IOSR Journal of Mechanical and Civil Engineering (IOSR-JMCE)
e-ISSN: 2278-1684,p-ISSN: 2320-334X, Volume 12, Issue 2 Ver. III (Mar - Apr. 2015), PP 78-87
www.iosrjournals.org

The Thick Orthotropic Plates Analysis Methods, Part II: State Space Equation Developments for Symmetric Clamped-Free Edges

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Abstract : *In this paper, State Space approach formulation is developed to obtain the three-dimensional solution of thick orthotropic plates with symmetric Clamped-Free edges. All equations of elasticity can be satisfied. All the elastic constants are taken into account in this approach. The system Matrix, which is one the main part of State Space solution, is derived for symmetric Clamped-Free boundaries.*

Keywords -*Exact 3D solution, Orthotropy, State Space method, Symmetric clamped-free edges, Thick plate analysis*

I. Introduction

As it has described in part I, Kirchhoff-Love theory can not estimate the exact stress-strain relationship in thick plate case. Ambartsumyan, Mindlin and Reissner analyses are also incapable to result in an exact relationship between stress and strain within thick plate sue to thickness effects nonexistent. Method of Initial function and State Space method are two analytical methods which could lead to exact 3-D behavior of thick plate under distributed load case and it is applicable to different boundary conditions[1, 2][3].

In early 90s, Fan developed State Space solution for different boundary conditions. In this part of research, the author considers Three-dimensional elasticity in this research. In addition, a state equation for an orthotropic body is used. The boundary condition which formulated in this dissertation refers to two opposite Clamped edges and two other edges Free (CFCF). In this paper, exact 3-D analytical solution for elasticity of orthotropic thick rectangular plate is used. The exact solution for the bending of static plates with arbitrary elastic constants and ratio between thickness and width will be obtained.

This paper investigates the exact system matrix in State Space solution of thick orthotropic plate with CFCF boundary conditions. By using system matrix, the initial lamina stresses and strains could lead to other stresses and strains in any point across the thickness. The boundary conditions for state space method equations derivations is expressed in Fig. 1.

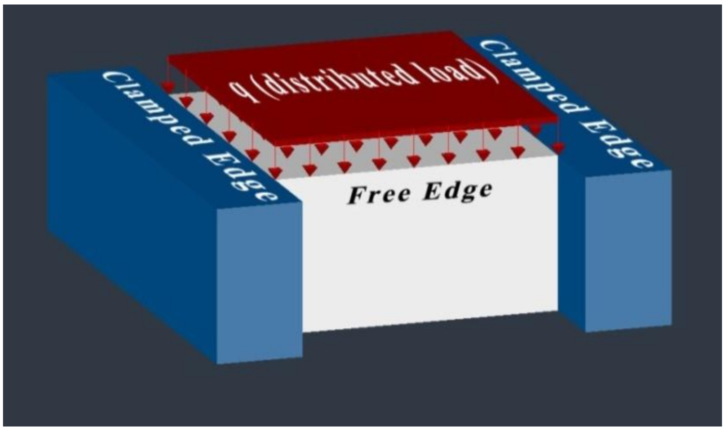

Fig. 1 Load & boundary conditions and general geometry of problem.

II. Jia-Rang Fan's State Equation Derivation for Simply Supported Orthotropic Plate

The State Space solution for thick orthotropic rectangular plate with simply supported edges developed by Fan in 1992. A thick rectangular plate of length a, width of b and uniform thickness of h considered in his solution, as shown in Fig. 2. Origin of co-ordinate was located at top corner point of the plate. U, V and W were three displacements in x, y and z direction, respectively and Plate was made of orthotropic material. The principle material axes and rectangular co-ordinate system, which is shown in Fig. 2, were coincided.

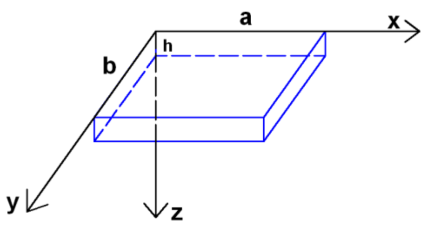

Fig. 2 General geometry of orthotropic thick rectangular plate.

The stress-strain relationship can be written as fallow:

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{Bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{bmatrix}\begin{Bmatrix}\frac{\partial U}{\partial x}\\ \frac{\partial V}{\partial y}\\ \frac{\partial W}{\partial z}\\ \frac{\partial V}{\partial z}+\frac{\partial W}{\partial y}\\ \frac{\partial W}{\partial x}+\frac{\partial U}{\partial z}\\ \frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\end{Bmatrix} \tag{1}$$

Das and Rao (1977) used matrix form in MIF which they called it “a mixed method“ [4]. They introduced Eq. (2) as the basic dynamic equations of equilibrium and stress-strain relations of an elastic body:

$$\frac{\partial}{\partial z}\begin{Bmatrix}U\\ V\\ Z\end{Bmatrix}=[A]\begin{Bmatrix}W\\ Y\\ X\end{Bmatrix} \qquad \frac{\partial}{\partial z}\begin{Bmatrix}W\\ Y\\ X\end{Bmatrix}=[B]\begin{Bmatrix}U\\ V\\ Z\end{Bmatrix} \tag{2}$$

