IOSR Journal of Mathematics (IOSRJM)
ISSN: 2278-5728 Volume 1, Issue 5 (July-Aug 2012), PP 05-12
www.iosrjournals.org

Fractional Integrals Involving Generalized Polynomials And Multivariable Function

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***Abstract**:Our aim of this paper is to find a Eulerian Integral and a main theorem based on the fractional operator associated with generalized polynomial and a multivariable I-function having general arguments. The theorem provides extension of various results. Some special cases are also given.*
***Keywords-**Fractionalintegral,Eulerian integral,multivariable I-function,Riemann-Liouville operator,Lauricella function.*

I. Introduction

The Riemann-Liouville operator of fractional integration $R^m f$ of order m is defined by

$$xD_y^{-m}[f(y)] = \frac{1}{\Gamma(m)} \int_x^y (y-t)^{m-1}.f(t).dt \qquad \ldots (1)$$

An equivalent form of Beta function is

$$\int_m^n (t-m)^{a-1}.(n-t)^{b-1}.dt = (n-m)^{a+b-1} B(a,b) \qquad \ldots (2)$$

Where m,n ∈ R (x<y),Re (a)>0 , Re(b)>0
To prove the Eulerian integrals, we use the following formula.

$$\int_x^y (t-x)^{a-1}.(y-t)^{b-1}.(p_1 t+q_1)^{\rho_1} \ldots.(p_h t+q_h)^{\rho_h}\, dt = (y-x)^{a+b-1}.B(a,b).(p_1 x+q_1)^{e_1} \ldots\ldots (p_h x + qh)eh$$
$$\times F_D^{(h)}\left[a,-\rho_1 \ldots.-\rho_h\,; a+b\,, \frac{-(y-x)\rho_1}{p_1 x+q_1} \ldots\ldots \frac{-(y-x)\rho_h}{p_h x+q_h}\right] \qquad \ldots (3)$$

Where $F_D^{(h)}$ is Lauricella function

Where x , y ∈ R (x<y) ; $p_j\,, q_j\ \rho_j\ \epsilon\ C\ (j = 1 \ldots\ldots h)$
$\min[R_e(a), R_e(b)] > 0\ and\ \max\left[\left|\frac{(y-x)\rho_1}{P_1 x+q_1}\right|, \ldots\ldots \left|\frac{(y-x)\rho_h}{P_h x+q_{1h}}\right|\right] <1$

The integral representation of $F_D^{(h)}$ is defined as

$$\frac{\Gamma(a).\Gamma(b_1)\ldots\ldots\ldots\Gamma(b_h)}{\Gamma(c)}.F_D^{(h)}[a, b_1 \ldots\ldots\ldots b_h, c, x_1 \ldots\ldots x_h] =$$
$$\frac{1}{(2\pi i)^h}\int_{-i\infty}^{i\infty} \ldots h \ldots \int_{-i\infty}^{i\infty} \frac{\Gamma(a+\xi_1+\cdots\xi_h).\Gamma(b+\xi_1)\ldots\ldots\ldots\Gamma(b+\xi_h)}{\Gamma(c+\xi_1+\cdots\xi_h)}, \Gamma(-\xi_1)\ldots.\Gamma(-\xi_h) \times$$
$$(-x_1)^{\xi_1}\ldots\ldots(-x_h)^{\xi_h}\, d\xi_1 \ldots.d\xi_h$$

Now $(p_1 t+q)^\alpha = (xp+q)^\alpha.\left[1+\frac{p(t-x)}{xp+q}\right]^\alpha = (xp+q)^\alpha.2F_1\left[-\alpha, 1,1,\frac{-p(t-x)}{xp+q}\right]$

Since $\frac{\Gamma(A).\Gamma(B)}{\Gamma(c)}.2F_1[A,B,C;Z] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(A+\varsigma).\Gamma(B+\varsigma)}{\Gamma(c+\varsigma)}\Gamma(-\varsigma).(-Z)^{\varsigma}\, d\varsigma$

$\therefore, (pt+q)^\alpha =$

$$\frac{(xp+q)^\alpha}{\Gamma(-\alpha)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(-\varsigma).\Gamma(\varsigma-\alpha).\left\{\frac{p(t-x)}{xp+q}\right\}^{\varsigma} d\varsigma \qquad \ldots (5)$$

