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The Interaction Forces between Two Identical Cylinders Spinning around their Stationary and ParallelAxes in an IdealFluid

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Abstract: *This paper derives the equations that describe the interaction forcesbetween two identical cylinders spinningaround their stationary and parallel axes in a fluid that isassumed to be in-viscous, steady, in-vortical, and in-compressible. The paper starts by deriving the velocity field from Laplace equation,governing this problem,and the system boundary conditions. It then determines the pressure field from the velocity field using Bernoulli equation. Finally, the paper integrates the pressure around either cylinder-surface to find the force acting on its axis.All equations and derivationsprovided in this paper are exact solutions. No numerical analysesor approximations are used.The paper finds that such identical cylinders repel or attract each other in inverse relation with separation between their axes, according to similar or opposite direction of rotation, respectively.*

Keywords: *Rotating identical cylinders, ideal fluid, Laplace equation,velocity field,Bernoulli equation, pressure, force, inverse law, repulsion, attraction.*

I. Introduction

Determining the force acting on an objectdue to its existence in a fluid is an important topic, and has several important applications. One of these applications is evaluating the lift force acting on an aeroplane wing due to the flow of the air. The solution to such a problem might be analytical or numerical, depending upon the complexity of the system and the required level of accuracy of the solution. Cylindrical Objects in fluid-flows constitute one category of such problems and have vast applications.

According to the literature reviewed, several such systems have already been studied both numerically and analytically,while other systems have attracted no attention. An example ofsuch studiedsystems[1]is the lift force acting on a cylinder rollingin aflow. Another example[2] is the interaction forces between two concentric cylinders with fluid internal and/or external to them. A third example[3]is the interaction forces between two cylinders rotating around two parallel floating axes. No study to the knowledge of both authors has been done on the interaction forceswhen the two parallel axes are fixed.

This paper is dedicated to find the interaction forcesbetween two identical cylindersspinning in an ideal fluid (in-viscous, steady, in-vortical, andin-compressible);when their two parallel axes of rotation are made fixed.These forces act on both axes. The system has never been studied before. For simplicity, the cylinders are assumed infinitely long, so asto have a two-dimensional problem in xy-plane, where rotational axes are parallel to the ignored z-axis.

II. Problem Statement

Fig.1 shows the top view of the system targeted by this paper. It depictstwo identical circles (for the two identical cylinders)of radius R. The distance between the two centres (for the two axes) is:$2a$, where: $a > R$.

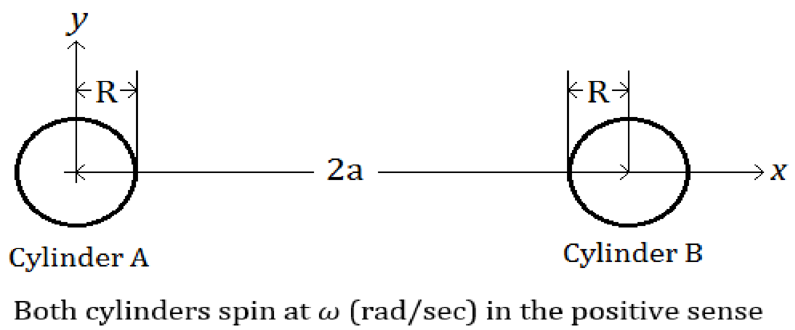

Fig.1: Top view of the system targeted by this paper

Each circle (cylinder) is rotating around its centre (axis) with a fixed angular velocity, ω, in a fluid that is in-viscous, steady, in-vortical, and in-compressible. The aim of the paper is to derive the exact equations describing the forcesboth cylindersexert on their fixed axes after the entire system reached steady state.

The next section uses the Laplace equation[4]governing such problems, to find the velocity field of the fluid, satisfying its boundary conditions, which are:

1. the velocity of the fluid at the circle (i.e. the surface of the cylinder)is tangential to it, with a magnitude of: $\omega \cdot R$; and:
2. the velocity of the fluid at infinity is zero.

III. Fluid Velocity

As the governing Laplace equation is linear, super-position can be applied to simplify the solution. Considering Cylinder-A alone,the steady-state fluid-velocity vector:

$VA(x,y) = \langle VA_x(x,y), VA_y(x,y) \rangle$,(1)

isknown [5] to be as shown in Fig.2, where its two components are givenby:

$VA_x(x,y) = -\omega \cdot R^2 \cdot \frac{y}{x^2+y^2}$, and:(2)

$VA_y(x,y) = \omega \cdot R^2 \cdot \frac{x}{x^2+y^2}$, provided:(3)

$x^2 + y^2 > R^2$.

