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"Enigmatic **exhibition of each real number on several sets**"

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Abstract: *In the most primitive times our ancestors employed the principle of one-one correspondence for counting purposes. In those days when the shepherd came back in the evening from the grazing fields he used to count his sheep by pebbles (or sticks) in his bag and he was satisfied if he had as many sheep as pebbles in his bag. Later on different vocal sounds were developed as word tally against the number of obstacles and later on symbols were evolved to stand for these numbers. In this way it took many thousands of years to come to 1, 2, 3, 4, 5,…… which are called natural numbers "ℕ". The properties of natural numbers were developed from a few of simpler properties known as Peano's Axioms after the Italian Mathematician Guiseppe Peano(1858-1932) who gave these axioms in 1899.We thus find that the operations of addition and multiplication are fully defined in the set of natural numbers and then union , intersection in different sets. After a long struggle we have been able to form a well-decorated set ℝ, the set of real numbers. This ℝ contains all kind of numbers namely natural numbers, negative integers, zero, rational numbers and irrational numbers. There are infinitely many real numbers in ℝ. Every real number has a specific name. In my research work, I would like to convey the concerned readers that every real number takes a new name when we consider that real number on different sets.*
Key words: *Enigmatic, visualization, Neighbourhood, real numbers, belongs to, traditional, various, cases*

I. Introduction:

Increasing interest is being shown now-a-days by students for studying pure mathematics. One of the most important tools in pure mathematics is the number theory. My intention is to define and describe the different types of names of a real number for different types of sets and how it related to others. From some point of view it might be said that a real number may have one more names in one set. Eventually the purpose of my work is to prove, every real number takes different identifications with different names when we realize the presence of it on several sets.

II. Objectives:

Numerous discussions have been done on sets and real numbers and still it is being run. If we think in different ways regarding real numbers, we will look every real number shows different names instead of their traditional name. Here I have tried to explain this matter clearly on various cases. For this propose I have used the real number "3" every time. However it should bear our mind that this visualization is not only true for 3 but also for all real numbers.

III. Methodology:

The study has been conducted mainly on the basis of secondary information. For this purpose, different types of definitions, axioms or properties have been followed.

IV. Literature Review:

Natural Numbers:

The set of natural numbers is denoted by ℕ and

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\ldots\ldots\ldots\}$$

Having the following properties

i. $1 \in \mathbb{N}$

ii. $n \in \mathbb{N} \Rightarrow n+1 \in \mathbb{N}$

The set of natural numbers is also called the set of positive integers. It is denoted by

$$Z^{+} = \{1, 2, 3, 4, \ldots\ldots\ldots\ldots\}$$

Negative Integers:

The set of negative integers is denoted by Z^{-} and

$Z^{-} = \{\ldots\ldots\ldots, -3, -2, -1\}$

Zero: Zero is denoted by 'O'.
Remark: The above three types of numbers are called integers. The set of integers is denoted by $\mathbb{Z}$ and

$\mathbb{Z}$ = {………..-3,-2,-1, 0, 1, 2, 3,……….}

Rational Numbers: The number which can be expressed in the form $\frac{p}{q}$, where p,q$\in$ $\mathbb{Z}$,q$\neq$ 0 and (p,q) =1is called rational number. The set of rational numbers is denoted by $\mathbb{Q}$. Rational numbers are also called fractional numbers or quotient numbers.
Remark: Fractions and integers are called rational numbers.

Irrational Numbers: The number which cannot be expressed in the form $\frac{p}{q}$, where p,q$\in$ $\mathbb{Z}$,q$\neq$ 0 and (p,q) =1is called irrational numbers. The set of rational numbers is denoted by I.

Example: $\sqrt{2}$, $\sqrt{3}$, e, π etc.