$p, q, \alpha\epsilon C,\ x, t\epsilon R$ and $\left|\arg\left(\frac{p}{xp+q}\right)\right| < \pi$ and path of integration is necessary in such a manner so as to separate the poles of $\Gamma(-\varsigma)$ from those if $\Gamma(\varsigma-\alpha)$.
Formula (3) , can be provided with the help of (2) , (4) and (5)
The generalized polynomial defined by Shrivastava[8] is as follows

$$S_{N_1\ldots\ldots N_k}^{M_1\ldots\ldots M_k}[x_1\ldots\ldots\ldots x_k] = \sum_{\alpha_1=0}^{[N_1/M_1]}\ldots\ldots\ldots\sum_{\alpha_k=0}^{[N_k/M_k]}\frac{(-N_1)\,M_1\alpha_1}{\llcorner\alpha_1}\ldots\frac{(-N_k)\,M_k\alpha_k}{\llcorner\alpha_k}\,B[N_1,\alpha_1,\ldots\ldots,N_k,\alpha_k]\,x_1^{\alpha_1}\ldots\ldots x_k^{\alpha_k} \quad \ldots(6)$$

Where, $N_i = 0,1,2,\ldots\ldots\ \forall\ i=(1\ldots\ldots\ldots k), M_1\ldots\ldots M_k$ are arbitrary positive integers and the coefficients $B[N_1,\alpha_1,\ldots\ldots,N_k,\alpha_k]$ are arbitrary constants.
Here h is a positive integer and 0……… 0 would mean h zeros.

II. MULTI-VARIABLE I-FUNCTION

It is defined and represented in the following manner:-

$$I[z_1\ldots\ldots z_1] = I^{0,n_2:0,n_3,\ldots\ldots,0,n_r,m^1,n^1\ldots\ldots m^{(r)},n^{(r)}}_{p_2,q_2:p_3,q_3,\ldots\ldots,p_r,q_r,p^1,q^1\ldots\ldots p^{(r)},q^{(r)}}\left[\begin{matrix}Z_1\\ \cdot\\ Z_r\end{matrix}\ \middle|\ \begin{matrix}(a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2}:(a_{3j};\alpha'_{3j},\alpha''_{3j})_{1,p_3}\ldots\ldots(a_{rj};\alpha'_{rj},\alpha''_{rj})_{1,p_r},(a'_j,\alpha'_j)_{1,p'},\ldots\ldots(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}}\\ (b_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2}:(\beta_{3j};\beta'_{3j},\beta''_{3j})_{1,q_3}\ldots\ldots(\beta_{rj};\beta'_{rj},\beta''_{rj})_{1,q_r},(b'_j,\beta'_j)_{1,q'},\ldots\ldots(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}}\end{matrix}\right]$$

$$= \left(\frac{1}{2\pi\omega}\right)^r \cdot \int_{L_1}\ldots\ldots\int_{L_1}\phi(\xi_1)\ldots\ldots\phi(\xi_r).\Psi(\xi_1\ldots\ldots\xi_r).\,z_1^{\xi_1}\ldots\ldots z_r^{\xi_r}\,d\xi_1\ldots\ldots d\xi_r \quad \ldots(7)$$

Where , $\phi_i(\xi_i) = \dfrac{\prod_{j=1}^{m_i}\Gamma(b_j^{(i)}-\beta_j^{(i)}\xi_i)\prod_{j=1}^{n_i}\Gamma(1-a_j^{(i)}+\alpha_j^{(i)}\xi_i)}{\prod_{j=m^{(i)}+1}^{q_i}\Gamma(1-b_j^{(i)}+\beta_j^{(i)}\xi_i)\prod_{j=n^{(i)}+1}^{p^{(i)}}\Gamma(a_j^{(i)}-\alpha_j^{(i)}\xi_i)}$ and

