

## Analysis of Electro-thermal Stress and Strain in a Functionally Graded Metal Line under Direct Current Field

Anjan Goswami<sup>1</sup>, Md. Rejaul Haque<sup>2,3</sup>, A. K. Ghosh<sup>3</sup>

<sup>1,2,3</sup>(Dept. of Mechanical Engineering, Bangladesh University of Engineering and Technology, Bangladesh)

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**Abstract:** In this study, the electro-thermal stress and strain of a thin, non-uniform functionally graded metal (FGM) line is analyzed under a direct current field. The material properties of the metal line are assumed to vary over the span following a linear functional relationship. The governing differential equations associated with the electro-thermal problems are derived in terms of variable electrical resistivity, thermal conductivity, co-efficient of thermal expansion and the modulus of elasticity of the material. The solution of the coupled boundary-value problem is then obtained numerically by using finite-difference technique. The thermal stress and strain distributions in the FGM line are determined for different environmental conditions and electric field intensities, which are discussed in light of comparison with those of individual constituent metal lines. The results are claimed to be accurate and reliable as good conformity is found with the theoretical predictions.

**Keywords** – DC field, Functionally graded metal, Finite difference method, Thermal stress, Thermal strain.

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### I. Introduction

Understanding the electro-thermal responses of different conducting materials facilitate the integrity assessment as well as effective manipulation capability in material selection for modern micro-electronic devices. To meet the requirements of extreme performance under various complex working conditions, pure metals are being replaced by different kinds of advanced materials, such as, alloys, composites and functionally graded materials (FGM) [1, 2]. FGM is characterized by a gradual change in material properties over volume. It is an anisotropic composite material where a material gradient has been deliberately introduced over two (or more) different materials and the overall properties of FGMs are clearly different from any of the constituent materials that form it.. FGMs offer great promise to meet a wide range of engineering applications since the compositional gradients can be tailored towards specific requirements [2].

When field current flows through an electrical conducting material, Joule heating is induced, which eventually leads to the generation of heat in the conductor. This electro-thermal conduction ultimately causes thermal stress in the materials, which is considered to be one of the major reasons of metal line failure in electronic packaging [3]. Carslaw and Jaeger [4], theoretically explained the problem of heat conduction in a wire under the influence of current flow. Considering temperature dependent material properties Greenwood and Williamson [5] treated the case of a conductor subjected to a direct current flow. The method was further extended by Jang *et al.* [6] to give a general solution to the coupled nonlinear problem of steady-state electrical and thermal conduction across an interface between two dissimilar half spaces. Introducing a new Joule heating residue vector, heat conduction in symmetrical electro-thermal problems has been analysed under the influence of direct current passing through symmetrical regions of the boundary [7]. Further, a nonlinear analysis has been reported for predicting the electro-thermal response of a metallic wire composed of dissimilar materials with temperature dependent thermal conductivity [8]. The resulting temperature field of a 2D electro-thermal problem near the corner composed of two dissimilar materials in an angled metal line has been analysed under a direct current flow [9]. Recently, the effect of material composition distributions on electro-thermal responses of FGM lines under a direct current field have been analyzed by Ghosh *et al.* [10]. The analytical solution for the thermal stresses of FGMs in the one-dimensional case for spheres and cylinders are given by Lutz and Zimmerman [11, 12].

This paper focuses on the of thermal stress and strain distribution in a thin, non-uniform FGM metal line under direct current field. The electrical, thermal and other metallic properties of the Cu-Al FGM line are assumed to vary over the line following a linear relationship. A metal line having converging-diverging shape with variable rectangular cross section is modeled to represent the geometrical non-uniformity. The numerical solutions of the present coupled multi-physics problem are obtained using a finite-difference computational scheme. The distributions of thermal stress and strain in the FGM line are presented for different environmental conditions and electric field intensities as well.