Remark: The above five types of numbers are called real numbers. The set of real numbers is denoted by $\mathbb{R}$.
Whole Numbers: The set of all non-negative integers is called whole numbers and it is denoted by **W** and

W = (0, 1, 2, 3, 4,………..}

It is clear from the above discussion that

$$\mathbb{N} \subset \mathbf{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Prime Number: An integer which is greater than 1 and has no divisor except 1 and the number itself is called a prime number.2,3,5,7,……..are prime numbers.

Composite Number: An integer which is greater than 1 and has at least one divisor except 1 and the number itself is called composite number.4,6,8,9,10,……..are composite numbers.

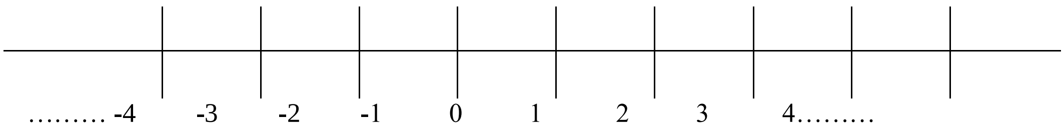
Even Number: The number of the form 2n, where n$\in$ $\mathbb{N}$ is called even number. The set of even numbers is denoted by E and

E = {2, 4, 6, 8,……..}

Odd Number:
The number of the form 2n-1, where n$\in$ $\mathbb{N}$ is called odd number. The set of odd numbers is denoted by O and

O = {1, 3, 5, 7,……..}

Real Line: All real numbers are considered to be arranged on a straight line which is called real line.



Set: A well-defined collection is called a set. A set is represented by second bracket and elements are represented by terns with commas.
Example:

A = {1, 2, 5, 6, 10} is a set.

B = {Pen, Book, 5} is a set.

X = {x: x is the number of stars in the sky} is not a set. As we are not confirm about the exact number of stars in the sky.

W = {x: x is the stars in the sky} is a set.

Note: Sets are represented by block letters namely A,B,C,X etc. and elements are represented by small letters namely a ,b ,c ,x etc. If x is a member of a set A, then we write x$\in$A and if not then x$\notin$A. The notation "$\in$" is known as belongs to.

Subset: Suppose A and B are two sets. If every element of B belongs to A, then B is called subset of A and denoted by B $\subset$ A. The notation "$\subset$" is called set inclusion. In this case A is called super set of B.

Example: If A = {1,2,3}, then B= {2,3} ,C = {3}are subsets of A.

Open and Closed Interval: If a,b $\in \mathbb{R}$, then {x $\in \mathbb{R}$: a < x < b} is called an open interval and it is denoted by $]a\,,b[$ and {x $\in \mathbb{R}$: a $\leq$ x $\leq$ b} is called closed interval and it is denoted by $[a, b]$. It is noted that, if a = b then $[a\,,b]$ = $[a\,,a]$ = {a} and $]a\,,b[\; = \;]a, a[$ = φ.(Real Analysis, P.N. Chatterji,page-52[2000])

Upper bound: Suppose

$S \subset \mathbb{R}$

Any u $\in \mathbb{R}$ is called upper bound of S if

$s \leq u\,;\ \forall s \in S$

e.g. (i) Let S = $[1,5] = \{x : 1 \leq x \leq 5\} \subset \mathbb{R}$

Here 5 is an upper bound of S

11.5 is an upper bound of S

4 is not an upper bound of S as 4.5 $\in$ S and 4 <4.5.

(ii) Let S = $\{1,2,3,4,5,6,7,8\} \subset \mathbb{R}$

Here 8 is an upper bound of S

100.5 is an upper bound of S

5 is not an upper bound of S as 7 $\in$ S and 5 < 7.

Lower bound: Suppose

$S \subset \mathbb{R}$

Any x $\in$ R is called lower bound of S if

$x \leq s\,;\ \forall s \in S$

e.g. (i) Let S = $[1,5] = \{x : 1 \leq x \leq 5\} \subset \mathbb{R}$

Here 1 is a lower bound of S

0 is a lower bound of S

4 is not a lower bound of S as 2 $\in$ S and 2 <4.