The way which Das formed the state vector was a bit different with the Fan's formulation. As Fan [5] rearranged Eq. (2), it can be written in a contracted form as:

$$\frac{d}{dz}[U\ \ V\ \ Z\ \ X\ \ Y\ \ W]^T=\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}[U\ \ V\ \ Z\ \ X\ \ Y\ \ W]^T \tag{3}$$

by simplification of Eq. (3), we can write this equation in the form of Homogenous linear z-invariant systems state equation as:

$$\frac{\partial}{\partial z}\{f\}=[D]\{f\} \tag{4}$$

where $[D]$ is the system matrix and $\{f\}$ is the state vector. $[A]$ and $[B]$ matrices are symmetric about the secondary diagonal. By Assuming $X=\tau_{xz}$, $Y=\tau_{yz}$, $Z=\sigma_{zz}$, $\alpha=\frac{\partial}{\partial x}$, $\beta=\frac{\partial}{\partial y}$, and eliminations of σ_{xx}, σ_{yy} and τ_{xy} from Eqs. (1), (3) and (4), the state equation can be obtained as:

$$\frac{\partial}{\partial z}\begin{Bmatrix}U\\ V\\ Z\\ X\\ Y\\ W\end{Bmatrix}=\begin{bmatrix}0 & 0 & 0 & C_8 & 0 & -\alpha\\ 0 & 0 & 0 & 0 & C_9 & -\beta\\ 0 & 0 & 0 & -\alpha & -\beta & 0\\ -C_2\alpha^2-C_6\beta^2 & -(C_3+C_6)\alpha\beta & C_1\alpha & 0 & 0 & 0\\ -(C_3+C_6)\alpha\beta & -C_6\alpha^2-C_4\beta^2 & C_5\beta & 0 & 0 & 0\\ C_1\alpha & C_5\beta & C_{10} & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}U\\ V\\ Z\\ X\\ Y\\ W\end{Bmatrix} \tag{5}$$

$C_i\ (1,2,3,\dots,9)$ are all the constants related to the 9 stiffness coefficients of the material [5] as fallow:

$$C_1=-\frac{C_{13}}{C_{33}} \qquad C_2=C_2-\frac{C_{13}{}^2}{C_{33}} \qquad C_3=C_{12}-\frac{C_{13}C_{23}}{C_{33}}C_4=C_{22}-\frac{C_{23}{}^2}{C_{33}}$$
$$C_5=-\frac{C_{23}}{C_{33}} \qquad C_6=C_{66} \qquad C_7=\frac{1}{C_{33}} \qquad C_8=\frac{1}{C_{55}} \qquad C_9=\frac{1}{C_{44}} \tag{6}$$

the elimination stress components can be derived as [5]:

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{Bmatrix}=\begin{bmatrix}C_2\alpha & C_3\beta & -C_1\\ C_3\alpha & C_4\beta & -C_5\\ C_6\beta & C_6\alpha & 0\end{bmatrix}\begin{Bmatrix}U\\ V\\ Z\end{Bmatrix} \tag{7}$$

To solve state Eqs. (5) and (7), the six state variables of the state vectors in these two equations can be written based on boundary conditions (Eqs. 9 and 10) as :

$$\mathrm{U} = \sum_m \sum_n \mathrm{U_{mn}}(z) \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{8}$$

$$\mathrm{V} = \sum_m \sum_n \mathrm{V_{mn}}(z) \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b}$$

$$\mathrm{Z} = \sum_m \sum_n \mathrm{Z_{mn}}(z) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$\mathrm{X} = \sum_m \sum_n \mathrm{X_{mn}}(z) \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$\mathrm{Y} = \sum_m \sum_n \mathrm{Y_{mn}}(z) \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b}$$

$$\mathrm{W} = \sum_m \sum_n \mathrm{W_{mn}}(z) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$\sigma_{xx} = \sum_m \sum_n \sigma_{xx_{mn}}(z) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$\sigma_{yy} = \sum_m \sum_n \sigma_{yy_{mn}}(z) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$\tau_{xy} = \sum_m \sum_n \tau_{xy_{mn}}(z) \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b}$$

All 9 equations should be satisfied boundary condition of simply supported thick rectangular plate:

$$\text{On} \quad x = 0, a \quad \rightarrow \quad \sigma_{xx} = \mathrm{W} = \mathrm{V} = 0 \tag{9}$$

$$\text{On} \quad y = 0, b \quad \rightarrow \quad \sigma_{yy} = \mathrm{W} = \mathrm{U} = 0 \tag{10}$$

substituting Eq. (8) into state Eq. (5) gives the fallowing result for each combination of m and n:

$$\frac{d}{dz}[\mathrm{U_{mn}}(z)\mathrm{V_{mn}}(z)\mathrm{Z_{mn}}(z)\mathrm{X_{mn}}(z)\mathrm{Y_{mn}}(z)\mathrm{W_{mn}}(z)]^T = \begin{bmatrix} 0 & \mathrm{A_{mn}} \\ \mathrm{B_{mn}} & 0 \end{bmatrix} [\mathrm{U_{mn}}(z)\mathrm{V_{mn}}(z)\mathrm{Z_{mn}}(z)\mathrm{X_{mn}}(z)\mathrm{Y_{mn}}(z)\mathrm{W_{mn}}(z)]^T \tag{11}$$