$$\psi(\xi_1\ldots\xi_r) = \frac{\prod_{j=1}^{n_2}\Gamma(1-a_{2j}+\sum_{i=1}^{2}\alpha_{2j}^{(i)}\xi_i)\prod_{j=1}^{n_3}\Gamma(1-a_{3j}+\sum_{i=1}^{3}\alpha_{3j}^{(i)}\xi_i)\ldots\prod_{j=1}^{n_r}\Gamma(1-a_{rj}+\sum_{i=1}^{r}\alpha_{rj}^{(i)}\xi_i)}{\prod_{j=n_2+1}^{p_2}\Gamma(a_{2j}-\sum_{i=1}^{2}\alpha_{2j}^{(i)}\xi_i)\prod_{j=n_3+1}^{p_3}\Gamma(a_{3j}-\sum_{i=1}^{3}\alpha_{3j}^{(i)}\xi_i)\ldots\prod_{j=n_r+1}^{p_r}\Gamma(a_{rj}-\sum_{i=1}^{r}\alpha_{rj}^{(i)}\xi_i)}$$

$$\times\frac{1}{\prod_{j=1}^{q_2}\Gamma(1-b_{2j}+\sum_{i=1}^{2}\beta_{2j}^{(i)}\xi_i)\ldots\prod_{j=1}^{q_r}\Gamma(1-b_{rj}+\sum_{i=1}^{r}\beta_{rj}^{(i)}\xi_i)}$$

The convergence and other details of multivariable I-function, see Prasad[4].

II Main Integral

The main Integral to be established here is

$$\int_m^n (t-m)^{a-1}(n-t)^{b-1}.\left\{\prod_{j=1}^{h}(p_j+q_j)^{\rho_j}\right\}$$

$$S_{N_1\ldots N_k}^{M_1\ldots M_k}\begin{bmatrix}x_1(t-m)^{\lambda_1}(n-t)^{\mu_1}.\prod_{j=1}^{h}(p_jt+q_j)^{\sigma_j^1}\\ \cdot\\ \cdot\\ \cdot\\ x_k(t-m)^{\lambda_k}(n-t)^{\mu_k}.\prod_{j=1}^{h}(p_jt+q_j)^{\sigma_j^k}\end{bmatrix}.I\begin{bmatrix}z_1(t-m)^{\gamma_1}(n-t)^{\tau_1}.\prod_{j=1}^{h}(p_jt+q_j)^{-c_j^1}\\ \cdot\\ \cdot\\ \cdot\\ z_r(t-m)^{\gamma_r}(n-t)^{\tau_r}.\prod_{j=1}^{h}(p_jt+q_j)^{-c_j^r}\end{bmatrix}dt =$$

$$W_1\sum_{\alpha_1=0}^{\left[\frac{N_1}{M_1}\right]}\ldots\ldots\sum_{\alpha_k=0}^{\left[\frac{N_k}{M_k}\right]}\frac{(-N_1)_{M_1\alpha_1}}{!\alpha_1}\ldots\ldots\frac{(-N_k)_{M_k\alpha_k}}{!\alpha_k}.$$

$$B[N_1,\alpha_1,\dots,N_k,\alpha_k].\,x_1{}^{\alpha_1}\dots x_k{}^{\alpha_k}.\,W_2\ \times I^{0,n_2:0,n_3\dots 0,n_r+h+2:(m^1,n^1)\dots(m^{(r)},n^{(r)}),(1,0)\dots(1,0)}_{p_2,q_2:p_3,q_3\dots p_r+h+2,q_r+h+1:(p^1,q^1)\dots(p^{(r)},q^{(r)}),(0,1)\dots(0,1)}$$