These velocity equations satisfyboth boundary conditions mentioned above. Furthermore, the velocity of the fluid due to the spinning of Cylinder-Ais seen to be directly proportional to ω, and inversely proportional tothe distance from the cylinder axis;i.e. the further from cylinder axis, the slower the fluid is.

Considering Cylinder-B alone, the steady-state fluid-velocity vector:

$VB(x,y) = \langle VB_x(x,y), VB_y(x,y) \rangle$,(4)

can be obtained from Eqs.2&3 (with a positive shift of: $2a$, along the x-axis) as:

$VB_x(x,y) = -\omega \cdot R^2 \cdot \frac{y}{(x-2a)^2+y^2}$, and:(5)

$VB_y(x,y) = \omega \cdot R^2 \cdot \frac{x-2a}{(x-2a)^2+y^2}$, provided:(6)

$(x-2a)^2 + y^2 > R^2$.

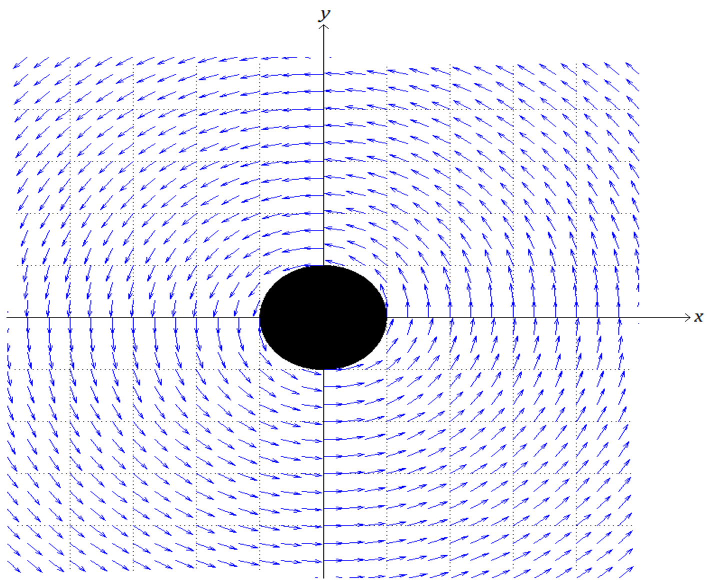

Fig.2: The velocity field of the ideal fluid due to the spinning of Cylinder-A

Hence, applying super-position;the fluid velocity for the system of two cylinders shown in Fig.1, can be found using Eqs.1-6 as:

$V(x,y) = VA(x,y) + VB(x,y) = \langle V_x(x,y), V_y(x,y)\rangle$,where:(7)

$V_x(x,y) = VA_x(x,y) + VB_x(x,y) = -\omega \cdot R^2 \cdot \left[\frac{y}{x^2+y^2} + \frac{y}{(x-2a)^2+y^2}\right]$, and:(8)

$V_y(x,y) = VA_y(x,y) + VB_y(x,y) = \omega \cdot R^2 \cdot \left[\frac{x}{x^2+y^2} + \frac{x-2a}{(x-2a)^2+y^2}\right]$,provided:(9)

$x^2 + y^2 > R^2$, and: $(x-2a)^2 + y^2 > R^2$; i.e. where fluid exists outside both cylinders.

The above fluid-velocity is plotted as shown in Fig.3 below.The next section uses the fluid velocity and Bernoulli equation to obtain the pressure field.