Eq. (11) can also be written as:

$$[\mathrm{U_{mn}}(z)\mathrm{V_{mn}}(z)\mathrm{Z_{mn}}(z)\mathrm{X_{mn}}(z)\mathrm{Y_{mn}}(z)\mathrm{W_{mn}}(z)] = e^{\begin{bmatrix} 0 & \mathrm{A_{mn}} \\ \mathrm{B_{mn}} & 0 \end{bmatrix}}[\mathrm{U_{mn}}(0)\mathrm{V_{mn}}(0)\mathrm{Z_{mn}}(0)\mathrm{X_{mn}}(0)\mathrm{Y_{mn}}(0)\mathrm{W_{mn}}(0)] \tag{12}$$

A_{mn} and B $_{mn}$ matrixes can be derived based on simply supported boundary conditions as fallow:

$$\mathrm{A_{mn}} = \begin{bmatrix} C_8 & 0 & -\xi \\ 0 & C_9 & -\eta \\ \xi & \eta & 0 \end{bmatrix} \tag{13}$$

$$B_{mn} = \begin{bmatrix} C_2\xi^2 + C_6\eta^2 & (C_3 + C_6)\xi\eta & C_1\xi \\ (C_3 + C_6)\xi\eta & C_6\xi^2 + C_4\eta^2 & C_5\eta \\ -C_1\xi & -C_5\eta & C_{10} \end{bmatrix} \tag{14}$$

where:

$$\xi = \frac{m\pi}{a} \qquad \eta = \frac{n\pi}{b}$$

From above state equation, six-order differential equation governing any of six components of (U_{mn}, V_{mn}, W_{mn}, X_{mn}, Y_{mn}, Z_{mn}) can be obtained. The differential equation for transverse displacement W_{mn} can be expressed in contracted form as:

$$\frac{d^6W_{mn}}{dz^6} + A_0\frac{d^4W_{mn}}{dz^4} + B_0\frac{d^2W_{mn}}{dz^2} + C_0\,W_{mn} = 0 \tag{15}$$

A_0 , B_0 and C_0 can be determined from the coefficient matrix in Eq. (15). All coefficients were provided by Wu and Wardenier (1997). The solution for this six-order differential equation was founded by Wu and Wardenier [6].

In other words, based on State Space Solution, the stresses and deflections in state vector $\{f(z)\}$ can be calculated by knowing the top surface values and the state system matrix $[D]$, as fallow:

$$\{f(z)\} = e^{[D]z}\{f(0)\} \tag{16}$$

III. State Equation for Symmetric Clamped-Free Edges

An orthotropic thick plate with CFCF boundary conditions is shown in Fig. 2. The free boundaries have the length of **a** and the clamped boundaries have the length of **b**. All equation derivations can be used for laminated thick plate analysis, too.

As its mention previously, the State Space solution for simply supported orthotropic plate has been derived by Fan [5]. To get the solution for CFCF boundary condition, we use the superposition technique. The superposition principle applied to get the same structural behavior in clamped boundary condition. This method applied by assuming the simply supported structure plus stress distribution at both the clamed boundary conditions. For two other free boundaries, the specific relation assumed which will be mentioned later. In fact, simply supported plate assumed as the first part of the solution and clamped effect applied on the solution by using stress distribution along the two end section (x=0, a). [7]

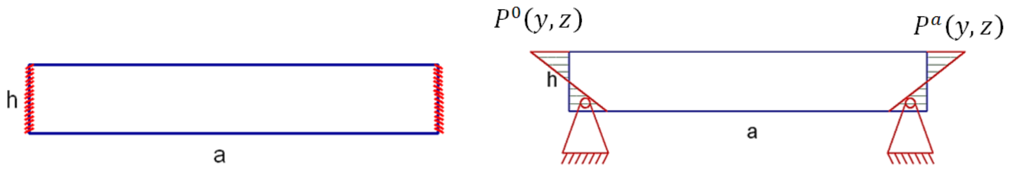

Fig. 3 (Left) Clamped; (right) simply supported with stress distribution.

In Fig. 4, an orthotropic rectangular plate with clamped condition at x=0, **a** and free conditions at y=0, **b** is shown. At y=0, $U^0(x,z)$ and $W^0(x,z)$ are the displacement in x, z directions, respectively. Also, $U^b(x,z)$ and $W^b(x,z)$ are the displacement in x, z directions at y= b, respectively.

