepsilon_x = \frac{du}{dx} = -z\frac{d^2w}{dx^2} + f(z)\frac{d\phi}{dx} \tag{2.2}$$

Shear Strain:

$$\gamma_{zx} = \frac{du}{dz} + \frac{dw}{dx} = \left[\cosh\left(\frac{1}{2}\right) - \cosh\left(\frac{z}{h}\right)\right]\phi \tag{2.3}$$

Where,

$$f(z) = z\cosh\left(\frac{1}{2}\right) - h\sinh\left(\frac{z}{h}\right)$$

2.2 Stress-strain relationships:

. The axial stress σ_x is related to strain ε_x and shear stress is related to shear strain by the following constitutive relations:

$$\sigma_x = E\varepsilon_x = -zE\frac{d^2w}{dx^2} + E\left[z\cosh\left(\frac{1}{2}\right) - h\sinh\left(\frac{z}{h}\right)\right]\frac{d\phi}{dx} \tag{2.4}$$

$$\tau_{zx} = G\gamma_{zx} = G\left\{\cosh\left(\frac{1}{2}\right) - \cosh\left(\frac{z}{h}\right)\right\}\phi \tag{2.5}$$

Where, *E* and *G* are the elastic constants of the beam material

2.3 Governing equations and boundary conditions

$$b\int_{x=0}^{x=L}\int_{z=-h/2}^{z=+h/2}\left(\sigma_x\delta\varepsilon_x+\tau_{zx}\delta\gamma_{zx}\right)dx\,dz-\int_{x=0}^{x=L}q(x)\,\delta w\,dx=0 \tag{2.6}$$

Further simplifying it leads to

$$\int_{x=0}^{x=L}\left[EI\frac{d^4w}{dx^4}-AEI\frac{d^3\phi}{dx^3}-q(x)\right]\delta w dx+\int_{x=0}^{x=L}\left[AEI\frac{d^3w}{dx^3}-BEI\frac{d^2\phi}{dx^2}+CGbh\phi\right]\delta\phi dx$$
$$+\left(EI\frac{d^2w}{dx^2}-AEI\frac{d\phi}{dx}\right)\frac{d\delta w}{dx}\bigg|_0^L+\left(-EI\frac{d^3w}{dx^3}+AEI\frac{d^2\phi}{dx^2}\right)\delta w\big|_0^L+\left(-AEI\frac{d^2w}{dx^2}+BEI\frac{d\phi}{dx}\right)\delta\phi\big|_0^L=0$$

(2.7)

The governing differential equations obtained are as follows:

$$EI\frac{d^4w}{dx^4}-AEI\frac{d^3\phi}{dx^3}-q(x)=0 \tag{2.8}$$

$$AEI\frac{d^3w}{dx^3}-BEI\frac{d^2\phi}{dx^2}+CGbh\phi=0 \tag{2.9}$$

Where,

A=0.102401712EI, B=0.010608504EI, C=0.008738524EI

The associated variationally consistent boundary conditions obtained at the ends x = 0 and x = L is of following form:

Either $V_x=EI\frac{d^3w}{dx^3}-\frac{4}{5}EI\frac{d^2\phi}{dx^2}=0$ or w is prescribed (3.0)

Either $M_x=EI\frac{d^2w}{dx^2}-\frac{4}{5}EI\frac{d\phi}{dx}=0$ or $\frac{dw}{dx}$ is prescribed (3.1)

Either $M_s=\frac{4}{5}EI\frac{d^2w}{dx^2}+\frac{68}{15}EI\frac{d\phi}{dx}=0$ or ϕ is prescribed (3.2)

where V_x, M_x are the shear force and bending moment resultants respectively analogous to elementary theory of beam bending and M_s is the moment resultant due to the effect of transverse shear deformation. All the left hand equations in Eqn. (3.0) to (3.2) are natural or forced boundary conditions, and all the right hand terms are rigid or kinematic boundary conditions.

Thus, the variationally consistent governing differential equations and boundary conditions are obtained. The static (flexural) behaviour of the beam is described by the solution of these equations and simultaneously satisfaction of the associated boundary conditions. The associated boundary conditions for static flexure of beam under consideration can be obtained directly from Eqns. (3.0) through (3.2).