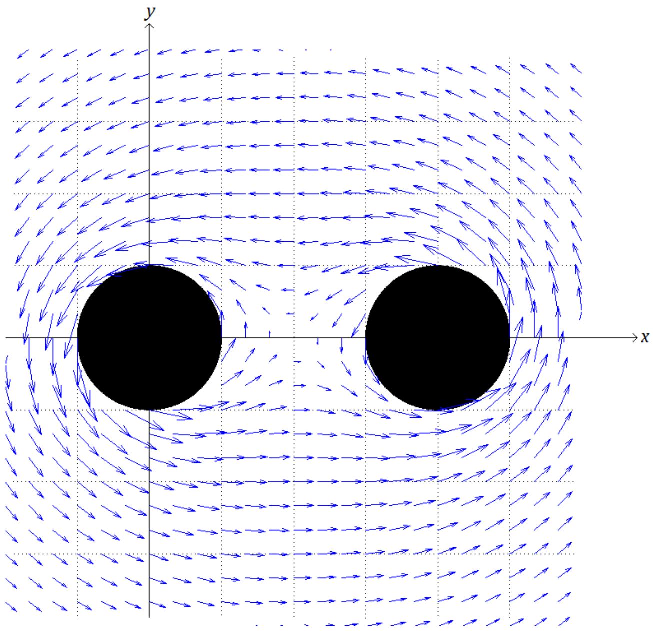

Fig.3: The velocity field of the ideal fluid due to the spinning of Cylinder-A and Cylinder-B in the same direction around their stationary and parallel axes

IV. Fluid Pressure

In this section, the pressure at the boundary of either cylinder is derived, in readiness to find the force exerted on its axes. Ignoring the effect of the gravitational force in the fluid, Bernoulli equation relates the pressure magnitude, $P(x,y)$, to the velocity field, $V(x,y)$, as:

$P(x,y) + \frac{1}{2}\rho \cdot |V(x,y)|^2 = Constant$, where: ρ is the density of the fluid. (10)

The above equation can be read as:the summation of both static and dynamic pressures is constant everywhere in the fluid. In this respect, it is the square of the magnitude of the fluid velocity, $|V(x,y)|^2$, is what really matters for the fluid pressure.

Applying Eq.10 at Cylinder-A border & infinity (where velocity diminishes), then:

$PA(x,y) + \frac{\rho}{2} \cdot \left|V(x,y)|_{@\ Cylinder\ -A\ boundary}\right|^2 = P_\infty$, where:

$PA(x,y)$: is the pressure at Cylinder-A boundary, and:
P_∞: is the fluid pressure at ∞. Hence:

$PA(x,y) = P_\infty - \frac{\rho}{2} \cdot \left|V(x,y)|_{@\ Cylinder\ -A\ boundary}\right|^2$.Using Eq.7, then:

$PA(x,y) = P_\infty - \frac{\rho}{2} \cdot \left[V^2{}_x(x,y) + V^2{}_y(x,y)\right]\Big|_{@\ Cylinder\ -A\ boundary}$.

This is simplified using Eqs.8&9with: $x^2 + y^2 = R^2$ to:

$$PA(x,y) = P_\infty - \frac{\rho \cdot \omega^2 \cdot R^4}{2} \cdot \left[\left(\frac{y}{R^2} + \frac{y}{R^2 - 4a \cdot x + 4a^2}\right)^2 + \left(\frac{x}{R^2} + \frac{x - 2a}{R^2 - 4a \cdot x + 4a^2}\right)^2\right].$$

This is reduced with: $x^2 + y^2 = R^2$ to:

$PA(x,y) = P_\infty - \frac{\rho \cdot \omega^2 \cdot R^2}{2} \cdot \left(1 + \frac{3R^2 - 4a \cdot x}{R^2 - 4a \cdot x + 4a^2}\right)$,where: $x \in$ Circle-A.

Converting to polar coordinates, with: $x = R \cdot \cos\theta$, then:

$PA(x,y) = PA(\theta) = P_\infty - \frac{\rho \cdot \omega^2 \cdot R^2}{2} \cdot \left[1 + \frac{3R^2 - 4a \cdot R \cdot \cos\theta}{R^2 - 4a \cdot R \cdot \cos\theta + 4a^2}\right], \quad \theta \in [0, 2\pi]$.(11)

$PA(\theta)$is seen to be symmetrical about the$x$-axis.

V. Force Acting on the Rotational Axis of Cylinder-A

The fluid pressure,$PA(\theta)$,expressed by Eq.11 is acting perpendicular to the surface of Cylinder-A as shown in Fig.4, and can be seen to cause infinitesimal repelling force, $dFA(\theta)$, in the same direction, given by:

$dFA(\theta) = PA(\theta) \cdot L \cdot R \cdot d\theta$, where:

L: is the Length of either cylinder, which is assumed to be infinitely long.

