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Super-Convergent Finite Element for Dynamic Analysis of Symmetric Composite Shear-deformable Beams under Harmonic Forces

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Abstract: *A super-convergent finite element is formulated for the dynamic flexural response of symmetric laminated composite beams subjected to various transverse harmonic forces. Based on the assumptions of Timoshenko beam theory, a one-dimensional finite beam element with two-nodes and four degrees of freedom per element is developed. The new beam element is applicable to symmetric laminated composite beams and accounts for the effects of shear deformation, rotary inertia, and Poison's ratio. The exact closed-form solution for flexural displacement functions developed in this study is employed to develop exact shape functions. Although the present finite element formulation is developed to obtain the steady state dynamic response but can be also used to capture the quasi-static analysis of the symmetric laminated composite beams. Moreover, it is also used to extract the natural frequencies and mode shapes for flexural response. The accuracy and efficiency of the present finite beam element are shown through comparisons with other established exact and Abaqus finite element solutions. The new element is demonstrated to be free from shear locking and mesh discretization errors occurring in conventional finite element solutions and illustrates an excellent agreement with those based on finite element solutions at a fraction of the computational and modeling cost.*
Keywords: *Symmetric laminates, exact shape functions, harmonic forces, super-convergent finite element, steady state response.*

I. Introductory Review on Finite Element Formulation

A number of researchers developed the finite element models technique to study the dynamic analysis of composite laminated beams. Most of the models are based on two approaches. In the first approach, formulations are based on approximate shape functions such as the work of [1]-[7], and recently [8]. In the second approach, exact shape functions which are based on the static solution of the governing equilibrium equations, such as the work of [9]. Based on first-order shear deformation, [1] developed a finite element model to study the free vibration characteristics of composite laminated beams including the effects of shear deformation and bi-axial bending and torsion. Reference [2] presented a finite element model to investigate the natural frequencies and mode shapes of laminated composite beams. Reference [3] analyzed the composite laminated beams using a two-noded curved composite beam element with three degrees of freedom per node. The formulation incorporated Poisson's effect and the coupled flexural and extensional deformations together with transverse shear deformation. Reference [4] developed a dynamic finite element for free vibration analysis of generally composite laminated beams. Their formulations incorporate the effects of Poisson's ratio, shear deformation, rotary inertia and coupling of extensional-bending and bending-torsional deformations. Recently, [8] developed a finite element model to predict the static and free vibration analyses for isotropic and orthotropic beams with different boundary conditions and length-to-thickness ratios. Based on higher-order shear deformation theory, [5] used the conventional finite element to analyze the free vibration behavior of laminated composite beams by considering the effects of rotary inertia, Poisson's effect, and coupled extensional and bending deformations. Reference [6] developed a two-noded C^1 finite element of eight degrees of freedom per node for flexural analysis of symmetric composite laminated beams. Reference [9] developed a two-noded beam element with four degrees of freedom per node based on higher-order shear deformation theory for coupled analysis of axial–flexural–shear deformation in asymmetric laminated composite beams. The beam element based on exact shape function matrix which is derived by satisfying the static solution of the governing equilibrium equations. Recently, [7] developed a two-noded C^1 finite beam element with five degrees of freedom per node to study the free vibration and buckling analyses of composite cross-ply laminated beams by using the refined shear deformation theory. Their formulations account for the parabolical variation of the shear strains through the beam depth and all coupling coming from the material anisotropy. A common feature to the previous studies is the use of approximate shape functions involving spatial discretization errors, and thus requiring fine meshes to converge to the exact results or/and exact shape functions based on static solution of the governing equilibrium equations.

Although a large number of finite element studies dealing with the dynamic analysis of composite laminated beams are developed, it should be noted that no work is reported in the literature on dynamic analysis of symmetric laminated composite beams under transverse harmonic forces using finite element formulation based on exact shape functions which exactly satisfy the homogeneous closed form solution of the dynamic flexural equations of motion. Thus, the aim of the present study is to develop such an efficient finite element solution based on the exact solution.

II. Governing Dynamic Bending Equations

2.1 Basic Assumptions

The following assumptions are considered in formulating the present theory:

- Cross-section is a rectangular section with the X axis taken as the longitudinal axis of composite beam,
- The formulation is restricted to composite beams with a symmetric laminates,
- The material of each layer is linearly elastic,
- Strains and rotations are assumed small,
- For transverse loading not involving twist, plane sections originally perpendicular to the longitudinal axis of the beam remain plane but not necessarily perpendicular to longitudinal axis after deformation, i.e. the transverse shear deformations are incorporated in the assumed kinematics (in a manner analogous to the Timoshenko beam theory),
- The layers are perfectly bonded together.
- The steady state component of the dynamic response is sought.

2.2 Displacement Fields

A straight composite beam of length L with a rectangular cross section of height h and width b, as shown in Fig (1), is considered. The composite beam is composed of many laminates of orthotropic materials in different fibre orientations with respect to the X axis. The theoretical developments presented in this paper required one set of coordinate system which is the orthogonal Cartesian coordinate system (X,Y,Z), for which the Y and Z axes are coincident with the principal axes of the beam, and X axis is coincident with the centroidal beam axis.

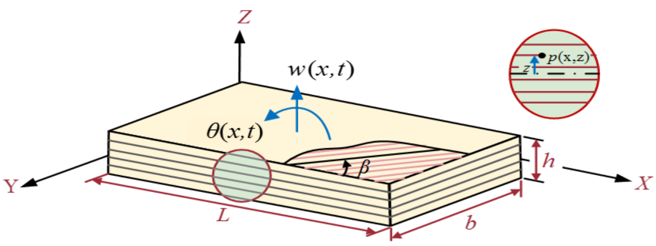


Figure (1): Geometry of a composite laminated beam

The assumed displacement field for a point $p(x,y,z)$ of height z from the centroidal axis of composite beam based on first-order shear deformation theory can be written as follows [4]:

$$u_p(x,z,t)=u(x,t)+z\theta(x,t) \tag{1}$$

$$v_p(x,z,t)=0 \tag{2}$$

$$w_p(x,z,t)=w(x,t) \tag{3}$$

where $u_p(x,z,t)$ is the longitudinal displacement of a point $p(x,z)$ along the X axis, $v_p(x,z,t)$ and $w_p(x,z,t)$ are the lateral and transverse displacements of a point $p(x,z)$, while $u(x,t)$ and $w(x,t)$ are the longitudinal and transverse displacements of a point located on the mid-plane $(z=0)$ in the X and Z directions, and $\theta(x,t)$ is the bending rotation of the cross-section about the Y axis.