## II. Mathematical Formulation

### 2.1 Electrical problem

The differential equation that governs the distribution of electric potential in a non-uniform metal line with variable electrical resistivity is as follows:

$$\frac{d^2\phi}{dx^2} = -\left(\frac{d\rho(x)}{dx} J(x) + \rho(x) \frac{dJ(x)}{dx}\right) \quad (1)$$

where,  $\phi$  is electric potential in V,  $\rho$  is the electrical resistivity in  $\Omega\text{m}$ ,  $J$  is current density in  $\text{Am}^{-2}$  respectively.

For uniform cross-sectional area and constant electrical resistivity, the derivatives in the right hand side of the Eq. (1) can be neglected. Equation (1) will then be reduced to the standard one dimensional Laplace equation. The end conditions of the metal line are simulated by the following relation of potential gradient:

$$\frac{d\phi}{dx} = \pm \rho \left(\frac{I}{A}\right) \quad (2)$$

where,  $I$  is the field intensity in *Amps* and  $A$  is the area in  $\text{m}^2$ . The negative sign of the equation (2) applies to the line end where current is being injected and the positive sign corresponds to the current outlet port.

### 2.1 Thermal problem

The general governing equation for steady state heat transfer in a metal line, the surface of which losses heat by convection to the surrounding atmosphere ( $T_\infty$ ) is as follows:

$$\frac{1}{A(x)} \frac{\partial}{\partial x} \left[ A(x)k(x) \frac{\partial T}{\partial x} \right] - \frac{HC(x)}{A(x)} [T - T_\infty] + G(x) = 0 \quad (3)$$

where,  $T$  denotes temperature in  $K$  and  $T_\infty$  is the ambient temperature in  $K$ ;  $k$  is the thermal conductivity in  $\text{Wm}^{-1}\text{K}^{-1}$ ,  $H$  is convective co-efficient in  $\text{Wm}^{-2}\text{K}^{-1}$ ,  $C$  is perimeter in  $\text{m}$  and  $G$  is volumetric heat generation in  $\text{Wm}^{-3}$  respectively.

For the present electro-thermal problem, the heat generation rate per unit volume ( $G$ ) is related to Joule heating caused by the current flow. For steady-state heat transfer in the metal line with variable thermal conductivity  $k(x)$ , subjected to Joule heating, the governing equation becomes:

$$\frac{1}{A(x)} \frac{d}{dx} \left[ A(x)k(x) \frac{dT}{dx} \right] - \frac{HC(x)}{A(x)} [T - T_\infty] + \frac{1}{A_m \rho(x)} \left(\frac{d\phi}{dx}\right)^2 = 0 \quad (4)$$

where,  $A_m$  is mechanical equivalent of heat in  $\text{jcal}^{-1}$ . For the thermal problem, the temperatures at the two ends of the line are assumed to be known. It is mentioned that all possible physical conditions at the ends can readily be accommodated in the present program.

Temperature changes cause the body to expand or contract. If the temperature deformation is not permitted to occur freely an internal stress is created defined as thermal stress and expressed as follows:

$$\sigma = E\varepsilon \quad (5)$$

where,  $\sigma$  represents the thermal stress in  $\text{MPa}$  and  $\varepsilon$  is the thermal strain in  $\text{mm/mm}$  and;  $E$  denotes the modulus of elasticity of the metal in  $\text{MPa}$ . The thermal strain is related to the change in temperature of the metal as:

$$\varepsilon = \alpha\Delta T \quad (6)$$

here,  $\alpha$  is the co-efficient of thermal expansion in  $\text{m/m-K}$

## III. Statement Of Thermal Problem Coupled With An Electrical Field

The analytical model of a variable cross-section FGM metal line with overall dimensions,  $L = 200 \text{ mm}$ ,  $w_1 = 5 \text{ mm}$ ,  $w_2 = 1 \text{ mm}$ ,  $t = 100 \mu\text{m}$ , which is subjected to a steady direct current field is shown in Fig. 1. The current flow is assumed to be,  $I = 1 \text{ A}$ . The FGM line is assumed to be composed of two metals (*Cu* and *Al*), the composition of which varies linearly over the line span ( $x = 0 \sim L$ ).