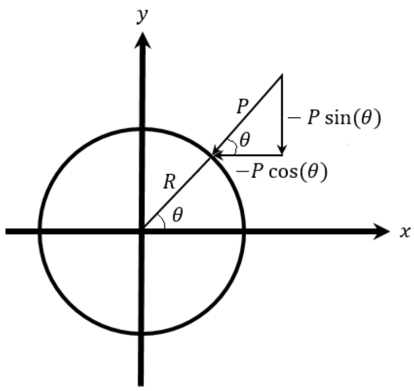

Fig.4: Top view of Cylinder-A showing fluid pressure

Decomposing: $dFA(\theta)$into two components, and ignoring its y-component due to the symmetry of $PA(\theta)$ about the x-axis; then:

$dFA_x(\theta) = -PA(\theta) \cdot L \cdot R \cdot \cos\theta \cdot d\theta$.(12)

Integrating $dFA_x(\theta)$around Circle-A (the surface of Cylinder-A) yields the force, F, exerted on the axis of rotation of Cylinder-A. Hence:

$F = \int_0^{2\pi} dFA_x(\theta)$. This is expressed using Eqs.11&12, as:

$$F = -L \cdot R \int_0^{2\pi} \left[P_\infty - \frac{\rho \cdot \omega^2 \cdot R^2}{2} \cdot \left(1 + \frac{3R^2 - 4a \cdot R \cdot \cos\theta}{R^2 - 4a \cdot R \cdot \cos\theta + 4a^2}\right)\right] \cdot \cos\theta \cdot d\theta.$$

The first two of the above three integrandswill integrate to zero, leaving Fas:

$F = \frac{1}{2} \cdot \rho \cdot L \cdot \omega^2 \cdot R^4 \cdot I$, where:(13)

$$I = \int_0^{2\pi} \frac{3R - 4a \cdot \cos\theta}{R^2 - 4a \cdot R \cdot \cos\theta + 4a^2} \cdot \cos\theta \cdot d\theta.$$

VI. Solving the Integral

The integral, I, can be solved [6] by usingthe complex transformation:

$z = e^{i\theta} = \cos\theta + i \cdot \sin\theta,$ where: $i = \sqrt{-1},$ $i^2 = -1,$ $d\theta = \frac{dz}{iz}$,and:

$\cos\theta = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z};$

whereby it converts to an integral over the closed contour, C, of the unit circle: $|z| = 1$, in the complex z-plane. Hence, removingθ:

$$I = \int_0^{2\pi} \frac{3R-4a\cdot\cos\theta}{R^2+4a^2-4a\cdot R\cdot\cos\theta} \cdot \cos\theta \cdot d\theta = \oint_C \frac{3R-4a\cdot\left(\frac{z^2+1}{2z}\right)}{R^2+4a^2-4a\cdot R\cdot\left(\frac{z^2+1}{2z}\right)} \cdot \frac{z^2+1}{2z} \cdot \frac{dz}{iz}.$$

This is simplified to:

$I = \frac{-i}{2}\oint_C \frac{2a\cdot z^4-3R\cdot z^3+4a\cdot z^2-3R\cdot z+2a}{z^2[2a\cdot R\cdot z^2-(R^2+4a^2)\cdot z+2a\cdot R]} \cdot dz$,and can be expressed as:

$I = \frac{-i}{4a\cdot R}\oint_C \frac{2a\cdot z^4-3R\cdot z^3+4a\cdot z^2-3R\cdot z+2a}{z^2\cdot(z-\alpha)\cdot(z-\beta)} \cdot dz$,where: α and: β are the roots of:

$2a \cdot R \cdot z^2 - (R^2 + 4a^2) \cdot z + 2a \cdot R = 0$, given by:

$\alpha = \frac{R}{2a}$,and: (14)

$\beta = \frac{2a}{R} = \frac{1}{\alpha}.$ (15)

Noting that:$a > R$, then: $\alpha < 1$, and is inside C, while: $\beta > 1$, and is outside C.Hence, the integrand in I has three poles within C, two at the origin and one at α.