(ii) Let S = $\{1,2,3,4,5,6,7,8\} \subset \mathbb{R}$

Here 1 is a lower bound of S

-5 is a lower bound of S

5 is not an upper bound of S as 2 $\in$ S and 2 < 5.

Bounded above set: Suppose

$S \subset \mathbb{R}$

S is called bounded above set if it has upper bound, i.e

$\exists x \in \mathbb{R}$ such that $s \leq x\,;\ \forall s \in S$

e.g. S = $[2,5]$ is a bounded above set.

Bounded below set: Suppose

$S \subset \mathbb{R}$

S is called bounded below set if it has lower bound, i.e

$\exists x \in \mathbb{R}$ such that $x \leq s\,;\ \forall s \in S$

e.g. S = $[3,8[$ is a bounded below set.

Supremum of a set: Suppose

$S \subset \mathbb{R}$

Any r $\in \mathbb{R}$ is called "Supremum of S" written "sup S" if

i. r is an ub of S [i.e. $s \leq r\,;\ \forall s \in S$]

ii. For some $\varepsilon > 0, \exists s_1 \in S$ s.t

$r - \varepsilon < s_1$ (Real Analysis, P.N. Chatterji,page-34[2000])

Note: Supremum of a set is the least upper bound of that set.

e.g. Let S = $[4,8]$.

sup S = 8

Infimum of a set: Suppose

$S \subset \mathbb{R}$

Any x $\in \mathbb{R}$ is called "Infimum of S" written "inf S" if

i. x is a lb of S [i.e. $x \leq s\,;\ \forall s \in S$]

ii For some $\varepsilon > 0, \exists s_1 \in S$ s.t

$s_1 < x + \varepsilon$ (Real Analysis, P.N. Chatterji,page-34[2000])

Note: Infimum of a set is the greatest lower bound of that set.

e.g. Let S = $[0,5[$.

inf S = 0

Bounded set: Suppose

$S \subset \mathbb{R}$

S is called bounded set if $\exists\, a \in \mathbb{R}$ s.t

$|x| \le a,\ \forall\, x \in S$ [i,e $-a \le x \le a,\ \forall\, x \in S$]

On the other hand, a set is called bounded set if it has both upper bound and lower bound.

e.g. i. S = $[4,8]$ is a bounded set.

ii. A = {-1, 0 4, 8, 10, 15} is a bounded set.

iii. B = $[0,\infty[$ is not bounded set.

iv. ℝ, ℕ, ℤ etc. are not bounded sets.

v. D = $]-5,2[\ \subset \mathbb{R}$ is a bounded set since

$|x| \le 5, \forall\, x \in S$ [i,e $-5 \le x \le 5,\ \forall\, x \in S$]

Unbounded set: Suppose

$S \subset \mathbb{R}$

S is called unbounded if it is not bounded.

e.g.

i. A = $[0,\infty[$ is an unbounded set.

ii. ℝ, ℕ, ℤ are unbounded sets.

Interior Point: A point x ∈ ℝ is called an interior point of a set S if S is nbd of x.

i,e if $\exists\, \varepsilon > 0$ s.t $]x-\varepsilon, x+\varepsilon[\, \subset S$ or $N_\varepsilon(x) \subset S$.

The set of all interior point of a set S is called "interior of S" and denoted by "intS" or S^i.

(Real Analysis, B.K. Lahari and K.C. Roy, page-47[1988])

e.g If S = [2 , 6] , then intS = $]2,6[$

Exterior Point: A point x ∈ ℝ is called an exterior point of a set S if ∃nbd N of x s.t

$N \cap S = \varphi$.

The set of all exterior point of a set S is called "exterior of S" and denoted by "extS" .

e.g If S = [2 , 6] , then extS = $]-\infty, 2[\ \cup\ ]6,\infty[$

Boundary Point: A point x ∈ ℝ is called is called a boundary point of a set S if it has neither an interior point nor an exterior point of S .Set of boundary point of S is called "boundary of S".

e.g 1. If S = [2 , 6] , then 2 and 6 are boundary points of S.