The entire metal line is assumed to be electrically insulated except for the two ends. For the solution of electrical problem, in addition to the given current densities at the two ends of the line, the zero potential condition is also satisfied at its mid-length position. For the thermal problem, the elevated temperature

Condition of the metal line was simulated by assigning a fixed temperature (313 K) at two ends of the metal line. The surface of the line is assumed to transfer heat by convection to the surrounding environment which is kept at a temperature of 310 K. The convection heat transfer co-efficient is assumed to be constant ( $10 \text{ Wm}^{-2}\text{K}^{-1}$ ) for the entire span of the FGM line.

The metal line is assumed to be fixed at both ends which do not permit the thermal expansion and results thermal stress inside the metal. Another consideration is that the metal was in thermal equilibrium with the environment before the starting of current flow. Hence, the initial temperature of the FGM line is equal to temperature of the environment,  $T_a$  (310K).

The individual electrical resistivity, thermal conductivity, thermal expansion co-efficient and elastic modulus of the two constituent metals (*Cu* and *Al*) assumed for the present analysis are listed in Table 1.

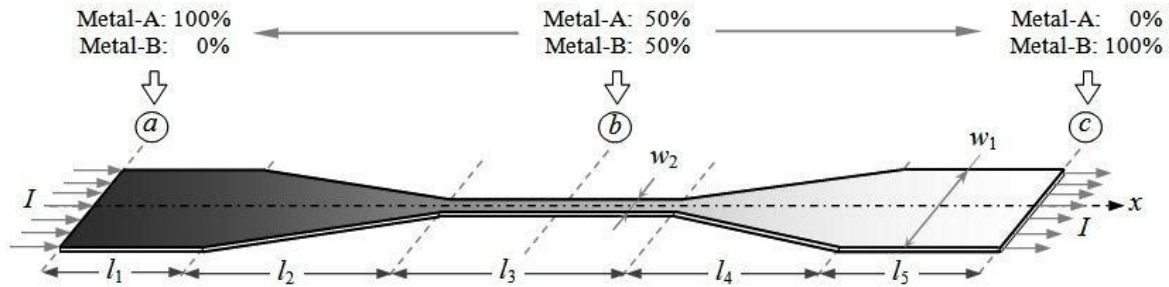


Figure 1: Model of a thin, non-uniform FGM line under direct current field

Both the material properties of interest are assumed to vary following a linear law along the length of the FGM line, which are as follows:

$$\rho(x) = \rho_{Cu} + (\rho_{Al} - \rho_{Cu})(x/L) \tag{7}$$

$$k(x) = k_{Cu} + (k_{Al} - k_{Cu})(x/L) \tag{8}$$

$$\alpha(x) = \alpha_{Cu} + (\alpha_{Al} - \alpha_{Cu})(x/L) \tag{9}$$

$$E(x) = E_{Cu} + (E_{Al} - E_{Cu})(x/L) \tag{10}$$

Table 1: The assumed electric, thermal and material properties of Copper and Aluminum at room temperature.

Metal	Electrical resistivity $\rho$ ( $\Omega\text{-m}$ )	Thermal conductivity $k$ ( $\text{Wm}^{-1}\text{K}^{-1}$ )	Co-efficient of thermal expansion $\alpha$ (m/m-k)	Modulus of elasticity $E$ (Pa)
Copper ( <i>Cu</i> )	$1.71 \times 10^{-8}$	400.35	$17 \times 10^{-6}$	$117 \times 10^9$
Aluminum ( <i>Al</i> )	$2.65 \times 10^{-8}$	238.97	$23.1 \times 10^{-6}$	$70 \times 10^9$

#### IV. Solution Methodology

Using finite-difference technique the present steady state heat conduction-convection boundary-value problem has been solved numerically. Both the governing differential equations associated with the electrical and thermal problems are discretized using the standard three-point central-difference scheme. The difference equations so developed for the electrical and thermal problems are respectively as follows:

$$\varphi_{i+1} - 2\varphi_i + \varphi_{i-1} = -\frac{h}{2} [J_i (\rho_{i+1} - \rho_{i-1}) + \rho_i (J_{i+1} - J_{i-1})] \tag{11}$$

$$[4k_i + k_i A_i^{-1} (A_{i+1} - A_{i-1}) + (k_{i+1} - k_{i-1})] T_{i+1} - [8k_i + 4A_i^{-1} H C_i h^2] T_i + [4k_i - k_i A_i^{-1} (A_{i+1} - A_{i-1}) - (k_{i+1} - k_{i-1})] T_{i-1} = -4A_i^{-1} H C_i h^2 T_\infty - \rho_i^{-1} A_m^{-1} (\varphi_{i+1} - \varphi_{i-1})^2 \tag{12}$$