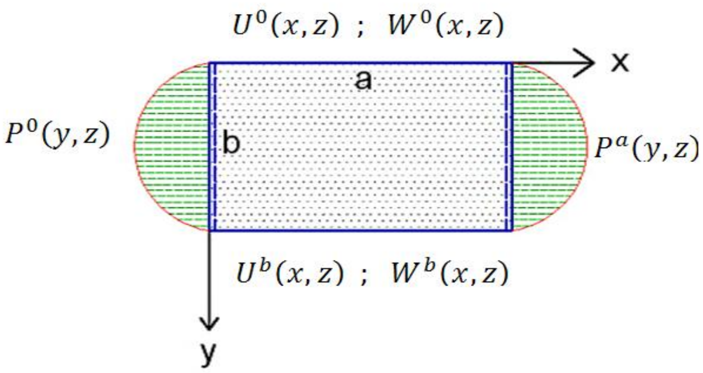

Fig. 4 Boundary conditions and stress distribution.

As we mentioned before changed the clamped boundaries to simply supported one, and add the longitudinal reactions $P^0(y,z)$ and $P^a(y,z)$ at x=zero and X= a, respectively (Fig. 4). Then, we use *Unit Impulse* function [5] as :

$$\mathrm{H}(\mathrm{x}-\mathrm{x}_0)=\begin{cases}1, & x=\mathrm{x}_0\\ 0, & x\neq \mathrm{x}_0\end{cases} \qquad \mathrm{x}\in[0,\mathrm{x}_0] \tag{17}$$

$$\mathrm{H}(\mathrm{x})=\begin{cases}1, & x=\mathrm{x}_0\\ 0, & x\neq \mathrm{x}_0\end{cases} \qquad \mathrm{x}\in[0,\mathrm{x}_0] \tag{18}$$

from above two equations (17-18), we can get Dirac Delta function as:

$$\frac{\mathrm{dH}\,(x)}{\mathrm{dx}}=-\delta(\mathrm{x})=\begin{cases}-\infty, & x=0\\ \ \ 0, & x\neq 0\end{cases} \tag{19}$$

$$\frac{\mathrm{d\,H}(\mathrm{x}-\mathrm{x}_0)}{\mathrm{dx}}=\delta(\mathrm{x}-\mathrm{x}_0)=\begin{cases}\infty, & x=\mathrm{x}_0\\ 0, & x\neq \mathrm{x}_0\end{cases} \tag{20}$$

Due to the reaction that applied on two edges and transverse loading, the in-plane direct stress within the material layer of the plate can be written as:

$$\sigma_{xx}=\bar{\sigma}_{xx}+H(x)P^{(0)}(y,z)+H(x\text{-}a)P^{(a)}(y,z) \tag{21}$$

The displacements in three x, y and z directions are U, V and W, respectively. The general equation of equilibrium for orthotropic material in rectangular coordinate system in bending case is:

$$\begin{gathered}\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}=0\\ \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}=0\\ \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}=0\end{gathered} \tag{22}$$

or by substituting Eq. (21) into Eq. (22), we can get:

$$\begin{gathered}\frac{\partial\bar{\sigma}_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}=\delta(x)\,P^{(0)}(y,z)-\ \delta(x-x_0)P^{(a)}(y,z)\\ \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}=0\\ \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}=0\end{gathered} \tag{23}$$

based on State Space method and superposition principle, we have:

$$\frac{\partial}{\partial z}\{\,f\,\}=[D]\{f\,\}+\{B\,\} \tag{24}$$

Fan also assumed the displacement of plate fallow below relations:

$$\begin{gathered}U(x,y,z)=\bar{U}(x,y,z)+f_u(x,y,z)\\ V(x,y,z)=\bar{V}(x,y,z)+f_v(x,y,z)\\ W(x,y,z)=\bar{W}(x,y,z)+f_w(x,y,z)\end{gathered} \tag{25}$$

These three equations state the idea that each displacement in each coordinate direction consists of two parts, first part is related to the displacement based on simply supported plate solution and the second part is going to define specific equations for different displacement. Those Eqs. (26) are assumed in Fan (1996) for symmetric Hinged-Free edges. Fan's function assumptions are appropriate for Clamped-Free boundaries because the effect of clamped conditions will be add to it by using unit impulse and dirac functions.
The second part of Eqs. (25) are assumed to be [5]:

$$\begin{gathered}f_u(x,y,z)=\left(1-\frac{y}{b}\right)U^{(0)}(x,z)+\ \frac{y}{b}U^{(b)}(x,z)\\ f_w(x,y,z)=\left(1-\frac{y}{b}\right)W^{(0)}(x,z)+\ \frac{y}{b}W^{(b)}(x,z)\end{gathered} \tag{26}$$

$$f_v(x,y,z) = \frac{bC_3}{2C_4}\ .\alpha\ .\ [\left(1-\frac{y}{b}\right)^2 U^{(0)}(x,z) - \left(\frac{y}{b}\right)^2\ U^{(b)}(x,z)\ ]$$