2.3 Strain Relations

The axial and shear strains of the composite beam based on small-displacement theory are written as:

$$\varepsilon_{xx}=\frac{\partial u_p(x,z,t)}{\partial x}=\bar{\varepsilon}_{xx}+z\kappa_x \tag{4}$$

$$\gamma_{xz} = \frac{\partial w_p(x,z,t)}{\partial x} + \frac{\partial u_p(x,z,t)}{\partial z} = w'(x,t) + \theta(x,t) \tag{5}$$

where $\bar{\varepsilon}_{xx}$ is the mid-plane axial strain defined by $\bar{\varepsilon}_{xx} = u'(x,t)$, and κ_x is the bending curvature defined by $\kappa_x = \theta'(x,t)$, and the prime denotes the differentiation with respect to x.

2.4 Constitutive Equations for Symmetric Laminates

The constitutive equations of the symmetric laminates (i.e., both geometric and material symmetry with respect to the mid-surface), in which the extension-bending coupling coefficients $B_{ij} = 0$ for $i, j = 1,2,6$, can be obtained using the classical lamination theory to give:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & & & \\ A_{12} & A_{22} & A_{26} & & B_{ij}=0 & \\ A_{16} & A_{26} & A_{66} & & & \\ & & & D_{11} & D_{12} & D_{16} \\ & B_{ij}=0 & & D_{12} & D_{22} & D_{26} \\ & & & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \bar{\varepsilon}_{xx} \\ \bar{\varepsilon}_{yy} \\ \bar{\gamma}_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{6}$$

where N_x, N_y are the normal forces, and N_{xy} is the in-plane force, while M_x, M_y are the bending moments, M_{xy} is the twisting moment, $\bar{\varepsilon}_{xx}$, $\bar{\varepsilon}_{yy}$ and $\bar{\gamma}_{xy}$ are normal and shear the strains, κ_x, κ_y are the bending curvatures, κ_{xy} is the twisting curvature, the extension A_{ij} and bending D_{ij} stiffness coefficients expressed as functions of laminate ply orientation β and material properties are denoted by:

$$A_{ij}, D_{ij} = \int_{-h/2}^{h/2} \bar{Q}_{ij}\left[1,\ z^2\right] dz\,, \quad (\text{for } i, j = 1,2,6)$$

where $\bar{Q}_{ij}$ for $i, j = 1,2,6$ are the transformed reduced stiffnesses and are given by Jone [10]. The effect of transverse shear deformation due to bending is captured by:

$$Q_{xz} = A_{55}\gamma_{xz} = A_{55}(w'+\theta) = k(w'+\theta)\int_{-h/2}^{h/2}\left(G_{13}\cos^2\beta + G_{23}\sin^2\beta\right)dz \tag{7}$$

where Q_{xz} is the transverse shear force per unit length, k is the correlation shear factor and is taken as 5/6 to account for the parabolic variation of the shear stresses, and the constants G_{13}, G_{23} are the shear moduli.

The laminated composite beam subjects only to axial and transverse forces and moments which cause a flexural deformation. In other words, forces or moments caused lateral and torsional deformations are negligible and set to zero, i.e., $N_y = N_{xy} = M_y = M_{xy} = 0$. In order to account for Poisson's ratio, the mid-plane strains $\bar{\varepsilon}_{yy}$, $\bar{\gamma}_{xy}$ and curvatures κ_y, κ_{xy} are assumed to be nonzero. For symmetric laminated beams, the extensional response is uncoupled from the flexural response of the beam, i.e., the bending stiffness coefficients B_{ij} are zero. Then, equation (6) is rewritten as:

$$\begin{Bmatrix} N_x \\ M_x \end{Bmatrix} = \begin{bmatrix} \bar{A}_{11} & 0 \\ 0 & \bar{D}_{11} \end{bmatrix} \begin{Bmatrix} \bar{\varepsilon}_{xx} \\ \kappa_x \end{Bmatrix} = \begin{bmatrix} \bar{A}_{11} & 0 \\ 0 & \bar{D}_{11} \end{bmatrix} \begin{Bmatrix} u' \\ \theta' \end{Bmatrix} \tag{8}$$

where $\bar{A}_{11} = A_{11} + \left[\dfrac{A_{12}(A_{66} - A_{26}) + A_{16}(A_{22} - A_{26})}{\left(A_{26}^2 - A_{22}A_{66}\right)}\right]$, and

$$\bar{D}_{11} = D_{11} + \left[\frac{D_{12}(D_{66} - D_{26}) + D_{16}(D_{22} - D_{26})}{\left(D_{26}^2 - D_{22}D_{66}\right)}\right].$$

If Poisson's ratio effect is ignored, the coefficients $\bar{A}_{11}, \bar{D}_{11}$ given in equation (8) are then replaced by the laminate stiffness coefficients A_{11}, D_{11}, respectively.

2.5 Expressions for Force-Moment Functions

The laminated composite beam illustrated in Fig (2) is assumed to be subjected to axial and transverse harmonic forces and moments with exciting frequency Ω . These forces are given as:

$$q_z(x,t), q_x(x,t), m_y(x,t) = \left[\bar{q}_z(x), \bar{q}_x(x), \bar{m}_y(x)\right] e^{i\Omega t} \tag{9}$$

$$P_x(x,t), P_z(x,t), M_y(x,t) = \left[\bar{P}_x(x), \bar{P}_z(x), \bar{M}_y(x)\right]_0^L e^{i\Omega t} \tag{10}$$

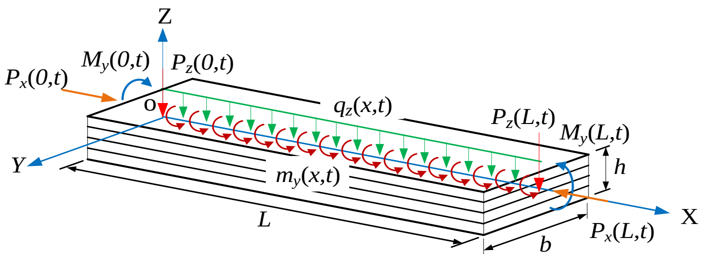


Figure (2): A laminated beam subjected to harmonic forces and moments

where Ω is the circular exciting frequency of the applied forces, $i = \sqrt{-1}$ is the imaginary constant, $q_x(x,t)$ and $q_z(x,t)$ are the distributed axial and transverse harmonic forces, $m_y(x,t)$ is the distributed harmonic bending moment, $P_x(x,t)$ and $P_z(x,t)$ are the concentrated axial and transverse harmonic forces, $M_y(x,t)$ is the concentrated harmonic bending moment, all concentrated forces and moments are applied at beam ends (i.e., $x = 0, L$).