Using Cauchy Theorem in complex integrals[6], I is found as:

$I = \frac{-i}{4a\cdot R} \cdot (2\pi \cdot i) \cdot \left\{\left[\frac{2a\cdot z^4-3R\cdot z^3+4a\cdot z^2-3R\cdot z+2a}{(z-\alpha)\cdot(z-\beta)}\right]'\Bigg|_{@z=0} + \left[\frac{2a\cdot z^4-3R\cdot z^3+4a\cdot z^2-3R\cdot z+2a}{z^2\cdot(z-\beta)}\right]\Bigg|_{@z=\alpha}\right\}$.This gives:

$I = \frac{\pi}{2a\cdot R} \cdot \left[\frac{-3R\cdot\alpha\cdot\beta+2a\cdot(\alpha+\beta)}{\alpha^2\cdot\beta^2} + \frac{2a\cdot\alpha^4-3R\cdot\alpha^3+4a\cdot\alpha^2-3R\cdot\alpha+2a}{\alpha^2\cdot(\alpha-\beta)}\right]$.Removing$\beta$using Eq.15, this becomes:

$I = \frac{\pi}{2a\cdot R} \cdot \left[-3R + 2a \cdot \left(\frac{\alpha^2+1}{\alpha}\right) + \frac{2a\cdot\alpha^4-3R\cdot\alpha^3+4a\cdot\alpha^2-3R\cdot\alpha+2a}{\alpha\cdot(\alpha^2-1)}\right].$

Substituting for:αusing Eq.14, and simplifying gives: $I = -\frac{2\pi}{a} \cdot \frac{2a^2-R^2}{4a^2-R^2}.$
Hence, putting this, in Eq.13, and simplifying gives:

$F = -\frac{\pi\cdot\rho\cdot L\cdot\omega^2\cdot R^4}{a} \cdot \frac{2a^2-R^2}{4a^2-R^2}$, remembering that: $a > R$. (16)

The negative sign means that the fluid actually repels Cylinder-A away from Cylinder-B.

VII. InterpretingThe Repulsion Force

Considering: a, to be given as: $a = n \cdot R$,then: $a > R$, implies that: $n > 1$;and Eq.16 for the force, F, can be simplified to:

$F = -\frac{R}{2a} \cdot \frac{n^2-0.5}{n^2-0.25} \cdot P = \frac{-1}{2n} \cdot \frac{n^2-0.5}{n^2-0.25} \cdot P,$ where: $n > 1$; and:

$P = \pi \cdot \rho \cdot L \cdot \omega^2 \cdot R^3$. (17)

When the two identical cylinders are made very near to each other, then: $n = 1^+$,and the repulsion force is:$F1$, given as:

$$F1 = -\frac{P}{3} = -0.33333P.$$

The maximum repulsion, Fp, can be found to occur at:$n = \sqrt{\frac{5+\sqrt{17}}{8}} = 1.06789$, with value of:

$$Fp = -0.33675P.$$

As the gap between the two identical cylinders ismade: $2R$, then:$n = 2$,and the repulsion force is reduced to:$F2$, given as:

$$F2 = -\frac{7P}{30} = -\frac{7}{15} \cdot \frac{R \cdot P}{a} = -0.23333P.$$

When gap is made: $4R$, then:$n = 3$,and the repulsion force is reduced to: $F3$, given as:

$$F3 = -\frac{17P}{105} = -\frac{17}{35} \cdot \frac{R \cdot P}{a} = -0.16190P.$$

When gap is made:$6R$, then:$n = 4$,and the repulsion force is reduced to: $F4$, given as:

$$F4 = -\frac{31P}{252} = -\frac{31}{63} \cdot \frac{R \cdot P}{a} = -0.12302P.$$

As the two identical cylinders are spaced far enough ($n > 4$) of each other, then the repulsion force goes asymptotically to:

$$Fn = -\frac{P}{2n} = -\frac{R \cdot P}{2a}.$$

This is an inverse relationship with the separation between axes of rotation, i.e. the identical cylinders rotating in the same directiontend to repel each other in inverse law with respect to their axial separation.

VIII. Effect of Opposing Direction

When Cylinder-B spins differentlyto Cylinder-A, namely in the negative sense, the fluid-velocity can be plotted as shown in Fig.5 below.