$$\left[\begin{matrix}R_1\\ \\ R_2\end{matrix}\left|\begin{matrix}X_1,X_2,X_3,\left(a_{2j}\,\alpha'_{2j},\alpha''_{2j}\right)_{1,p_2},\left(a_{3j}\,\alpha'_{3j},\alpha''_{3j},\alpha'''_{3j}\right)_{1,p_3},\dots\left(a_{rj},\alpha'_{rj}\dots\dots,\alpha^{(r)}_{rj},0,\dots\dots0\right)_{1,p_r},\left(a'_j,\alpha'_j\right)_{1,p^1}\dots\left(a^{(r)}_j,\alpha^{(r)}_j\right)_{1,p^{(r)}};--;--;--\\ \\ \left(b_{2j}\,\beta'_{2j},\beta''_{2j}\right)_{1,q_2},\left(b_{3j}\,\beta'_{3j},\beta''_{3j},\beta'''_{3j}\right)_{1,q_3},\dots\left(b_{rj},\beta'_{rj}\dots,\beta^{(r)}_{rj},0,\dots0\right)_{1,q_r}\ X_4,X_5,(b'_j,\beta'_j)_{1,q^1}\dots\left(b^{(r)}_j,\beta^{(r)}_j\right)_{1,q^{(r)}},(0,1)\dots(0,1)\end{matrix}\right|\right]$$

……………….(8)

Where $W_1\ =(n-m)^{a+b-1}\left\{\prod_{j=1}^{h}(p_j m+q_j)^{p_j}\right\}$

$$W_2=\ (n-m)^{\sum_{i=1}^{k}(\lambda_i+\mu_i)\alpha_i}\quad .\left\{\prod_{j=1}^{h}(p_j m+q_j)^{\sum_{i=1}^{k}\sigma_j^{(i)}.\alpha_i}\right\}$$

$$X_1\ =[1-a-\textstyle\sum_{i=1}^{k}\ \lambda_i.\alpha_i\ \ ;\ \gamma_i\ \dots\dots\gamma_{r\ ,1\dots\dots1}]$$

$$X_2=\left[1-b-\sum_{i=1}^{k}\ \mu_i.\alpha_i\ \ ;\ \tau_1\dots\dots\tau_{r\ ,0\dots\dots0}\right]$$

$$X_3=\left[1+\rho_j+\sum_{i=1}^{k}\ \sigma_j^{(i)}.\alpha_i\ \ ;\ C_j^1\dots\dots C_j^{(r)},0\dots\dots\overset{j}{1}\ \dots\dots.0\right]_{1,h}$$

$$X_4=\left[1+\rho_j+\sum_{i=1}^{k}\ \sigma_j^{(i)}.\alpha_i\ \ ;\ C_j^1\dots\dots C_j^{(r)},0\dots\dots\dots\dots.0\right]_{1,h}$$

$$X_5=[1-a-b-\textstyle\sum_{i=1}^{k}\ (\lambda_i+\mu_i).\alpha_i\ \ ;\ \gamma_1+\ \tau_1\dots\dots,1\dots\dots.\gamma_r+\ \tau_r,1\dots.1]$$

$$R_1=\begin{cases}Z_1(n-m)^{\gamma_1+\tau_1}\Big/\prod_{j=1}^{h}(p_j m+q_j)^{c_j^1}\\ \vdots\\ Z_r(n-m)^{\gamma_r+\tau_r}\Big/\prod_{j=1}^{h}(p_j m+q_j)^{c_j^{(r)}}\end{cases}$$

$$R_2=\begin{cases}\dfrac{(n-m)p_1}{p_1 m+q_1}\\ \vdots\\ \dfrac{(n-m)p_h}{p_h m+q_h}\end{cases}$$

The following are the conditions of validity of integral (8)

1) $m,n\,\epsilon\,R\ (m<n),\ \gamma_i,\ \tau_i\,;c_j^{(i)},\lambda_i,\mu_i,\sigma_j^{(i)}\epsilon\,R^+,\rho_j\epsilon R\ \ p_j,q_j\,\epsilon\,C\ ,z_i\epsilon C\ (i=1\dots\dots r,j=1\dots..h)$

2) $\max_{1\le j\le h}\left[\ \left|\dfrac{(n-m)p_j}{p_j m+q_j}\right|\right]<1$

3) $R_e\left[a+\sum_{i=1}^{r}\gamma_i\ \dfrac{b_j^{(i)}}{B_j^{(i)}}\right]>0\ ,j=1\dots.m^{(i)},R_e\left[b+\sum_{i=1}^{r}\ \tau_i\ \dfrac{b_j^{(i)}}{B_j^{(i)}}\right]>0\ ,j=1\dots.m^{(i)}$