Formerly, the State Space method can be written a contracted form as:

$$\frac{\partial}{\partial z}[\overline{U}\quad \overline{V}\quad Z\quad X\quad Y\quad \overline{W}] = \overline{D}[\overline{U}\quad \overline{V}\quad Z\quad X\quad Y\quad \overline{W}] + \ \overline{B} \tag{27}$$

by using $X = \tau_{xz}$, $Y = \tau_{yz}$, $Z = \sigma_{zz}$, $\alpha = \frac{\partial}{\partial x}$, $\beta = \frac{\partial}{\partial y}$, $\xi = \frac{m\pi}{a}$, $\eta = \frac{n\pi}{b}$ and eliminating σ_{xx} , σ_{yy} , τ_{xy} from Eqs. (22) and (1) we can derived each components of system matrix.

$$\overline{D} = \begin{bmatrix} 0 & 0 & 0 & C_8 & 0 & -\alpha \\ 0 & 0 & 0 & 0 & C_9 & -\beta \\ 0 & 0 & 0 & -\alpha & -\beta & 0 \\ -C_2\alpha^2 - C_6\beta^2 & -(C_3+C_6)\alpha\beta & C_1\alpha & 0 & 0 & 0 \\ -(C_3+C_6)\alpha\beta & -C_6\alpha^2 - C_4\beta^2 & C_5\beta & 0 & 0 & 0 \\ C_1\alpha & C_5\beta & C_{10} & 0 & 0 & 0 \end{bmatrix} \tag{28}$$

C_i $(1, 2, 3, \dots, 9)$ are all the constants related to the 9 stiffness coefficients of the material [5] :

(29)

$$C_1 = -\frac{C_{13}}{C_{33}} \qquad C_2 = C_2 - \frac{{C_{13}}^2}{C_{33}} \qquad C_3 = C_{12} - \frac{C_{13}C_{23}}{C_{33}} C_4 = C_{22} - \frac{{C_{23}}^2}{C_{33}}$$
$$C_5 = -\frac{C_{23}}{C_{33}} \qquad C_6 = C_{66} \qquad C_7 = \frac{1}{C_{33}} \qquad C_8 = \frac{1}{C_{55}} \qquad C_9 = \frac{1}{C_{44}}$$

by using Eq.1 and considering Eqs. (23-29) gives:

(30)

$$\frac{\partial\overline{U}}{\partial z} = \text{Simply supported} - \left[\left(1-\frac{y}{b}\right)\frac{\partial U^{(0)}}{\partial z} + \frac{y}{b}\frac{\partial U^{(b)}}{\partial z}\right] - \alpha\left[\left(1-\frac{y}{b}\right)W^{(0)} + \frac{y}{b}W^{(b)}\right]$$

$$\frac{\partial\overline{U}}{\partial z} = \text{Simply supported} - \left[\left(1-\frac{y}{b}\right)\frac{\partial U^{(0)}}{\partial z} + \frac{y}{b}\frac{\partial U^{(b)}}{\partial z}\right] - \alpha\left[\left(1-\frac{y}{b}\right)W^{(0)} + \frac{y}{b}W^{(b)}\right]$$

$$\frac{\partial\sigma_{zz}}{\partial z} = \text{Simply Supported} - \left(\frac{1}{C_8} - \frac{C_3}{C_4C_9}\right)\ \alpha\ \left[\left(1-\frac{y}{b}\right)\frac{\partial U^{(0)}}{\partial z} + \frac{y}{b}\frac{\partial U^{(b)}}{\partial z}\right] - \frac{1}{C_8}\alpha^2\ \left[\left(1-\frac{y}{b}\right)W^{(0)} + \frac{y}{b}W^{(b)}\right]$$

$$\frac{\partial\tau_{xz}}{\partial z} = \text{Simply supported} - \left[\left(1-\frac{y}{b}\right)\frac{\partial W^{(0)}}{\partial z} + \frac{y}{b}\frac{\partial W^{(b)}}{\partial z}\right] + \left(C_1 - \frac{C_3C_5}{C_5}\right)\ \alpha\ \left[\left(1-\frac{y}{b}\right)U^{(0)} + \frac{y}{b}U^{(b)}\right]$$
$$+\ \delta(x)P^{(0)}(y,z) - \delta(x-a)P^{(a)}(y,z)$$

$$\frac{\partial\tau_{yz}}{\partial z} = \text{Simply supported} + \frac{C_6}{b}\ \alpha\left[U^{(0)} - U^{(b)}\right] - \frac{C_3C_6b}{2C_4}\alpha^3\left[\left(1-\frac{y}{b}\right)^2 U^{(0)} - \left(\frac{y}{b}\right)^2 U^{(b)}\right]$$

$$\frac{\partial\overline{W}}{\partial z} = \text{Simply supported} - \left(1-\frac{y}{b}\right)\frac{\partial W^{(0)}}{\partial z} - \frac{y}{b}\frac{\partial W^{(b)}}{\partial z} + \left(C_1 - \frac{C_3C_5}{C_5}\right)\ \alpha\ \left[\left(1-\frac{y}{b}\right)U^{(0)} + \frac{y}{b}U^{(b)}\right]$$

and for three eliminated stresses σ_{xx} , σ_{yy} , τ_{xy} ,it can be derived as :