III. Illustrative Example

Example 1: A cantilever beam with varying load, $q(x)=q_0\frac{x}{L}$

The beam has its origin at left hand side fixed support at x = 0 and free at x = L. The beam is subjected to varying load, $q(x)=q_0\frac{x}{L}$ on surface z = +h/2 acting in the downward z direction with maximum intensity of load q_0.

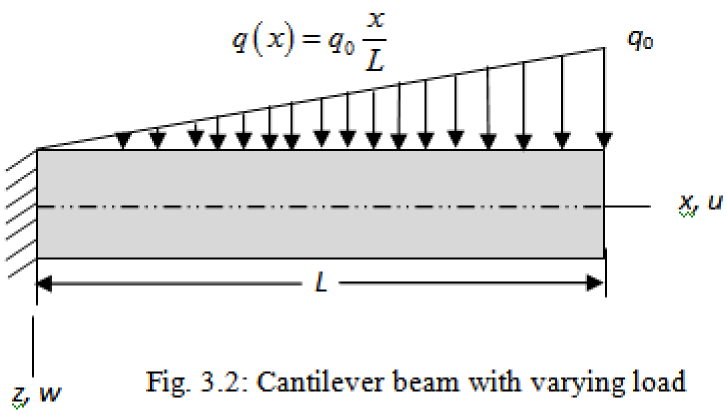

Fig. 3.2: Cantilever beam with varying load

The material properties for the beam used are: E=210Gpa, μ =0.3 and □ = 7800 kg/m³ as young's modulus, Poisson's ratio, density respectively

Boundary conditions associated with this problem are as follows:

At Free end: $EI\frac{d^2w}{dx^2} = EI\frac{d\phi}{dx} = EI\frac{d^3w}{dx^3} = EI\frac{d^2\phi}{dx^2} = 0$ at $x = L$ and

At Fixed end: $\frac{dw}{dx} = \phi = w = 0$ at $x = 0$

Using general solutions for $\phi(x)$ and $w(x)$ from Eqn. (3.13) and (3.14) the complete solution for a beam is obtained by imposing natural (forced) and / or geometric or kinematical boundary /end conditions of beam as mentioned in Eqn. (3.15) through Eqn. (3.17). A detailed analytical solution of this beam problem is given in Appendix C. The final expressions for transverse displacement $w(x)$ and $\phi(x)$ obtained from this solution are as follows:

$$w(x) = \left[\frac{x^5}{L^5} - 3\frac{x^3}{L^3} + 2\frac{x^2}{L^2} - \frac{3}{2}\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(\begin{array}{l}\dfrac{\sinh\lambda x - \cosh\lambda x + 1}{\lambda L} - \\ \dfrac{x}{L} + \dfrac{\cosh\lambda x}{\lambda L \sinh\lambda L}\end{array}\right)\right] + \frac{B_0}{A_0}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{1}{3}\frac{x^3}{L^3} - \frac{1}{2}\frac{x^2}{L^2}\right) \tag{3.24}$$

Substituting expressions for w given by Eqns. (3.24) into Eqns. (3.2), (3.3), (3.7) and (3.8), the final expressions for axial displacement u, transverse displacement w, axial stresses σ_x and transverse shear stress τ_{zx} can be obtained respectively.

Expression for axial displacement, u

$$\bar{u} = -\frac{z}{h}\frac{L^3}{h^3}\left\{\begin{array}{l}\dfrac{1}{2}\dfrac{x^4}{L^4} + \dfrac{2}{5}\dfrac{x}{L} - \dfrac{9}{10}\dfrac{x^2}{L^2} - \dfrac{3}{20}\dfrac{A_0}{C_0}\dfrac{E}{G}\dfrac{h^2}{L^2} \\ (\cosh\lambda x - \sinh\lambda x - 1) - \dfrac{1}{2}\dfrac{A_0^2}{C_0}\dfrac{E}{G}\dfrac{h^2}{L^2}\left(\dfrac{x}{L}\right) \\ -\dfrac{1}{10}\dfrac{B_0}{A_0}\dfrac{E}{G}\dfrac{h^2}{L^2}\left(\dfrac{x^2}{L^2} - \dfrac{x}{L}\right)\end{array}\right\} + \left(\begin{array}{l}\dfrac{z}{h}\cosh\left(\dfrac{\alpha}{2}\right) \\ -\dfrac{1}{\alpha}\sinh\left(\dfrac{z\alpha}{h}\right)\end{array}\right)\frac{3}{20}\frac{A_0}{C_0}\frac{E}{G}\frac{L}{h}\left[\begin{array}{l}\sinh\lambda x - \cosh\lambda x \\ -\dfrac{7}{3} + \dfrac{x^3}{L^3}\end{array}\right]$$