2.6 Expressions for Steady State Displacements Functions

Under the given transverse harmonic forces and moments, the steady state displacements are assumed to take the exponential form:

$$u(x,t), w(x,t), \theta(x,t) = \left[U(x), W(x), \Theta(x)\right] \mathrm{e}^{i\Omega t} \tag{11}$$

where $U(x)$, $W(x)$ and $\Theta(x)$ are the amplitude space functions for longitudinal, bending translation, and related rotation responses, respectively. As the present formulation is intended to capture only the steady state dynamic response of the system, the displacement functions postulated in equation (11) neglect the transient component of the response.

III. Hamilton Variational Formulation

The dynamic governing differential equations of motion for forced vibrations of symmetric laminated composite beams can be easily derived using Hamilton's principle. According to the Hamilton's principle the integration of the Lagrangian of a dynamical system on any arbitrary interval of time is stationary, i.e.,

$$\int_{t_1}^{t_2} \delta L\, dt = \int_{t_1}^{t_2} \delta(T - \Pi)\, dt = 0 \tag{12}$$

where T is the total kinetic energy for the symmetric laminated composite beams, given by:

$$T = \frac{1}{2}\int_0^L \int_{-h/2}^{h/2} \rho\left(\dot{u}_p^2 + \dot{w}_p^2\right) bdzdx$$

From equations (1-3), by substituting into above equation, kinetic energy can be written as:

$$T = \frac{1}{2}\int_0^L \left[I_1\left(\dot{u}^2 + \dot{w}^2\right) + I_2\left(\dot{\theta}\right)^2 \right] bdx \tag{13}$$

in which the dot denotes the derivative with respect to time t, the densities of the composite beam (I_1, I_2) are given by: $I_1, I_2 = \int_{-h/2}^{h/2} \rho\left[1, z^2\right] dz = \sum_{k=1}^{m} \rho_k \left[\left(z_k - z_{k-1}\right), \left(z_k^3 - z_{k-1}^3\right) / 3\right]$, where ρ_k is the k^{th} layer mas density.

The total potential energy of symmetric laminated beam denotes by Π and is the sum of the internal strain energy U and potential energy V gained by the applied forces. The internal strain energy is expressed by:

$$U = \frac{1}{2} \int_0^L \int_A \left[\sigma_x \varepsilon_{xx} + \tau_{xz} \gamma_{xz}\right] dA dx \tag{14}$$

From equations (4-5) and (8) into equation (14), the strain energy can be stated as:

$$U = \frac{1}{2} \int_0^L \left[N_x \varepsilon_{xx_o} + M_x \kappa_x + Q_{xz} \gamma_{xz}\right] b\, dx = \frac{1}{2} \int_0^L \left[\bar{A}_{11} u'^2 + \bar{D}_{11} (\theta')^2 + A_{55}\left(w'^2 + 2w'\theta + \theta^2\right)\right] b\, dx \tag{15}$$

The potential energy V of the applied harmonic axial and flexural forces, as shown in Fig (2), can be given by:

$$\begin{aligned} V = &\int_0^L \left[q_z(x,t) w(x,t) + q_x(x,t) u(x,t) + m_x(x,t) \theta(x,t)\right] dx \\ &+ \left[P_x(x_e,t) u(x_e,t)\right]_0^L + \left[P_z(x_e,t) w(x_e,t)\right]_0^L + \left[M_x(x_e,t) \theta(x_e,t)\right]_0^L \end{aligned} \tag{16}$$

From equations (9-11), by substituting into energy expressions (13), (15) and (16), and the resulting expressions into Hamilton's principle in equation (12), performing integration by parts, the governing dynamic equations are found to take the matric form:

$$\begin{bmatrix} \left(\bar{A}_{11} \mathcal{D}^2 + I_1 \Omega^2\right) & 0 & 0 \\ 0 & \left(I_1 \Omega^2 + A_{55}\right) & A_{55} \mathcal{D} \\ 0 & A_{55} \mathcal{D} & \left(A_{55} - I_2 \Omega^2 - \bar{D}_{11} \mathcal{D}^2\right) \end{bmatrix}_{3\times3} \begin{Bmatrix} U(x) \\ W(x) \\ \Phi(x) \end{Bmatrix}_{3\times1} = \begin{Bmatrix} \bar{q}_x(x)/b \\ -\bar{q}_z(x)/b \\ \bar{m}_x(x)/b \end{Bmatrix}_{3\times1} \tag{17}$$

The boundary terms arising from integration parts of the Hamiltonian functional provide the possibly boundary conditions of the problem. They take the form:

$$\left[b \bar{A}_{11} U'(x)\right] \delta U(x)\Big|_0^L = \bar{P}_x(x)\Big|_0^L \tag{18}$$

$$b A_{55} \left[W'(x) + \Phi(x)\right] \delta W(x)\Big|_0^L = \bar{P}_z(x)\Big|_0^L \tag{19}$$

$$\left[b \bar{D}_{11} \Phi'(x)\right] \delta \Phi(x)\Big|_0^L = \bar{M}_x(x)\Big|_0^L \tag{20}$$

The first partition in equation (17) with equation (18) provides the governing equation for longitudinal vibration of the symmetric composite beam which is uncoupled from the remaining field equations and can be solved independently. The second partition with boundary equations (19-20) presents the coupled equations for bending vibration of the symmetric laminated beams. The present study thus focuses on developing an exact finite beam element for the steady state dynamic analysis of shear-deformable composite beams with symmetrical laminates provided in the second partition.