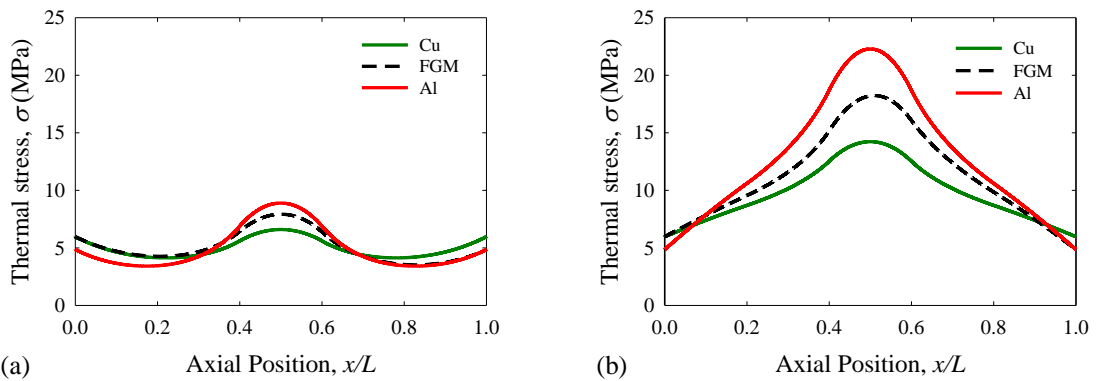
$$\varepsilon_i = \alpha_i (T_i - T_a) \tag{13}$$

$$\sigma_i = E_i \varepsilon_i \tag{14}$$

A computer code was developed using MATLAB to solve the present coupled electro-thermal model. The resulting tri-diagonal systems of algebraic equations are solved by the matrix decomposition method. For the calculation of secondary parameters of density, electrical heat generation, etc., both the three-point forward and backward as well as central differencing schemes were adopted to keep the order of error the same ( $O(h^2)$ ). A total of 1000 nodal points have been considered to discretize the computational domain. The convergence as well as the stability of the numerical solution has however been verified by varying the nodal points from 25 to 1500.