(3.25)

Expression for axial stress, σ_x

$$\bar{\sigma}_x = \frac{z}{h}\frac{L^2}{h^2}\left\{\begin{array}{l} 2\frac{x^3}{L^3}-\frac{9}{5}\frac{x}{L}+\frac{2}{5}- \\ \frac{3}{20}\frac{A_0}{C_0}\frac{E}{G}\frac{h^2}{L^2} \\ \left(\lambda L\left(\sinh\lambda x-\cosh\lambda x\right)\right) \end{array}\right\} - \frac{1}{2}\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2} - \frac{1}{10}\frac{B_0}{A_0}\frac{E}{G}\frac{h^2}{L^2}\left(2\frac{x}{L}-1\right)\left(\frac{z}{h}\cosh\left(\frac{\alpha}{2}\right)-\frac{1}{\alpha}\sinh\left(\frac{z\alpha}{h}\right)\right)\frac{3}{20}\frac{A_0}{C_0}\frac{E}{G}$$

$$\left(\lambda L\cosh\lambda x-\lambda L\sinh\lambda x-\frac{20}{3}\frac{x}{L}\right)$$

(3.26)

Expression for transverse shear stress using constitutive relationship τ_{zx}^{CR}

$$\tau_{zx}^{CR} = \left(\cosh\left(\frac{\alpha}{2}\right)-\cosh\left(\frac{z\alpha}{h}\right)\right)\frac{3}{20}\frac{A_0}{C_0}\frac{L}{h}\left(\sinh\lambda x-\cosh\lambda x-\frac{7}{3}\frac{x^2}{L^2}\right)$$

(3.27)

Expression for transverse shear stress τ_{zx}^{EE} **obtained from equilibrium equation**

The alternate approach to determine the transverse shear stresses is the use of equilibrium equations. Integrating the first equation with respect to the thickness coordinate and satisfying the boundary conditions at the bounding surfaces of the beam on can obtain the final expressions of transverse shear stresses. The first stress equilibrium equation of two dimensional theory of elasticity is as follows:

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{zx}}{\partial z}=0 \qquad (3.28)$$

The expression of obtained for transverse shear stress for this loading case is as follows:

$$\bar{\tau}_{zx} = \frac{1}{8}\left(4\frac{z^2}{h^2}-1\right)\frac{L}{h}\left\{\begin{array}{l} 6\frac{x^2}{L^2}-\frac{39}{20}+\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2} \\ \left(\lambda^2L^2\left(\begin{array}{l}\cosh\lambda x- \\ \sinh\lambda x\end{array}\right)\right)-\frac{1}{5}\frac{B_0}{A_0}\frac{E}{G}\frac{h^2}{L^2} \end{array}\right\} + \left[\begin{array}{l} \frac{1}{8}\left(\left(4\frac{z^2}{h^2}-1\right)\right)\cosh\left(\frac{\alpha}{2}\right) \\ -\frac{1}{\alpha^2}\cosh\left(\frac{\alpha}{2}\right) \end{array}\right]\frac{3}{20}\frac{A_0}{C_0}\frac{E}{G}\frac{h}{L}\left(\begin{array}{l} \lambda^2L^2\left(\begin{array}{l}\sinh\lambda- \\ \cosh\lambda x\end{array}\right)- \\ \frac{20}{3} \end{array}\right)$$

(3.29)

IV. Graphs and Tables

For beams subjected to various types load, *q(x)*

$$\bar{u} = \frac{Ebu}{q_0 h}, \quad \bar{w} = \frac{10Ebh^3 w}{q_0 L^4}$$

The transverse shear stresses ($\bar{\tau}_{zx}$) are obtained directly by constitutive relation and, alternatively, by integration of equilibrium equation of two dimensional elasticity and are denoted by ($\bar{\tau}_{zx}^{CR}$) and ($\bar{\tau}_{zx}^{EE}$) respectively. The transverse shear stress satisfies the stress free boundary conditions on the top $(z=-h/2)$ and bottom $(z=+h/2)$ surfaces of the beam when these stresses are obtained by both the above mentioned approaches.