IV. Exact Solution For Coupled Bending Equations

The exact homogeneous solution of the coupled bending equations governed by the second partition in (18) is obtained by setting the right-hand side of the equations to zero, i.e. $\bar{q}_z(x) = \bar{m}_x(x) = 0$. The homogeneous solution of the displacements is then assumed to take the exponential form:

$$\{\chi(x)\}_{2\times1} = \begin{Bmatrix} W(x) \\ \Phi(x) \end{Bmatrix}_{2\times1} = \begin{Bmatrix} C_{1,i} \\ C_{2,i} \end{Bmatrix}_{2\times1} e^{m_i x}, \ for\ i = 1,2,3,4 \tag{22}$$

in which $\langle \chi(x) \rangle_{1\times2} = \langle W(x) \ \ \Phi(x) \rangle_{1\times2}$ is the bending displacements, and $\langle C \rangle_{1\times2} = \langle C_{1,i} \quad C_{2,i} \rangle_{1\times2}$ is a vector of unknown integration constants corresponding to root. From equation (22), by substituting into second partition in equation (18), yields:

$$\begin{bmatrix} \left(I_1\Omega^2 + A_{55}m_i^2\right) & A_{55}m_i \\ A_{55}m_i & \left(A_{55} - I_2\Omega^2 - \bar{D}_{11}m_i^2\right) \end{bmatrix}_{2\times2} \begin{bmatrix} e^{m_i x} & 0 \\ 0 & e^{m_i x} \end{bmatrix}_{2\times2} \begin{Bmatrix} C_{1,i} \\ C_{2,i} \end{Bmatrix}_{2\times1} = \{0\}_{2\times1} \tag{23}$$

For a non-trivial solution of equation (23), the determinant of the bracketed matrix is set to zero, leading to the quartic equation of the form:

$$A_{55}\bar{D}_{11}\, m_i^4 + \Omega^2 (A_{55}I_2 + I_1\bar{D}_{11})\, m_i^2 + \Omega^2 I_1 (I_2\Omega^2 - A_{55}) = 0 \tag{24}$$

The characteristic equation (24) which depends upon section properties, orthotropic material constants and exciting frequency has the following four distinct roots:

$$m_{1,2} = \pm\sqrt{\frac{1}{2A_{55}\bar{D}_{11}}\left[-\Omega^2(A_{55}I_2 + I_1\bar{D}_{11}) + \sqrt{\Omega\left[4A_{55}^2\bar{D}_{11}I_1 + \Omega^2(A_{55}I_2 - \bar{D}_{11}I_1)^2\right]}\right]}, \text{ and}$$

$$m_{3,4} = \pm i\sqrt{\frac{1}{2A_{55}\bar{D}_{11}}\left[\Omega^2(A_{55}I_2 + \bar{D}_{11}I_1) + \sqrt{\Omega\left[4A_{55}^2\bar{D}_{11}I_1 + \Omega^2(A_{55}I_2 - \bar{D}_{11}I_1)^2\right]}\right]}$$

For each root m_i, there corresponds a set of unknown constants $\langle C\rangle_{i,1\times2} = \langle C_{1,i} \;\; C_{2,i}\rangle_{i,1\times2}$. By back-substitution into the original system of bending equations, one can relate constants $C_{1,i}$ to $C_{2,i}$ through $C_{1,i} = G_i C_{2,i}$, where $G_i = -A_{55}m_i \big/ \left[I_1\Omega^2 + A_{55}m_i^2\right]$, for $i = 1,2,3,4$.

Thus, the exact homogeneous solution for the transverse displacement $W(x)$ and related bending rotation $\Phi(x)$ are found by:

$$\{\chi(x)\}_{1\times2} = [\bar{G}]_{2\times4}[E(x)]_{4\times4}\{\bar{C}\}_{4\times1} = [\psi(x)]_{2\times4}\{\bar{C}\}_{4\times1} \tag{25}$$

where $[\psi(x)]_{2\times4} = [\bar{G}]_{2\times4}[E(x)]_{4\times4}$ consists of functions which exactly satisfy the coupled dynamic bending equations, $[\bar{G}]_{2\times4} = \left[\begin{Bmatrix} G_1 \\ 1 \end{Bmatrix} \begin{Bmatrix} G_2 \\ 1 \end{Bmatrix} \begin{Bmatrix} G_3 \\ 1 \end{Bmatrix} \begin{Bmatrix} G_4 \\ 1 \end{Bmatrix}\right]_{2\times4}$, $[E(x)]_{4\times4}$ is a diagonal matrix consisting of exponential functions $e^{m_i x}$ (for $i=1,2,3,4$), and $\langle\bar{C}\rangle_{1\times4} = \langle C_{2,1} \;\; C_{2,2} \;\; C_{2,3} \;\; C_{2,4}\rangle_{1\times4}$ is the vector of unknown constants.

V. Finite Element Formulation

In this section, a new finite beam element is proposed for dynamic analysis of composite shear-deformable beams with symmetrical laminates under various transverse harmonic forces and moments. Figure (3) shows the proposed two-noded finite composite beam element with four degrees of freedom per element. A family of shape functions which exactly satisfy the homogeneous solution of the coupled dynamic bending equations is used to formulate the exact stiffness and mass matrices as well as the load potential vector.

5.1 Formulating of Exact Displacement Functions

In the present formulation, the vector of unknown integration constants $\{\bar{C}\}_{4\times1}$ can be expressed in terms of the nodal displacements $\langle S_N\rangle_{1\times4} = \langle W_1 \;\; \Theta_{x1} \;\; W_2 \;\; \Theta_{x2}\rangle_{1\times4}$ by enforcing the conditions $W(0)=W_1$, $\Theta(0)=\phi_{x1}$, $W(L)=W_2$ and $\Theta(L)=\phi_{x2}$, (Fig. 3), one obtains:

$$\{S_N\}_{4\times1} = \begin{Bmatrix} \{\chi(0)\}_{2\times1} \\ \{\chi(L)\}_{2\times1} \end{Bmatrix}_{4\times1} = \begin{bmatrix} [\psi(0)]_{2\times4} \\ [\psi(L)]_{2\times4} \end{bmatrix}_{4\times4} \{\bar{C}\}_{4\times1} = [\Psi]_{4\times4}\{\bar{C}\}_{4\times1} \tag{26}$$

From equation (26), by substituting into equation (25), yields:

$$\{\chi(x)\}_{1\times2} = [\psi(x)]_{2\times4}[\Psi]_{4\times4}^{-1}\{S_N\}_{4\times1} = [H(x)]_{2\times4}\{S_N\}_{4\times1} \tag{27}$$

in which $[H(x)]_{2\times4}=[\psi(x)]_{2\times4}[\Psi]_{4\times4}^{-1}$ is a matrix of eight shape functions for the bending response, where $[H(x)]_{4\times2}^{T}=\left[H_{1,j}(x) \;\vdots\; H_{2,j}(x)\right]_{4\times2}^{T}$. It is noted that the interpolation shape functions provided in equation (27) exactly satisfy the homogeneous solution of the coupled bending equations are dependent on the beam span, cross-section geometry, and the exciting frequency of the applied harmonic forces.