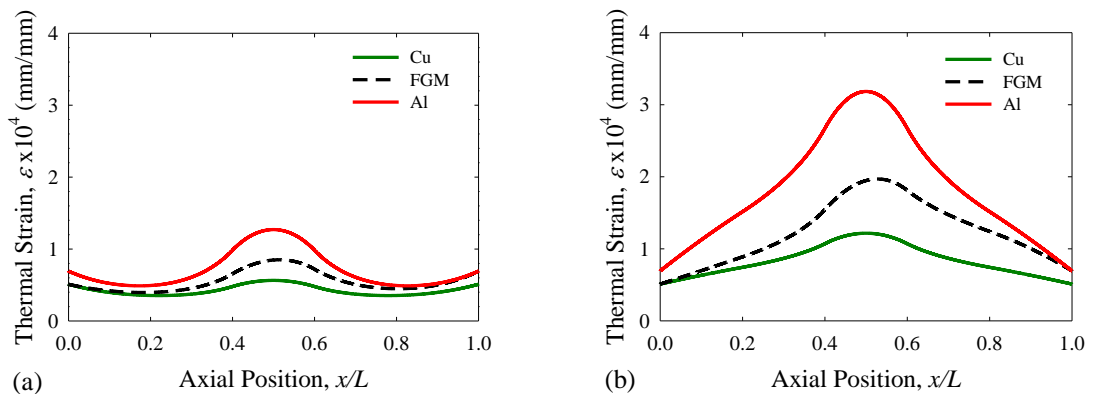
**V. Results And Discussion**

The electro-thermal responses of a FGM metal line (*Cu-Al*) subjected to a direct current field is described in this section. Fig. 2 shows the variation of thermal stress along the metal lines under two different conditions, which are in fact bare and buried lines. In attempt to compare the thermal response of FGM metal line with those of individual *Cu* and *Al* lines, the corresponding thermal stress distribution is presented together with those of the individual *Al* and *Cu* lines (see Fig. 2). For the individual metal lines, the distributions are found to be symmetric about the mid-length position, at which the maximum stress is developed. However, this is not the case for the FGM line, in which the maximum stress position is shifted slightly towards the right from its mid-length position. Fig. 2(b) describes the corresponding thermal stress variations for the case of buried metal lines which are found to be very similar to those of bare lines in terms of nature of variation, but they differ quite significantly in the sense of magnitude. This is quite logical because, in case of buried lines, no heat loss is allowed from their surfaces through convection to surroundings, thereby shifting the overall state of thermal stress to a higher level compared to the case of bare lines.



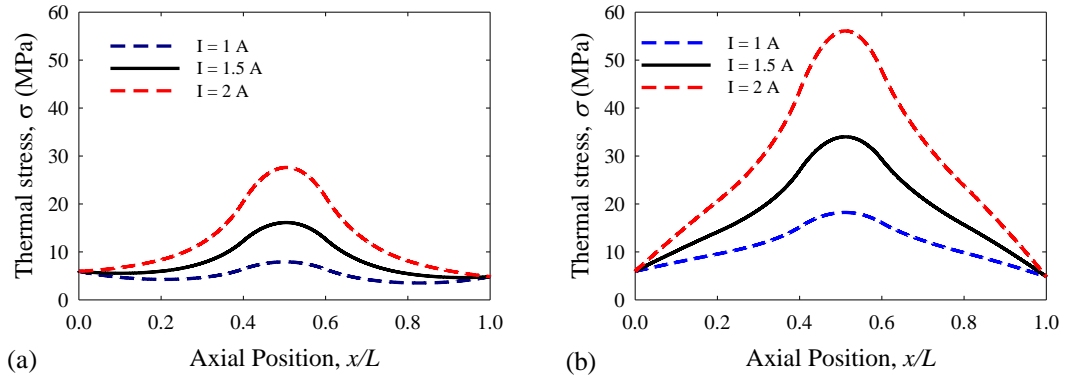
**Figure 2:** Thermal stress distribution along the FGM lines: (a) bare lines; (b) buried lines

The variation of generated thermal strain is presented in Fig. 3 as a function of axial location of lines. The strain distributions for the FGM metal line resides in between those of the parent metals, and maintain higher similarities with that of *Cu* for the starting section and *Al* for the end section. This is because the proportion of *Cu* is higher for the first half section and *Al* is higher for the last half section. Like thermal stress distribution in case of FGM metal line the maximum strain does not occur at the mid-length position, shifted slightly toward the right from its mid-length. Also the overall state of thermal strain for buried lines (see Fig. 3) shifted to a higher level compared to the case of bare lines.