2. If S = {1 , 2, 3, 4, 5, 6} , then each point of S are boundary point.

Neighbourhood of a real number: Suppose x ∈ ℝ. Then N ⊂ ℝ is called "Neighbourhood of x" if $\exists\, \varepsilon > 0$ s.t

$$]x-\varepsilon, x+\varepsilon[\, \subset N$$

"Neighbourhood of x" is symbolically written as "nbd of x". (Real Analysis, P.N. Chatterji,page-59[2000])

Open Set: Suppose G ⊂ ℝ . G is called open set iff ∀ x ∈ G , ∃nbd N of x s.t

x ∈ N ⊂ G. (Real Analysis, B.K. Lahari and K.C. Roy, page-47[1988])

i. G = $]2,5[$ is an open set.

ii. S = [2 , 5] is not an open set.

iii. A = {3, 4, 5, 6, 7} is not an open set.

Adherent or Closure point of a set: A point x ∈ ℝ is called an adherent point of a set S if every nbd of x contains at least one point of S.(P.N. Chatterji,page-68[2000])

e.g 1. Let S = {2, 3, 4, 5},then 3 is an adherent point of a set S. But 3 is not limit point of S.

2. Let S = [1 , 3], then 2 is an adherent point of a set S. Also 2 is a limit point of S.

Limit point of a set: A point x ∈ ℝ is called an limit point of a set S if ∃nbd N of x s.t

$N \cap S$ = Infinite set.(P.N. Chatterji,page-68[2000])e.g

Let S = [1 , 3], then 2 is a limit point of S.

Cartesian product of two set: Suppose A add B be two sets. The Cartesian product of A and B is denoted by A×B and defined as follows:

$$A\times B = \{(a,b) : a\in A , b\in B\}$$

e,g Let A={4,5} and B= {1,2,3}

$\therefore$ A×B= {(4,1),(4,2),(4,3),(5,1),(5,2),(5,3)}

Relation: Suppose A and B be two sets. A relation "R" from A to B is a subset of A×B.(Lattices and Boolean algebras by V.K Khanna,page-3).

e,g From the immediate previous example we get

R = {(a,b) : a$\in$ A , b$\in$ B and a = b+2}
= {(4,2),(5,3)} is a relation.

H = {(a,b) : a$\in$ A , b$\in$ B and a=b^2}
= {(4,2)} is a relation.

S = {(a,b) : a$\in$ A , b$\in$ B and b< a}
= {(4,1),(4,2),(4,3),(5,1),(5,2),(5,3) is a relation.

T = {(a,b) : a$\in$ A , b$\in$ B and a<b}
= $\emptyset$ is a relation.[As empty set is a subset of all sets]

Binary relation: Suppose A is a non-empty set. A binary relation "R" is a subset of A×A.

Let R be a binary relation in A. Then R is said to be

i. Reflexive if $\forall$a$\in$A ;
aRa
ii. Symmetric if $\forall$a , b$\in$A ;
aRb $\Longrightarrow$ bRa
iii. Anti-symmetric if $\forall$a , b$\in$A ;
aRb and bRa $\Longrightarrow$ a=b
iv. Transitive if $\forall$a , b , c$\in$A ;
aRb and bRc $\Longrightarrow$ aRc

A binary relation R is called partially ordered relation if it is Reflexive, Anti-symmetric and Transitive.
A binary relation R is called equivalence relation if it is Reflexive, symmetric and Transitive. .(Lattices and Boolean algebras by V.K Khanna,page-3).

Poset: A non-empty set P together with a binary relation R is said to be a poset if R is Reflexive, Anti-symmetric and Transitive. Usually we represent a poset by (P, R).We generally use the symbol "$\leq$" in place of R. (Lattices and Boolean algebras by V.K Khanna, page-11).

e,g

i. (P= {2, 3, 4, 6}, $|$) is a poset, where the relation "$|$" indicates divisibility.
ii. (P(X), $\subseteq$) is a poset, where X= {1,2,3} and the relation "$\subseteq$" indicate set inclusion.