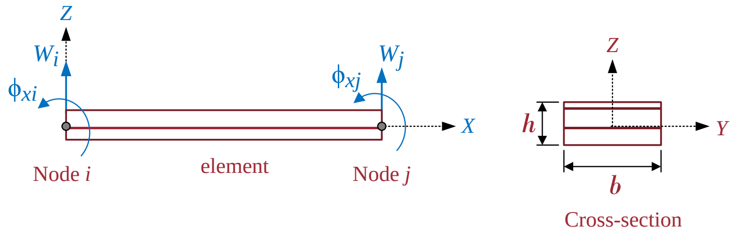


Figure (3): Composite Shear-deformable two-noded beam element

5.2 Energy Expressions in Terms of Nodal Displacements

The variations of the total kinetic energy δT , internal strain energy δU , and work done by the applied harmonic forces δV for symmetric laminated composite beam are, respectively, obtained in terms of nodal degrees of freedom using equation (27) as:

$$\delta T=-\langle\delta S_N\rangle_{1\times4}\left(\Omega^2\int_0^L\left[[H(x)]_{4\times2}^T\left[\bar{Y}_m\right]_{2\times2}[H(x)]_{4\times2}\right]dx\right)\{S_N\}_{2\times1}e^{i\Omega t} \tag{28}$$

$$\delta U=\langle\delta S_N\rangle_{1\times4}\left(\int_0^L\left[[H'(x)]_{4\times2}^T\left[\bar{Z}_m\right]_{2\times2}[H'(x)]_{2\times4}+[H(x)]_{4\times2}^T\left[\bar{Z}_s\right]_{2\times2}[H(x)]_{2\times4}\right.\right.$$
$$\left.\left.+[H_s(x)]_{4\times2}^T[H_r(x)]_{2\times4}\right]dx\right)\{S_N\}_{4\times1}e^{i\Omega t} \tag{29}$$

and,

$$\delta V=-\langle\delta S_N\rangle_{1\times4}\left(\int_0^L[H(x)]_{4\times2}^T\{\bar{Q}_F\}_{2\times1}dx+\left[[H(x)]_{4\times2}^T\{\bar{Q}_C\}_{2\times1}\right]_0^L\right)e^{i\Omega t} \tag{30}$$

in which $\left[\bar{Y}_m\right]_{2\times2}=diag[bI_1 \quad bI_2]$, $\left[\bar{Z}_m\right]_{2\times2}=Diag\left[bA_{55} \quad b\bar{D}_{11}\right]$, $\left[\bar{Z}_s\right]_{2\times2}=Diag[0 \quad bA_{55}]$,
$[H_s(x)]_{4\times2}^T=\left[\{H'_{1,j}(x)\}_{4\times1} \;\vdots\; \{H_{2,j}(x)\}_{4\times1}\right]_{4\times2}^T$, $[H_r(x)]_{2\times4}=bA_{55}\left[\{H_{2,j}(x)\}_{4\times1} \;\vdots\; \{H_{1,j}(x)\}_{4\times1}\right]_{4\times2}$,
$\langle\bar{Q}_F\rangle_{1\times2}=\langle\bar{q}_z \quad \bar{m}_x\rangle_{1\times2}$ and $\langle\bar{Q}_C\rangle_{1\times2}=\langle\bar{P}_z \quad \bar{M}_x\rangle_{1\times2}$.

5.3 Matrix Formulation

From equations (28-30), by substituting into Hamilton's principle in (13), performing integration by parts, one obtains:

$$\left([K_e]_{4\times4}-\Omega^2[M_e]_{4\times4}\right)\{S_N\}_{4\times1}=\{F_e\}_{4\times1} \tag{31}$$

in which the element stiffness matrix $[K_e]_{4\times4}$ is given by:

$$[K_e]_{4\times4}=\int_0^L\left([H'(x)]_{4\times2}^T\left[\bar{Z}_m\right]_{2\times2}[H'(x)]_{2\times4}+[H(x)]_{4\times2}^T\left[\bar{Z}_s\right]_{2\times2}[H(x)]_{2\times4}+[H_s(x)]_{4\times2}^T[H_r(x)]_{2\times4}\right)dx.$$

The element mass matrix $[M_e]_{4\times4}$ is obtained as:

$$[M_e]_{4\times4}=\int_0^L\left([H(x)]_{4\times2}^T\left[\bar{Y}_m\right]_{2\times2}[H(x)]_{4\times2}\right)dx$$

and, the element load vector $\{F_e\}_{4\times1}$ is given by:

$$\{F_e\}_{4\times1} = \int_0^L [H(x)]_{4\times2}^T \{\bar{Q}_F\}_{2\times1} dx + \left[[H(x)]_{4\times2}^T \{\bar{Q}_C(x)\}_{2\times1} \right]_0^L .$$

VI. Numerical Examples And Discussion

In this section, two examples are presented in order to show the validity, accuracy and applicability of the new finite element developed in the present study. The new finite beam element can (a) capture the quasi-static response when adopting a very low exciting frequency Ω compared to the first bending natural frequency ω_1 of the given beam (i.e., $\Omega \approx 0.01\omega_1$), and (b) predict the bending natural frequencies and mode shapes of the composite beam under harmonic force from the steady state dynamic response. The present finite element formulation is based on the shape functions which exactly satisfy the exact homogeneous solution of the coupled bending field equations. This treatment offer two advantages: (1) it eliminates mesh discretization errors arising in conventional interpolation schemes used in the finite element solutions and thus converge to the solution using a minimal number of degrees of freedom, and (2) it leads to elements that are free from shear locking. As a result, it is observed that, the of the present results obtained based on a new finite beam element using a single two-noded finite element per span yielded the corresponding results which exactly matched with those based on the exact closed-form solutions provided by Hjaji et al. [11] up to five significant digits. The results based on the present finite beam element (with two degrees of freedom per node) which accounts for shear deformation, rotary inertia, Poisson ratio and fibre orientation are compared with exact solutions available in the literature and Abaqus finite beam B13 element (with six degrees of freedom per node, i.e., three translation and three rotations) which accounts for the effects of shear deformation effects. The examples are investigated for symmetric laminated cross-ply and angle-ply composite beams with a rectangular cross-section and a variety of loading and boundary conditions.