(31)

$$\sigma_{xx} = \text{Simply Supported} + \left(C_2 - \frac{{C_3}^2}{C_4}\right)\ \alpha\ \left[\left(1-\frac{y}{b}\right)U^{(0)} + \frac{y}{b}U^{(b)}\right]$$

$$\sigma_{yy} = \text{Simply Supported} + \underbrace{C_3\ \alpha\left[\left(1-\frac{y}{b}\right)U^{(0)} + \frac{y}{b}U^{(b)}\right] - C_4\frac{C_3}{C_4}\alpha\left[\left(1-\frac{y}{b}\right)U^{(0)} + \frac{y}{b}U^{(b)}\right]}_{\text{Zero}}$$

$$\tau_{xy} = \text{Simply supported} + C_6\ \left(\frac{-1}{b}U^{(0)} + \frac{1}{b}U^{(b)}\right) + C_6\ \frac{bc_3}{2c_4}\ .\alpha^2.\left[\left(1-\frac{y}{b}\right)^2 U^{(0)} - \left(\frac{y}{b}\right)^2 U^{(b)}\right]$$

After derivation of all 9 stresses and displacements through Eqs. (30) and (31), additional matrix $[\overline{\mathrm{B}}]$ can be written as:

$$\overline{\mathbf{B}} = \left\{ \begin{array}{c} -\left[\left(1-\frac{y}{b}\right)\frac{\partial U^{(0)}}{\partial z}+\frac{y}{b}\frac{\partial U^{(b)}}{\partial z}\right]-\alpha\left[\left(1-\frac{y}{b}\right)W^{(0)}+\frac{y}{b}W^{(b)}\right] \\ \frac{1}{b}\left[W^{(0)}-W^{(b)}\right]-\frac{C_3 b}{2C_4}\,\alpha\left[\left(1-\frac{y}{b}\right)^2\frac{\partial U^{(0)}}{\partial z}-\left(\frac{y}{b}\right)^2\frac{\partial U^{(b)}}{\partial z}\right] \\ -\left(\frac{1}{C_8}-\frac{C_3}{C_4C_9}\right)\alpha\left[\left(1-\frac{y}{b}\right)\frac{\partial U^{(0)}}{\partial z}+\frac{y}{b}\frac{\partial U^{(b)}}{\partial z}\right]-\frac{1}{C_8}\alpha^2\left[\left(1-\frac{y}{b}\right)W^{(0)}+\frac{y}{b}W^{(b)}\right] \\ \left[\frac{C_3(C_3+C_6)}{C_4}-C_2\right]\alpha^2\left[\left(1-\frac{y}{b}\right)U^{(0)}+\frac{y}{b}U^{(b)}\right]+\delta(x)P^{(0)}(y,z)-\delta(x-a)P^{(a)}(y,z) \\ \frac{C_6}{b}\,\alpha\left[U^{(0)}-U^{(b)}\right]-\frac{C_3C_6 b}{2C_4}\alpha^3\left[\left(1-\frac{y}{b}\right)^2U^{(0)}-\left(\frac{y}{b}\right)^2U^{(b)}\right] \\ -\left(1-\frac{y}{b}\right)\frac{\partial W^{(0)}}{\partial z}-\frac{y}{b}\frac{\partial W^{(b)}}{\partial z}+\left(C_1-\frac{C_3C_5}{C_5}\right)\alpha\left[\left(1-\frac{y}{b}\right)U^{(0)}+\frac{y}{b}U^{(b)}\right] \end{array} \right\} \tag{32}$$

and:

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{Bmatrix}=\begin{bmatrix}C_2\alpha & C_3\beta & -C_1\\ C_3\alpha & C_4\beta & -C_5\\ C_6\beta & C_6\alpha & 0\end{bmatrix}\begin{Bmatrix}U\\ V\\ Z\end{Bmatrix}+\left\{\begin{array}{c}\left(C_2-\frac{C_3^{\,2}}{C_4}\right)\alpha\left[\left(1-\frac{y}{b}\right)U^{(0)}+\frac{y}{b}U^{(b)}\right]\\ 0\\ \frac{-C_6}{b}\left[U^{(0)}-U^{(b)}\right]+C_6\,\frac{bC_3}{2C_4}\alpha^2\left[\left(1-\frac{y}{b}\right)^2U^{(0)}-\left(\frac{y}{b}\right)^2U^{(b)}\right]\end{array}\right\} \tag{33}$$

Fan (1996) in Symmetric Hinged-Free orthotropic thick plate solution set $\frac{\partial\sigma_{zz}}{\partial z}$ equal to just the simply support response of the plate, which is not true. In this paper, the equation for superposition part for Symmetric Clamped-Free condition provided by the author. However, Fan assumed this additional part for his Symmetric Hinged-Free Boundaries as zero. In both boundary conditions (i. e.Fan and current paper), the equation for $\frac{\partial\sigma_{zz}}{\partial z}$ derived in this paper, should be used.