Chain: A poset (P, $\leq$) is said to be a chain iff every two elements of P are comparable.i,e

$\forall$ a, b $\in$L ; a $\leq$ b or b $\leq$ a(Lattice Theory, George Grätzer,page-2, 1970)

e,g

i. (P= {1,2, 4, 8}, $|$) is a chain, where the relation "$|$" indicates divisibility.
ii. (P= {1,2, 3, 4, 6,12}, $|$) is not a chain, where the relation "$|$" indicates divisibility as 2 and 3 are not comparable.

Lattice: A poset (L, $\leq$) is said to be a lattice iff $\forall$ a, b $\in$L

a$\wedge$b = inf {a,b} $\in$ L
a$\vee$b = sup {a,b} $\in$ L.(Introduction to Lattice Theory , D.E. Rutherford,page-3, 1965) (Lattice Theory, George Grätzer,page-3, 1970.

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Note: The symbols $\wedge$ *and* $\vee$ are called **meet** and **join** respectively.

e,g

i. (P= {1,2, 3, 4, 6,12}, $|$) is a lattice , where the relation "$|$" indicates divisibility.
ii. (P(X), $\subseteq$) is a lattice, where X= {1,2,3} and the relation "$\subseteq$" indicate set inclusion.

iii. (P= {1, 2, 3},≤) is a lattice, where the relation "≤" is usual.

Maximal Element of a Poset: Suppose (P,≤) is a poset.Any x∈P is called maximal element of P if x < p, ∃ no p∈P. (Lattices and Boolean algebras by V.K Khanna, page-14)

Minimal Element of a Poset: Suppose (P,≤) is a poset.Any x∈P is called minimal element of P if p < x, ∃ no p∈P. (Lattices and Boolean algebras by V.K Khanna, page-15)

Maximum Element of a Lattice: Suppose (L,≤) is a lattice. Any m∈L is called maximum element of L iff ∀x ∈L;

$$x \leq m$$

Minimum Element of a Lattice: Suppose (L,≤) is a lattice. Any m∈L is called minimum element of L iff ∀x ∈L;

$$m \leq x$$

Cover of an Element in a Lattice: Suppose (L,≤) is a lattice and a and b be two elements of L. Then we say "a covers b" iff ∃ no c∈L s.t b < c < a. .(Lattices and Boolean algebras by V.K Khanna,page-52) (Lattice Theory, George Grätzer,page-12, 1970).

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Atom: Suppose (L,≤) is a lattice. An element x of L is said to be an atom of L if it covers the minimum element of L. .(Lattices and Boolean algebras by V.K Khanna,page-55) (Lattice Theory, George Grätzer,page-59, 1970).

Dual Atom: Suppose (L,≤) is a lattice. An element x of L is said to be an dual atom of L if the maximum element of L covers x. (Lattices and Boolean algebras by V.K Khanna,page-55) (Lattice Theory, George Grätzer,page-60, 197

V. Discussion:

Now we take the real number 3 .It is a natural number .Also it is a prime number and odd number. Consider the set S = $[2,5]$ of S .Clearly 3∈S.

Case-1: If we discuss the definition of interior point of a set viz any real number m is said to be an interior point of a set A iff A itself is a nbd of m i,e $\exists\ \varepsilon > 0$ such that $]m-\varepsilon, m+\varepsilon[\ \subset$ A, then 3 is an interior point of S as 3 satisfies the above definition on S.

Case-2: Again if we consider the set T = [4, 6], then it is clear that ***3∉T***.

In this case 3 is an exterior point of T, as ∃a nbd N of 3 s.t **N∩T= ∅.**

Case-3: Now we are going to consider a finite set namely S = {1, 2, 3, 4, 5}.Here 3 itself is an element of S. Obviously 3 is a boundary point of S. Because if we see the above two definitions, 3 will neither be interior point of S nor exterior point of S i,e S is not nbd of S and also ∃no nbd N of 3 s.t **N∩S** = ∅.