6.1 Example 1 – Cantilever Beam under Distributed Harmonic Force - Verification

A 3000mm composite cantilever beam with symmetric four cross-ply $(0^o, 90^o, 90^o, 0^o)$ laminates subjected to distributed transverse harmonic force $q_z(x,t) = 8.0e^{i\Omega t} kN/m$, as shown in Fig (4), is considered. The cantilever beam has a rectangular cross-section of width $b = 100mm$ and height $h = 200mm$. The details of the orthotropic composite properties are: $E_{11} = 144.9GPa$, $E_{22} = 9.65GPa$, $G_{12} = G_{13} = 4.140GPa$, $G_{23} = 3.450GPa$, $\rho = 1389 kg/m^3$, and $\upsilon_{12} = 0.30$. For verification purposes, it is required to (a) extract the flexural natural frequencies from a steady state dynamic analysis, (b) conduct a quasi-static analysis by adopting a very low exciting frequency $\Omega \approx 0.01\omega_1$, and (c) investigate the steady state dynamic response for an exciting frequency $\Omega = 1.68\omega_1$, where the first natural frequency of the beam is $\omega_1 = 33.38Hz$.

The symmetric cross-ply composite cantilever is modelled in Abaqus by using 100 beam elements B31 along the longitudinal axis of the cantilever beam (606 dof) to yield the accuracy of this example. In contrast, the present finite beam element uses a single beam element (4 dof) to approach the exact solution.

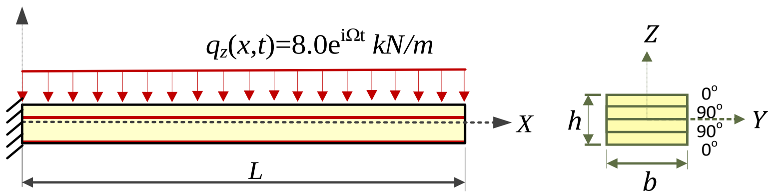


Figure (4): Symmetric cross-ply laminated cantilever beam under distributed harmonic force

6.1.1 Extracting Flexural Natural Frequencies and Mode Shapes

Under the distributed transverse harmonic force, the flexural natural frequencies are extracted from the steady state dynamic response analyses in which the exciting frequency Ω varying from nearly zero to 700Hz. The flexural natural frequencies are extracted from the peaks of the displacement-frequency relationships. The nodal transverse displacement W_2 and bending rotation ϕ_{x2} at the cantilever tip against the exciting frequency are shown in Fig (5). For comparison, Abaqus model based on 100 beam B31 elements are plotted on the same diagrams. Peaks on both diagrams indicate resonance and are then identify the natural flexural frequencies of the given cantilever composite beam. Thus, the first three flexural natural frequencies extracted from the peaks are given in Table 1. Table 1 provides the first three non-dimensional flexural natural frequencies extracted from the steady state dynamic analyses of symmetric cross-ply $(0^o, 90^o, 90^o, 0^o)$ laminated composite cantilever beam, in which the non-dimensional relation used is defined by $\bar{\omega} = \omega L^2 \sqrt{\rho / E_{11}} / h$. The present finite element model based on a single beam element (4 dof) predicts the flexural natural frequencies in excellent agreement with those based on Vo and Inman [10], Shi and Lam [12] and Marur and Kant [13] and Abaqus beam model using B31 elements (606 dof). This is a natural outcome that the finite element developed in the present study based on the exact shape functions eliminates the mesh discretization errors.

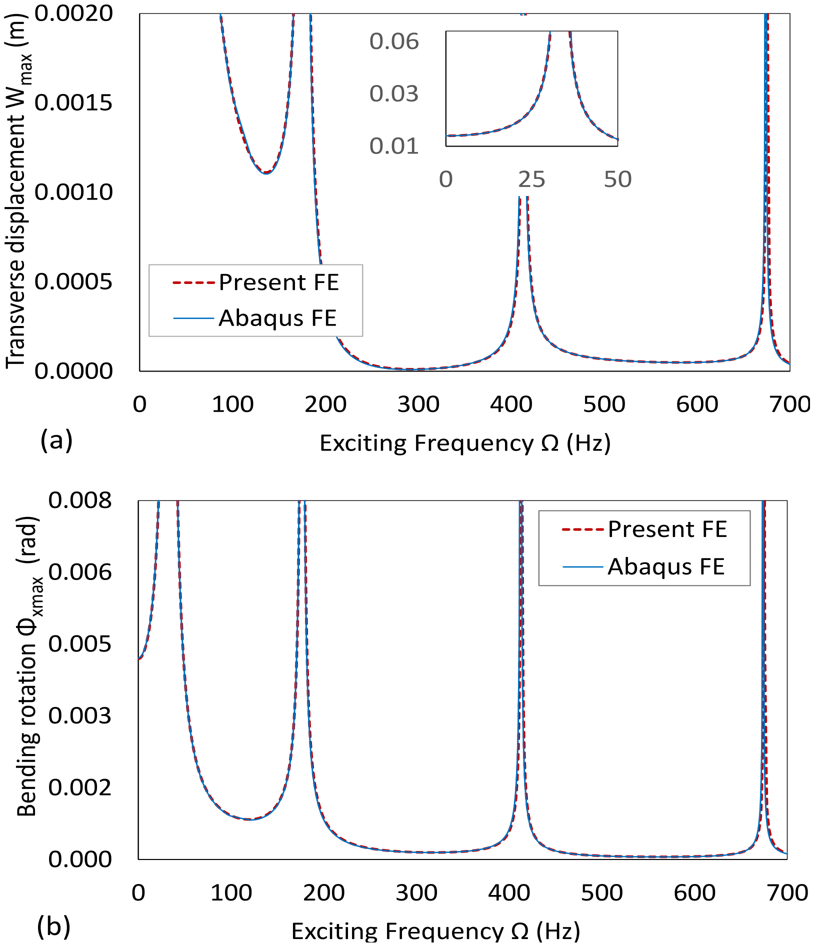


Figure (5): Natural flexural frequencies for symmetric cross-ply laminated beam under harmonic force