Table 4.1: Non-Dimensional Axial Displacement ($\bar{u}$) at (x = L, z = h/2), Transverse Deflection ($\bar{w}$) at (x = L, z = 0.0), Axial Stress ($\bar{\sigma}_x$) at (x = 0, z = h/2) Maximum Transverse Shear Stresses $\bar{\tau}_{zx}^{CR}$ (x= 0.01L, z =0.0) and $\bar{\tau}_{zx}^{EE}$ (x = 0, z =0) of the Cantilever Beam Subjected to Varying Load for Aspect Ratio 4

Source	Model	$\bar{u}$	$\bar{w}$
Present HPSDT	HPSDT	54.2771	12.6187
Dahake [54]	TSDT	54.2767	12.6172
Krishna Murty [50]	HSDT	54.2771	12.6191
Timoshenko [53]	FSDT	48.0000	11.3250
Bernoulli-Euler	ETB	48.0000	11.0000

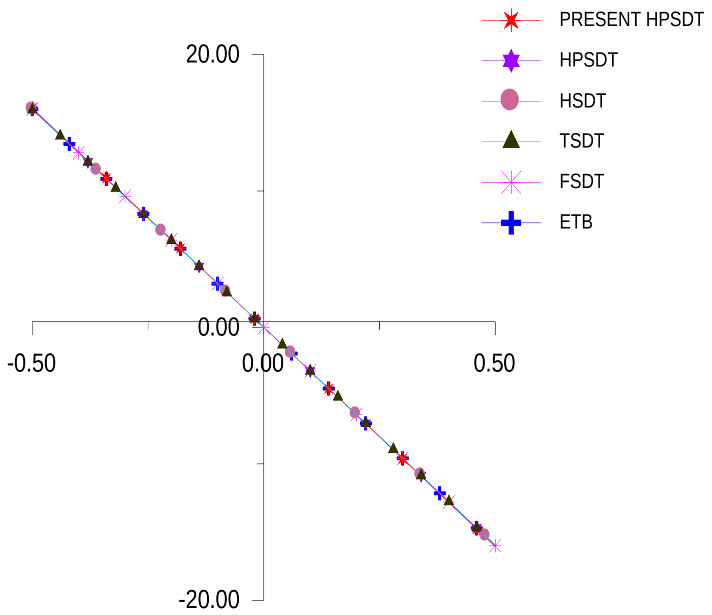

Fig. 4.1: Variation of axial displacement ($\bar{u}$) through the thickness of cantilever beam at (x = L, z) when subjected to varying load for aspect ratio 4.

Table 4.2: Non-Dimensional Axial Displacement ($\bar{u}$) at (x = L, z = h/2), Transverse Deflection ($\bar{w}$) at (x = L, z =0.0) Axial Stress ($\bar{\sigma}_x$) at (x = 0, z = h/2) Maximum Transverse Shears Stresses $\bar{\tau}_{zx}^{CR}$ (x = 0.01L, z = 0.0) and $\bar{\tau}_{zx}^{EE}$ (x = 0, z = 0) of the Cantilever Beam Subjected to Varying Load for Aspect Ratio 10

Source	Model	$\bar{u}$	$\bar{w}$
Present HPSDT	HPSDT	765.6593	11.2600
Dahake [54]	TSDT	765.6917	11.2601
Krishna Murty [50]	HSDT	765.6928	11.2603
Timoshenko [53]	FSDT	750.0000	11.0520
Bernoulli-Euler	ETB	750.0000	11.0000