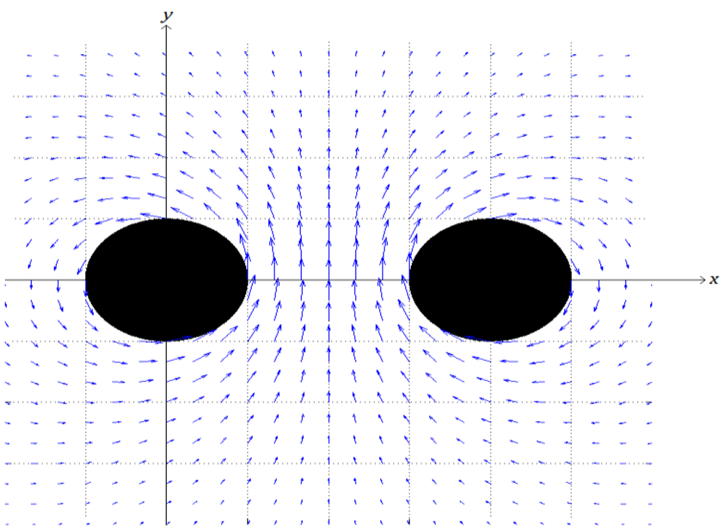

Fig.5: The velocity field of the ideal fluid due to the spinning of Cylinder-A and Cylinder-B in different directionsaround their stationary and parallel axes

A similar treatment as above, gives the fluid force acting on the axis of Cylinder-A, as:

$F = \frac{\pi \cdot \rho \cdot L \cdot \omega^2 \cdot R^4}{a} \cdot \frac{2a^2}{4a^2 - R^2}$, remembering: $a > R$. Using Eq.17, this can be simplified to:

$F = \frac{R}{2a} \cdot \frac{a^2}{a^2 - 0.25R^2} \cdot P = \frac{R}{2a} \cdot \frac{n^2}{n^2 - 0.25} \cdot P = \frac{1}{2n} \cdot \frac{n^2}{n^2 - 0.25} \cdot P$, where: $n > 1$.

This is an attraction force, reduced asymptotically for far enough spacing ($n > 4$), to:

$Fn = \frac{P}{2n} = \frac{R \cdot P}{2a}$.

This is also an inverse relationship with the separation between axes of rotation. Since both cylinders are identical, thenthe results obtained are valid for both of them.

IX. Conclusion

This paper derived the force acting on twoidentical cylinders spinning atconstant angular velocity around their stationary and parallel axes in anin-viscid, steady,in-vortical, and in-compressible fluid.The obtained equations showed that each cylinder axis,in such a system,experiences a repelling or attracting force according to similar or opposite sense of rotation respectively.The magnitude of that force is inversely proportional to the separation between the axes. It is also proportional to the density of the fluid, the cylinder volume, its radius, and the product of the two angular velocities of the cylinders.

Nomenclature:

This section summarizes the symbols used in the paper in alphabetical order as follows:

ρ: Density of the fluid
ω: Angular speed of spinning of either cylinder
a: Half the distance between axes of cylinders
$dFA(\theta)$: Infinitesimal force acting on Cylinder-A
$dFA_x(\theta)$: Component of: $dFA(\theta)$ along the x-axis
F: Interaction force acting on the axle of Cylinder-A
$F1$: Value of: F when both cylinders are about to touch each other
$F2$: Value of: F when both cylinders are spaced $2R$ apart
$F3$: Value of: F when both cylinders are spaced $4R$ apart
$F4$: Value of: F when both cylinders are spaced $6R$ apart
Fp: Peak value of: F
L: Length of either cylinder
$P(x, y)$: Pressure magnitude of the fluid
P_∞: Pressure magnitude of the fluidat ∞
$PA(x, y)$: Pressuremagnitude of the fluidat Cylinder-A boundary in xy-coordinates
$PA(\theta)$: Pressure magnitude of the fluidat Cylinder-A boundary in $r\theta$-coordinates
R: Radius of either cylinder
$V(x, y)$: Velocity vector of the fluid due to the spinning of both cylinders
$V_x(x, y)$: Component of: $V(x, y)$ along the x-axis
$V_y(x, y)$: Component of: $V(x, y)$ along the y-axis
$VA(x, y)$: Velocity vector of the fluid due to the spinning of Cylinder-A
$VA_x(x, y)$: Component of: $VA(x, y)$ along the x-axis
$VA_y(x, y)$: Component of: $VA(x, y)$ along the y-axis
$VB(x, y)$: Velocity vector of the fluid due to the spinning of Cylinder-B
$VB_x(x, y)$: Component of: $VB(x, y)$ along the x-axis
$VB_y(x, y)$: Component of: $VB(x, y)$ along the y-axis

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