Case-4: Again, Consider S = [2,4]⊂ ℝ.Clearly 3∈S and 3 is a limit point of S.As ∃a nbd N of 3 s.t N∩S is an infinite set.

On the other hand, if S =]3 5[⊂ ℝ.Here 3∉S but 3 is a limit6 point of S. As ∃a nbd N of 3 s.t N∩S is an infinite set.

From the above two experiment it is clear that, a limit point of a set may be or may not be the member of that set.

Case-5: Let us consider my first example S= [2,5].Here 3 is an adherent(Closure)point of S.As every nbd N of 3contains at least one point of S.

Observing the limit point and an adherent point of a set we conclude that, every limit point of a set is also an adherent point of that set but the converse is not true.

We shall be clear the above two statements by the following two examples:

Let S = {2, 3, 4, 6}

Here 3 is an adherent point of S. But 3 is not a limit point of S.

Again, Let T= [3,8]

Here 3 is a limit point of T. Also 3 is an adherent point of T.

Case-6: Consider S= [0,3].Here 3 is supremum of S.

As i. 3 is an ub of S [i.e. $s \leq 3$; $\forall s \in S$]

ii. For some $\varepsilon > 0, \exists s_1 \in S$ s.t

$$3-\varepsilon < s_1$$

Case-7: If we take S = [3,6],then 3 is infimum of S.

As i. 3 is a lb of S [i.e. $3 \leq s$; $\forall s \in S$]

ii For some $\varepsilon > 0, \exists s_1 \in S$ s.t

$$s_1 < 3 + \varepsilon$$

Case-8: Consider (P= {2, 3, 4}, |) is a poset.

Here 3 is a maximal element of P.As ∃no x∈P s.t 3 < x. Note that in this case 3 is not maximum element of P.

Case-9: Let (P= {2, 3, 4, 6}, |) is a poset.

Here 3 is a minimal element of P.As ∃ no x∈P s.t x < 3. Note that in this case 3 is not minimum element of P.

Case-10: Consider (P={1,2,3},≤)be a poset,where ≤ is usual.

Here 3 is maximum element or greatest element of P.As ∀x∈P; x ≤ 3.Note that in this case 3 is maximal element of P also.

Case-11: Consider (P={3,4,5,6,7,8,9,10},≤)be a poset,where ≤ is usual.

Here 3 is minimum element or least element of P.As ∀x∈P; 3 ≤ x.Note that in this case 3 is minimal element of P also.

From the above four definitions it can be clear that, every maximum element of a poset must be a maximal element of that poset and the converse is not true and every minimum element of a poset must be a minimal element of that poset and the converse is not true.

Case-12: Consider (P= {1, 2, 3, 4, 6, 12}, |) is a lattice. Here 3 is cover of 2 because ∃no k∈P s.t 2 < k < 3

Case-13: Consider (L= {1, 2, 3, 4, 6, 12}, |) is a lattice. Here 1 is the least element of L as ∀x∈L; 1 ≤ x and 3 is a cover of 1 as ∃no m∈L s.t 1 < m < 3.By the definition of atom we can say 3 is an atom of L.

Case-14: Consider (L= {1, 2, 3, 6}, |) is a lattice. Here 6 is the maximum element of L as ∀x∈L; x ≤ 6 and 6 covers 3 as ∃no n∈L s.t 3 < n < 6.By the definition of dual atom we can say 3 is dual atom of L.

Case-15: Let S = $\{2 + (-1)^n\}$, where n∈N be a sequence .Then 3 is a limit point of the sequence S.

VI. Conclusion and Findings:

At the eleventh our of my research work, it can be seen that, “3” can be defined and described the different types of names for different types of sets. Further it might be said that “3” has taken one more names in one set. The above discussion is not true only for 3.It is a fact that, every real number plays the same characteristics on several well-defined collections. Ultimately it is shown that, every real number takes different identifications with different names when we realize the presence of it on several sets.

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