Table 1: The first three non-dimensional flexural natural frequencies $\bar{\omega}$ of symmetric laminated beam

Mode	Vo and Inam [10]	Shi and Lam [12]	Marur and Kant [13]	FE Abaqus (606 dof)	Present FE (4 dof)
1	0.9222	0.9199	0.9214	0.9238	0.9240
2	4.9165	4.9054	4.8919	4.8860	4.8924
3	11.600	11.489	11.476	11.419	11.439

The first three steady state transverse displacement and related bending rotation modes for symmetric four-layered cross-ply $(0^o, 90^o, 90^o, 0^o)$ laminated composite cantilever beam subjected to distributed transverse harmonic force using three different exciting frequencies at the peaks (i.e., $\omega_1 = 33.38 Hz$, $\omega_2 = 176.7 Hz$, $\omega_3 = 413.2 Hz$) are shown in Figs (6a) and (6b), respectively. The normalized steady state flexural modes based on the present solution (8 beam elements with 18 dof are used for comparison purpose) and those based on Abaqus beam model plotted on the same diagrams exhibit excellent agreement.

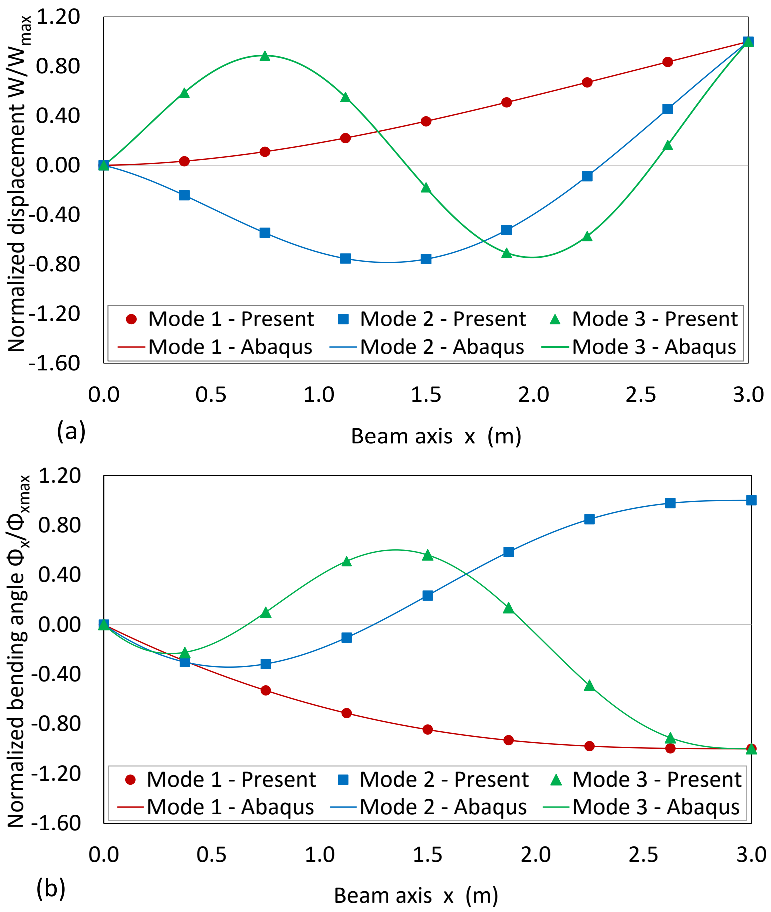


Figure (6): Steady state flexural modes for symmetric cross-ply cantilever under harmonic force

6.1.2 Quasi-Static Flexural Solution

Table 2 shows the quasi-static flexural response results for the maximum displacement $W_{max}(=W_2)$ and bending rotation $\phi_{xmax}(=\phi_{x2})$. Results based on the finite beam element developed in the present study are observed to exactly match with those based on the Abaqus beam model. As a general observation, the present finite element is successful at capturing the quasi-static response of the given beam.

6.1.3 Steady State Flexural Dynamic Solution

The steady state flexural response for symmetric cross-ply $(0^o, 90^o, 90^o, 0^o)$ laminated composite cantilever beam under the distributed harmonic force with exciting frequency $\Omega = 1.68\omega_1 = 56.08\,Hz$ is presented in Table 2. The nodal flexural displacement and bending angle results at the cantilever tip based on the present formulation are compared with those based on Abaqus beam model solution. It is observed that results obtained from the finite element formulation developed using one beam element with 4 dof provide excellent agreement with Abaqus beam model using 100 B31elements (606 dof).

Table 2: Quasi-static and steady state analyses of symmetric laminated cantilever under harmonic force

Type of Response	Variable	Abaqus FE [1] (606 dof)	Present FE [2] (4 dof)	%Difference =[1-2]/1
Static	W_{max} (mm)	-10.04	-10.01	0.33%
$\Omega \approx 0.01\omega_1$	ϕ_{xmax} (10^{-3} rad)	4.212	4.194	0.43%
Steady state	W_{max} (mm)	5.783	5.779	0.07%
$\Omega = 1.68\omega_1$	ϕ_{xmax} (10^{-3} rad)	-2.692	-2.698	-0.20%

6.2 Example 2-Clamped-clamped symmetric composite beam - Finite Element Formulation

This example is presented to demonstrate the ability of the finite element developed in the present study by comparing the results for quasi-static and dynamic responses with those based on the Abaqus solution.