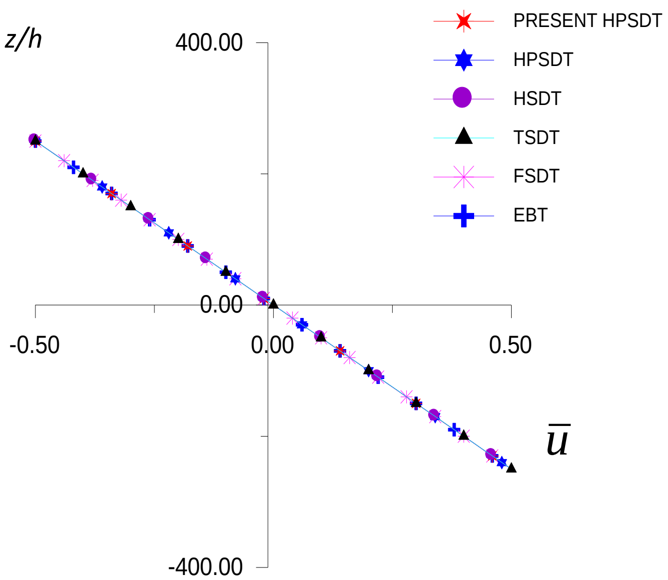

Fig. 4.2: Variation of axial displacement ($\bar{u}$) through the thickness of cantilever beam at (x = L, z) when subjected to varying load for aspect ratio 10.

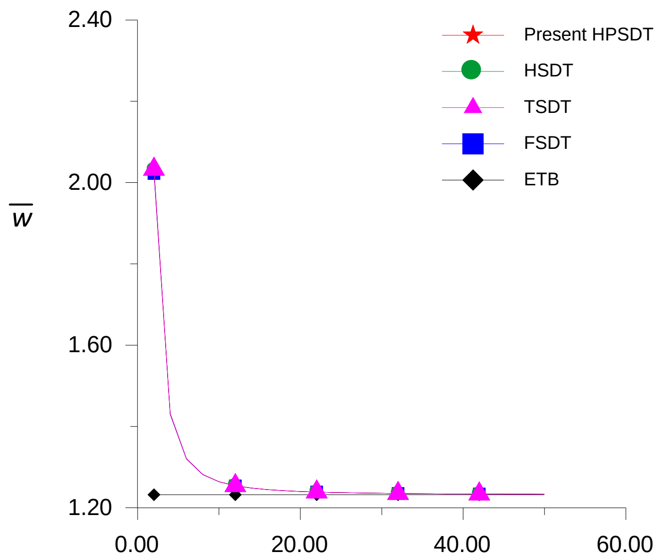

Fig. 4.3: Variation of maximum transverse displace t ($\bar{w}$) of cantilever beam at (x = L, z = 0) when subjected to varying load with aspect ratio S.

INTERPRETATION:

Among the results of all the other theories, the values of axial displacement given by present theory are in close agreement with the values of other refined theories for aspect ratio 4 and 10.The through thickness distribution of this displacement obtained by present theory is in close agreement with other refined theories except the one given by classical and first order shear deformation theory (FSDT) as shown in Figures 4.1, 4.2, for aspect ratio 4 and 10.

The comparison of results of maximum non-dimensional transverse displacement ($\bar{w}$) for the aspect ratios is presented in Tables 4.1 through for cantilever beams subjected to linearly varying load and parabolic load. Among the results of all the other theories, the values of present theory are in excellent agreement with the values of other refined theories for aspect ratio 4 and 10 except those of classical beam theory (ETB) and FSDT of Timoshenko. The variation of $\bar{w}$ with aspect ratio (S) is shown in Figure 4.3. For the aspect ratios greater than 20 all the refined theories converges to the values of classical beam theory.

$\bar{\sigma}_x$

V. Conclusion

The variationally consistent theoretical formulation of the theory with general solution technique of governing differential equations is presented. The general solutions for beam with uniformly varying load are obtained in case of cantilever beam. The displacements and stresses obtained by present theory are in excellent agreement with those of other equivalent refined and higher order theories. The present theory yields the realistic variation of axial displacement and stresses through the thickness of beam. The theory is shown to be capable of predicting the effects of stress concentration on the axial and transverse stresses in the vicinity of the built-in end of the beam which is the region of heavy stress concentration. Thus the validity of the present theory is established.

Acknowledgements

I am greatly indebted forever to my guide Dr. M. N. Mangular, Asso. Prof. Marathwada Institute of Technology, Aurangabad for her continuous encouragement, support, ideas, most constructive suggestions, valuable advice and confidence in me. I sincerely thank to Mr. R.L Shirale, Prof. Government Polytechnic Aurangabad, for their encouragement and kind support and stimulating advice.

References

[1] Averill, R.C., Reddy, J.N. (1992) An assessment of four-noded plate finite elements based on a generalized third order theory, *International Journal of Numerical Methods in Engineering*, **33**, 1553-1572.

[2] Bhimaraddi, A., Chandrashekhara, K. (1993) Observations on higher order beam Theory, *ASCE Journal of Aerospace Engineering*, **6(4)**, 408-413.