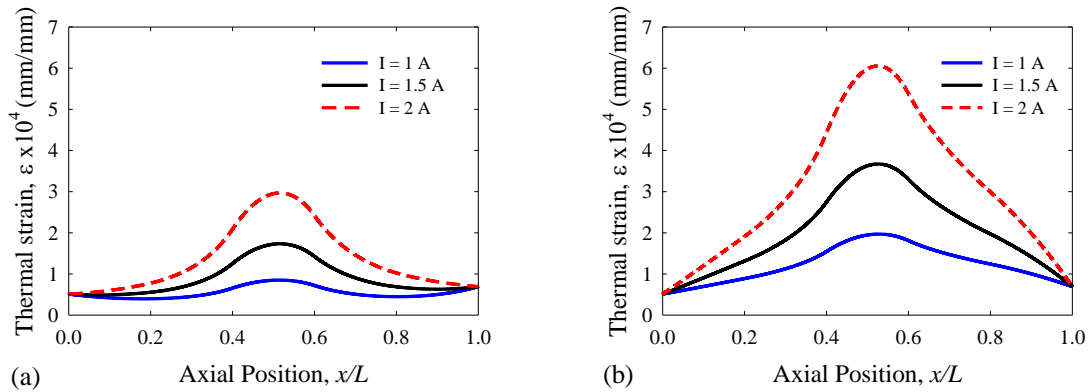


**Figure 3:** Thermal strain distribution along the FGM lines: (a) bare lines; (b) buried lines

The effect of electric field intensity on the thermal stress generation for both bare and buried metal lines is demonstrated in Fig. 4. Both the conditions show a significant increase in developed thermal stress with the increase of current flow. It is also found that the stress development under buried condition (Fig. 4(b)) is much higher compared to that of bare line (Fig. 4(a)).



**Figure 4:** Effect of field intensity on thermal stress distribution: (a) bare lines; (b) buried lines



**Figure 5:** Effect of field intensity on thermal strain distribution: (a) bare lines; (b) buried lines

Fig. 5 shows how electric field intensity affects thermal strain along metal lines for both bare and buried conditions. Like thermal stress distribution a significant increase in developed thermal strain due to increase in field intensity is revealed. The numerical results are found to be in good conformity with the expectation, because thermal stresses are proportional to corresponding thermal strains. And also it is to be noted that in our metal line, for all cases maximum stress and strain values are found near the mid-section of the metal lines where the cross sectional area is minimum.

The mesh sensitivity analysis of the present computational scheme is shown in Fig. 6. The same computational domain of the FGM line is discretized with various numbers of nodes varying from 25 to 1500. The solution of the problem, i.e. the maximum thermal stress developed in the FGM line is plotted against the number of the nodes used to discretize the domain. It is observed that, for lower values of nodes, the magnitude of maximum thermal stress decreases with the increase of node numbers to a lowest value. Then for a range of node numbers thermal stress varies in gradual increasing and decreasing nature. But as the number of nodes goes to a value over 290, the value of maximum thermal stress generation becomes independent of the node numbers and the curve becomes flat. This convergence is an indication of stable solution for these values of node numbers.

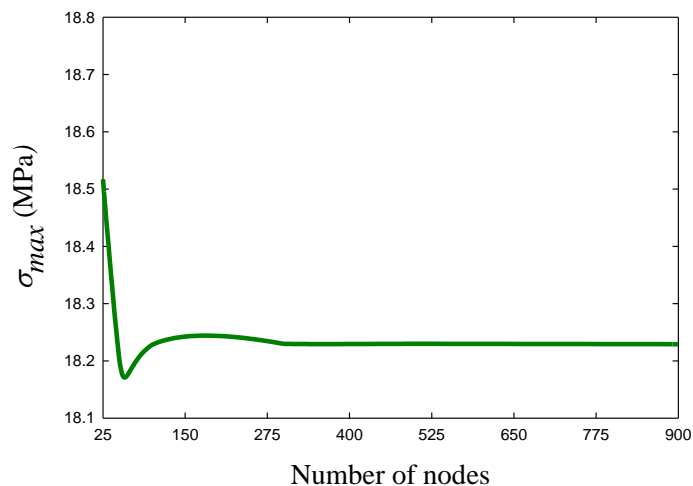


Figure 6: Maximum stress developed in the FGM line as a function of nodal points

## VI. Conclusion

Variation of thermal stress and strain due the effect of direct current field in a thin, non-uniform FGM metal composed of two metals (*Al* and *Cu*) is investigated. It is observed that the distribution of the associated material properties of the resulting metal line is assumed to be linear functions of the spatial coordinate. The magnitude of the thermal stress and strain as well as their nature of distributions differs significantly from those of the individual constituent metal lines. The intensity of the electric field plays a quite substantial role in stress and strain development. Based on the findings it can be concluded that this analysis provides a reliable guideline to predict the performance of functionally graded metal lines in electronic devices and to develop effective FGM metal lines as well.

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