A clamped-clamped composite beam with symmetric $(0^o, 90^o, 0^o, 90^o, 0^o)$ cross-ply laminates under various harmonic transverse forces; $P_{z1}(2m,t) = 1.0e^{i\Omega t}\,kN$, $P_{z2}(4m,t) = 2.0e^{i\Omega t}\,kN$ and $P_{z3}(6m,t) = 4.0\,e^{i\Omega t}\,kN$, and uniformly distributed force $q_z(x,t) = 4.0\,e^{i\Omega t}\,kN/m$ is considered as shown in Fig. (7). The composite beam has a rectangular cross-section of width $b = 100mm$ and height $h = 100mm$, while the mechanical composite properties are: $E_{11} = 37.41GPa$, $E_{22} = 13.67\,GPa$, $G_{12} = G_{13} = 5.478\,GPa$, $G_{23} = 6.666GPa$, $\upsilon_{12} = 0.30$ and $\rho = 1969kg/m^3$. It is required to assess the accuracy and efficiency of the present finite element in evaluating the nodal degrees of freedom for quasi-static and steady state analyses of the composite beam under the given harmonic forces with exciting frequencies $\Omega \approx 0.01\omega_1$ and $\Omega = 1.80\omega_1$ are investigated, respectively. The first natural frequency of the system is $\omega_1 = 6.619Hz$.

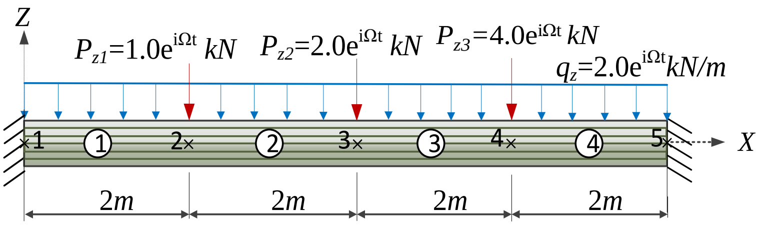
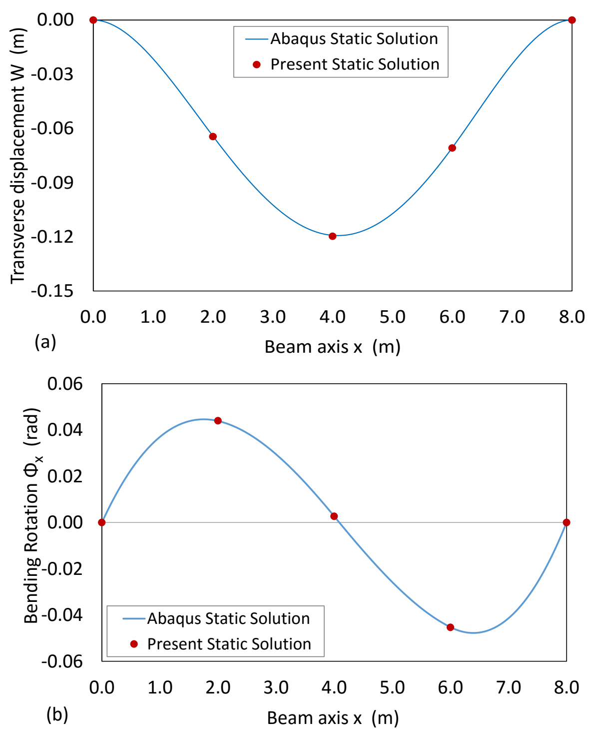


Figure (7): Clamped-clamped symmetric laminated beam under various transverse harmonic forces

In order to demonstrate the accuracy and capability of the finite element developed in the present study, the nodal degrees of freedom results for static and steady state transverse dynamic response are obtained and compared against the results based on Abaqus beam solution. Under the present finite element solution, the clamped-clamped composite beam is analyzed based on four beam elements with a total of 10 degrees of freedom, while in Abaqus beam solution, the model is consisted of 200 B31 elements with 1206 dof..

Static and Steady State Dynamic Flexural Solutions

The quasi-static and steady state dynamic results for the nodal transverse displacement and related bending rotation plotted against the beam coordinate axis are presented in Figs. (8a-b) and (8c-d), respectively. It is observed that, the developed finite element solution results based on four beam elements with 10 degrees of freedom shows an excellent agreement with those results based on Abaqus model using 200 beam B31 elements with 1206 degrees of freedom.



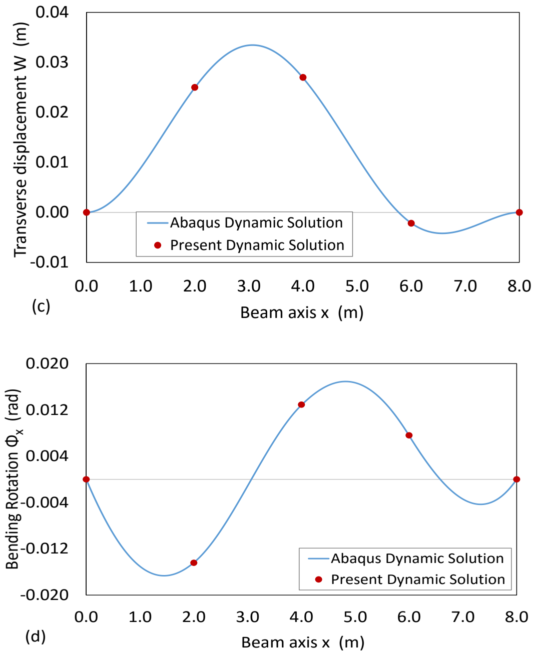


Figure (8): Static and dynamic analyses for clamped-clamped symmetric beam under harmonic forces

VII. Conclusions

- A super-convergent finite beam element is developed for symmetric composite laminated beams subjected to transverse harmonic forces. The finite element is based on the exact shape functions which exactly satisfy the homogeneous form of the transverse field equations.
- The new beam element is based on the first order shear deformation theory in which the formulation captures the effects of shear deformation, rotary inertia, Poison's ratio.
- The present finite element formulation successfully captures the quasi-static and steady state flexural dynamic responses of symmetric composite laminated beams under various transverse harmonic forces. The solution is also able to capture the eigen-frequencies and eigen-modes of the composite beams.
- The new beam element contains no discretization errors and usually provides excellent results with Abaqus beam solution while keeping the number of degrees of freedom a minimum.
- Comparison with exact solutions and established finite element Abaqus solutions shows the validity and accuracy of the present finite element formulation.

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