[3] Bickford, W.B. (1982) A consistent higher order beam theory, *International Proceeding of Dev. in Theoretical and Applied Mechanics (SECTAM)*.**11**, 137-150.

[4] Vaibhav B.Chavan and Dr. Ajay G. Dahake, 2014,"Analysis of Thick Beam Bending Problem by using a New Hyperbolic Shear Deformation Theroy.Vol. 2,No.5,ISSN 2091-2730.

[5] Mr. Mithun.K.Sawant and Dr. Ajay G. Dahake,2016, "Transverse Shear Deformation for Deep Beam using New Hyperbolic Shear Deformation Theory, Vol. 2,No.6,ISSN 2454-1362

[6] Bresse, J.A.C. (1859) Cours de Mechanique Applique, Mallet-Bachelier, Paris, 1859.

[7] Cowper, G.R. (1966) The shear coefficients in Timoshenko beam theory, *ASME Journal of Applied Mechanic*, **33(2)**, 335-340.

[8] Cowper, G.R. (1968) On the accuracy of Timoshenko beam theory, *ASCE Journal of Engineering Mechanics Division*. **94 (EM6),** 1447-1453.

[9] Ghugal, Y.M., Shmipi, R.P. (2001) A review of refined shear deformation theories for isotropic and anisotropic laminated beams, *Journal of Reinforced Plastics And Composites*, **20(3)**, 255-272.

[10] Kant,T. Gupta, A.(1988) A finite element model for higher order shears deformable beam theory, *Journal of Sound and Vibration*, **125 (2)**, 193-202.

[11] Krishna Murthy, A.V. (1984)Towards a consistent beam theory, *AIAA Journal*, **22(6)**,811-816.

[12] Lord Rayleigh, J.W.S. (1877)The Theory of Sound, *Macmillan Publishers*, London.

[13] Reddy, J.N. (1993) An Introduction to Finite Element Method. 2nd Edition, *McGraw-Hill, Inc*., New York, 1993.

[14] Stein, M. (1989) Vibration of beams and plate strips with three dimensional flexibility, *ASME Journal of Applied Mechanics*, **56(1)**, 228-231.

[15] Timoshenko, S.P. (1921) On the correction for shear of the differential equation for transverse vibrations of prismatic bars, *Philosophical Magazine*, **41 (6)**, (1921), 742-746.

[16] Timoshenko, S.P., Goodier, J.N. (1970) Theory of Elasticity, Third International Edition, *McGraw-Hill, Singapore*.

[17] Ghugal Y. M. and Dahake A. G. (2012) "Flexural Analysis of Deep Beam Subjected to Parabolic Load Using Refined Shear Deformation Theory", *Applied and Computational Mechanics*, **6(2)**, 163-172.

Appendix A

Notations

A : Cross sectional area of beam = bh

b : Width of beam in y -direction

E, G, μ : Elastic constants of the beam material

h : Deepness of beam

I : Moment of inertia of cross-section of beam

L : Span of the beam

q_0 : Intensity of parabolic transverse load

S : Aspect ratio of the beam = L / h

w : Transverse displacement in z direction

$\bar{w}$: Non-dimensional transverse displacement

$\bar{u}$: Non-dimensional axial displacement
x, y , z : Rectangular Cartesian coordinates
$\bar{\sigma}_x$: Non-dimensional axial stress in *x* -direction
$\bar{\tau}_{ZX}^{CR}$: Non-dimensional transverse shear stress via constitutive relation
$\bar{\tau}_{ZX}^{EE}$: Non-dimensional transverse shear stress via equilibrium equation
$\phi(x)$: Unknown function associated with the shear slope

List of abbreviations

CR : Constitutive Relations
EE : Equilibrium Equations
TSDT : Trigonometric Shear Deformation Theory
HPSDT : Hyperbolic Shear Deformation Theory
HSDT : Third Order Shear Deformation Theory
FSDT : First Order Shear Deformation Theory
ETB : Elementary Theory of Beam

Shaikh Nagma Saba "Bending Analysis of Isotropic Beam using “Hyperbolic Shear Deformation Theory”." IOSR Journal of Mechanical and Civil Engineering (IOSR-JMCE), vol. 15, no. 1, 2018, pp. 31-39.