In order to solve Eq. (24) , by using Eq. (27) and Eq. (32) we can expressed the each components of state vector {f}in terms of Fourier series expansion by introducing [5] :

$$\begin{aligned} \overline{U} &= \sum_m\sum_n \overline{U}_{mn}(z)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\ \overline{V} &= \sum_m\sum_n \overline{V}_{mn}(z)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\ Z &= \sum_m\sum_n \overline{Z}_{mn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\ X &= \sum_m\sum_n \overline{X}_{mn}(z)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\ Y &= \sum_m\sum_n \overline{Y}_{mn}(z)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\ \overline{W} &= \sum_m\sum_n \overline{W}_{mn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \end{aligned} \tag{34}$$

All six above equations are related to simply support thick plate conditions and can satisfy all boundaries condition that Fan introduced in 1996.

Other assumptions which we have to be introduced in Fourier series due to solving the Eq. (27) and (32) are related to f_u, f_v and f_w. On the other hand, all variables in these three parts should be written in Fourier series, as fallow:

(35)

$$U^{(0)} = U^{(0)}(x,z) = \sum_m U_m{}^{(0)}(z)\cos\frac{m\pi x}{a}$$

$$W^{(0)} = W^{(0)}(x,z) = \sum_m W_m{}^{(0)}(z)\sin\frac{m\pi x}{a}$$

$$U^{(b)} = U^{(b)}(x,z) = \sum_m U_m{}^{(b)}(z)\cos\frac{m\pi x}{a}$$

$$W^{(b)} = W^{(b)}(x,z) = \sum_m W_m{}^{(b)}(z)\sin\frac{m\pi x}{a}$$

(36)

$$\frac{y}{b} = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi}{n}\sin\frac{n\pi y}{b}$$

$$\left(\frac{y}{b}\right)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2}\cos\frac{n\pi y}{b}$$

$$1 = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n}\sin\frac{n\pi y}{b}$$

$$\left(1-\frac{y}{b}\right)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi y}{b}$$

(37)

$$\delta(x)P^{(0)}(y,z) = \left(\frac{1}{a} + \frac{2}{a}\sum_{m=1}^{\infty}\cos\frac{m\pi x}{a}\right)\sum_{n=1}^{\infty}P_n{}^{(0)}(z)\sin\frac{n\pi y}{b}$$

$$\delta(x-a)P^{(a)}(y,z) = \left(\frac{1}{a} + \frac{2}{a}\sum_{m=1}^{\infty}(-1)^m\,.\cos\frac{m\pi x}{a}\right)\sum_{n=1}^{\infty}P_n{}^{(a)}(z)\sin\frac{n\pi y}{b}$$

by using Eqs. (35-39) and Eq. (27) and (32) for each combination of *m* and *n*, it can be written as:

$$\frac{\partial}{\partial z}\{f\} = [D]\{f\} + \{B\} \tag{38}$$

(39)

$$\frac{d}{dz}[\bar{U}_{mn}(z) \quad \bar{V}_{mn}(z) \quad Z_{mn}(z) \quad X_{mn}(z) \quad Y_{mn}(z) \quad \bar{W}_{mn}(z)]^T = \begin{bmatrix} 0 & A_{mn} \\ B_{mn} & 0 \end{bmatrix}[\bar{U}_{mn}(z) \quad \bar{V}_{mn}(z) \quad Z_{mn}(z) \quad X_{mn}(z) \quad Y_{mn}(z) \quad \bar{W}_{mn}(z)]^T + B_{mn}(z)$$

where:

$$A_{mn} = \begin{bmatrix} C_8 & 0 & -\xi \\ 0 & C_9 & -\eta \\ \xi & \eta & 0 \end{bmatrix} \tag{40}$$

$$B_{mn} = \begin{bmatrix} C_2\xi^2 + C_6\eta^2 & (C_3 + C_6)\xi\eta & C_1\xi \\ (C_3 + C_6)\xi\eta & C_6\xi^2 + C_4\eta^2 & C_5\eta \\ -C_1\xi & -C_5\eta & C_{10} \end{bmatrix} \tag{41}$$

$$\underset{n\neq 0}{\overset{m=0}{}}\ \mathbf{B_{mn}(z)} = \begin{Bmatrix} \frac{2}{n\pi}\left[\cos n\pi \frac{d}{dz}U_m{}^{(b)}(z) - \frac{d}{dz}U_m{}^{(0)}(z)\right] \\ 0 \\ 0 \\ \frac{1}{a}\left[P_n{}^{(0)}(z) + P_n{}^{(a)}(z)\right] \\ 0 \\ \frac{2}{n\pi}\frac{d}{dz}\left[\cos n\pi\ W_m{}^{(b)}(z) - W_m{}^{(0)}(z)\right] \end{Bmatrix} \tag{42}$$

$$\underset{n=0}{\overset{m\neq 0}{}}\ \mathbf{B_{mn}(z)} = \begin{Bmatrix} 0 \\ -\frac{1}{b}\left[W_m{}^{(b)}(z) - W_m{}^{(0)}(z)\right] - \frac{C_3\xi b}{6C_4}\frac{d}{dz}\left[U_m{}^{(b)}(z) - U_m{}^{(0)}(z)\right] \\ 0 \\ 0 \\ \frac{C_6\xi}{b}\left[U_m{}^{(b)}(z) - U_m{}^{(0)}(z)\right] + \frac{C_3C_6b\,\xi^3}{6C_4}\left[U_m{}^{(b)}(z) - U_m{}^{(0)}(z)\right] \\ 0 \end{Bmatrix} \tag{43}$$

$$\underset{n\neq 0}{\overset{m\neq 0}{}}\ \mathbf{B_{mn}(z)} = \begin{Bmatrix} \frac{2}{n\pi}\left[\xi\cos n\pi\ W_m{}^{(b)}(z) - \xi W_m{}^{(0)}(z) + \cos n\pi \frac{d}{dz}U_m{}^{(b)}(z) - \frac{d}{dz}U_m{}^{(0)}(z)\right] \\ -\frac{2C_3\xi b}{C_4n^2\pi^2}\left[\cos n\pi \frac{d}{dz}U_m{}^{(b)}(z) - \frac{d}{dz}U_m{}^{(0)}(z)\right] \\ \left(\frac{1}{C_8} - \frac{C_3}{C_4C_9}\right)\left[\frac{2m}{na}\frac{d}{dz}U_m{}^{(0)}(z) - \frac{m}{a}\frac{2\ \cos n\pi}{n}\frac{d}{dz}U_m{}^{(b)}(z)\right] + \\ \frac{1}{C_8}\left[\frac{2m^2\pi}{n\ a^2}W_m{}^{(0)}(z) - \frac{2m^2\pi}{a^2}\frac{\cos n\pi}{n}W_m{}^{(b)}(z)\right] \\ \left[\frac{C_3(C_3+C_6)}{C_4} - C_2\right]\frac{2\xi^2}{n\pi}\left[\cos n\pi\ U_m{}^{(b)}(z) - U_m{}^{(0)}(z)\right] + \\ \frac{2}{a}\left[P_n{}^{(0)}(z) - \left((-1)^m P_n{}^{(a)}(z)\right)\right] \\ \frac{2\ C_3C_6b\,\xi^3}{C_4n^2\pi^2}\left[\cos n\pi\ U_m{}^{(b)}(z) - U_m{}^{(0)}(z)\right] \\ \frac{2}{n\pi}\frac{d}{dz}\left[\cos n\pi\ W_m{}^{(b)}(z) - W_m{}^{(0)}(z)\right] + 2\left(C_1 - \frac{C_3C_5}{C_4}\right)\frac{\xi}{n\pi}\left[\cos n\pi\ U_m{}^{(b)}(z) - U_m{}^{(0)}(z)\right] \end{Bmatrix} \tag{44}$$

$$\underset{n=0}{\overset{m=0}{}}\ B_{mn}(z) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T \tag{45}$$

Eq. (39) is called a variable coefficient nonhomogeneous state equation. If the solution of Eq. (39) is found and the boundary conditions are satisfied, all mechanical quantities can be determined from Eqs. (21), (25), (26), (33) and (34).

The six unknowns ${P_n}^{(0)}(z)$, ${P_n}^{(a)}(z), {U_m}^{(0)}(z)$, ${U_m}^{(b)}(z){W_m}^{(0)}(z)$ and ${W_m}^{(b)}(z)$ could be found by applying boundary conditions in table. 1.

Table 1.Boundary condition that should be satisfied

x=0	Need to Check?	x=a	Need to Check?	y=0	Need to Check?	y=0	Need to Check?
U=0	YES	U=0	YES	$\tau_{yz} = Y = 0$	YES	$\tau_{yz} = 0$ $Y = 0$	YES
V=0	Automatically Satisfied	V=0	Automatically Satisfied	$\tau_{xy} = 0$	YES	$\tau_{xy} = 0$	YES
W=0	Automatically Satisfied	W=0	Automatically Satisfied	σ_{yy}=0	Automatically Satisfied	$\sigma_{yy} = 0$	Automatically Satisfied

IV. Conclusion

Study of thick orthotropic plate shows that two-dimensional analysis based on CLPT plate assumptions can not be true for thick orthotropic plate with symmetric Clamped-Free boundary conditions[8]. As it is explained in this paper, the State Space equations show that all mechanical behaviors in thick orthotropic plate should be changed with the location. The equation for vertical displacement (w) is related to variable z. However, thickness effect was neglected in Kirchhoff two-dimensional plate analysis method.

In the derivation of the equation of state space (Eq. (11)) for $\frac{\partial \sigma_{zz}}{\partial z}$ component, Fan (1996) set it equal to the simply support response of the plate for Symmetric Hinged-Free orthotropic thick plate, which is not true. Fan assumed this additional part for his Symmetric Hinged-Free Boundaries as zero. In this paper, the equation for superposition part for symmetric Clamped-Free condition provided by the author. However, in both boundary conditions, Fan and current paper equation for $\frac{\partial \sigma_{zz}}{\partial z}$ should be the same.

Further numerical analysis based on Eqs.(32-33) and Eqs. (40-45) , which developed by the author in this paper, would help in identifying the unknowns and improving the usages of State Space exact solution in case of thick plate analysis.

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