4) $\left|\arg(z_i)\prod_{j=1}^{h}(p_j+q_j)^{-c_j^i}\right| < \frac{T_i\pi}{2}\ (m \le t \le n\,, i = 1 \ldots\ldots r)$

Where

$$T_i = \sum_{j=1}^{n^{(i)}}\alpha_j^{(i)} - \sum_{j=n^{(i)}+1}^{p^{(i)}}\alpha_j^{(i)} + \sum_{j=1}^{m^{(i)}}\beta_j^{(i)} - \sum_{j=m^{(i)}+1}^{q^{(i)}}\beta_j^{(i)} + \left(\sum_{j=1}^{n_2}\alpha_{2_j}^{(i)} - \sum_{j=n_2+1}^{p_2}\alpha_{2_j}^{(i)}\right) + \left(\sum_{j=1}^{n_3}\alpha_{3_j}^{(i)} - \sum_{j=n_3+1}^{p_3}\alpha_{3_j}^{(i)}\right) + \cdots + \left(\sum_{j=1}^{n_r}\alpha_{r_j}^{(i)} - \sum_{j=n_r+1}^{p_r}\alpha_{r_j}^{(i)}\right) + \left(\sum_{j=1}^{q_2}\beta_{2_j}^{(i)} + \sum_{j=1}^{q_3}\beta_{3_j}^{(i)} + \cdots + \sum_{j=1}^{q_r}\beta_{r_j}^{(i)}\right)$$

III. PROOF

In order to prove integral (8),expand multivariable I-function in terms of mellin-Barnes type of contour integral by (7) , generalized polynomial by(6).Now interchanging the order of summation and integration (which is permissible under the conditions of validity stated above) , We get the following form:-

$$\sum_{\alpha_1=0}^{[N_1/M_1]}\ldots\ldots\sum_{\alpha_k=0}^{[N_k/M_k]}\frac{(-N_1)_{M_1\alpha_1}}{!\,\alpha_1}\ldots\ldots\frac{(-N_k)_{M_k\alpha_k}}{!\,\alpha_k}\;B[N_1\alpha_1,\ldots\ldots,N_k\alpha_k]\,.\,x_1^{\alpha_1}\ldots\ldots x_k^{\alpha_k}$$

$$\mathrm{X}\left(\frac{1}{2\pi w}\right)^{r}\int_{L_1}\ldots\ldots\int_{L_r}\emptyset_1(\xi_1)\ldots\ldots\emptyset_r(\xi_r)\;.Z_1^{\xi_1}\ldots\ldots Z_r^{\xi_r}\left\{\int_m^n (t-m)^{a+\sum_{i=1}^{k}\lambda_i\alpha_i+\sum_{s=1}^{r}\gamma_s\xi_s-1}.(n-t)^{b+\sum_{i=1}^{k}\mu_i\alpha_i+\sum_{s=1}^{k}\tau_s\xi_s-1}\prod_{j=1}^{h}(p_jt+q_j)^{\rho_j+\sum_{i=1}^{k}\sigma_j^{(i)}\alpha_i-\sum_{s=1}^{r}c_j^{(s)}\xi_s}\,dt\right\}d\xi_1\ldots\ldots d\xi_r$$

Now using the formula (3) for inner integral i.e,

$\int_x^y (t-x)^{a-1}.(y-t)^{b-1}.(p_1t+q_1)^{\rho_1}\ldots\ldots(p_ht+q_h)^{\rho_k}\ dt =$
y−xa+b−1.Ba,b.(p1x+q1)e1……(phx+qh)en ×FD(h) a,−ρ1…−ρh ;a+b , −y−xp1p1x+q1……−y−xphphx+qh

And converting the Lauricella function $F_D^{(h)}$ in integral form from equation (4) and after simplification,we get the required result:

IV. SPACIAL CASES:-

1.If we set $\gamma_1 = 0 = \cdots = \gamma_r$ *and* $\lambda_1 = 0 = \cdots = \lambda_k$, the integral (8) reduces to

$$\int_m^n (t-m)^{a-1}(n-t)^{b-1}.\left\{\prod_{j=1}^{h}(p_jt+q_j)^{\rho_j}\right\} S_{N_1\ldots N_k}^{M_1\ldots M_k}\begin{bmatrix} x_1(n-t)^{\mu_1}.\prod_{j=1}^{h}(p_jt+q_j)^{\sigma_j^1} \\ \cdot \\ \cdot \\ \cdot \\ x_k(n-t)^{\mu_k}.\prod_{j=1}^{h}(p_jt+q_j)^{\sigma_j^k}\end{bmatrix}.I\begin{bmatrix} z_1(n-t)^{\tau_1}.\prod_{j=1}^{h}(p_jt+q_j)^{-c_j^1} \\ \cdot \\ \cdot \\ \cdot \\ z_r(n-t)^{\tau_r}.\prod_{j=1}^{h}(p_jt+q_j)^{-c_j^r}\end{bmatrix}$$

$= E_1\sum_{\alpha_1=0}^{\left[\frac{N_1}{M_1}\right]}\ldots\ldots$
$\sum_{\alpha_k=0}^{\left[\frac{N_k}{M_k}\right]}\frac{(-N_1)_{M_1\alpha_1}}{!\alpha_1}\ldots\ldots\frac{(-N_k)_{M_k\alpha_k}}{!\alpha_k}\;.B[N_1,\alpha_1,\ldots,N_k,\alpha_k].\,x_1^{\alpha_1}\ldots x_k^{\alpha_k}.E_2\ \times$
$I_{p_2,q_2:p_3,q_3\ldots p_r+h+2,q_r+h+1:(p^1,q^1)\ldots(p^{(r)},q^{(r)}),(0,1)\ldots(0,1)}^{0,n_2:0,n_3\ldots 0,n_r+h+2:(m^1,n^1)\ldots(m^{(r)},n^{(r)}),(1,0)\ldots(1,0)}$

$$\left[\begin{matrix} L_1 \\ L_2 \end{matrix}\middle| \begin{matrix} A_1,A_2,A_3,\left(a_{2j}\alpha'_{2j},\alpha''_{2j}\right)_{1,p_2},\left(a_{3j}\alpha'_{3j},\alpha''_{3j},\alpha'''_{3j}\right)_{1,p_3},\ldots\left(a_{rj},\alpha'_{rj}\ldots\ldots,\alpha^{(r)}_{rj},0,\ldots\ldots 0\right)_{1,p_r},(a'_j,\alpha'_j)_{1,p^1}\ldots\left(a^{(r)}_j,\alpha^{(r)}_j\right)_{1,p^{(r)}};--;--;-- \\ \left(b_{2j}\beta'_{2j},\beta''_{2j}\right)_{1,q_2},\left(b_{3j}\beta'_{3j},\beta''_{3j},\beta'''_{3j}\right)_{1,q_3},\ldots\left(b_{rj},\beta'_{rj}\ldots,\beta^{(r)}_{rj},0,\ldots 0\right)_{1,q_r}A_4,A_5,(b'_j,\beta'_j)_{1,q^1}\ldots\left(b^{(r)}_j,\beta^{(r)}_j\right)_{1,q^{(r)}},(0,1)\ldots(0,1) \end{matrix}\middle|\right]$$

Where, $W_1 = (n-m)^{a+b-1}\left\{\prod_{j=1}^{h}\left(p_j m+q_j\right)^{\rho_j}\right\}$

$$W_2 = (n-m)^{\sum_{i=1}^{k}\mu_i.\alpha_i} \cdot \left\{\prod_{j=1}^{h}\left(p_j m+q_j\right)^{\sum_{i=1}^{k}\sigma_j^{(i)}.\alpha_i}\right\}$$

$$A_1 = [1-a;\overbrace{0\ldots\ldots 0}^{r},1\ldots\ldots 1]$$

$$A_2 = \left[1-b-\sum_{i=1}^{k}\mu_i.\alpha_i \ ;\ \zeta_i\ldots\ldots\zeta_r\,,_{0\ldots\ldots 0}\right]$$

$$A_3 = \left[1+\rho_j+\sum_{i=1}^{k}\sigma_j^{(i)}.\alpha_i \ ;\ C_j^1\ldots\ldots C_j^{(r)},0\ldots\ldots 1^j\ldots\ldots.0\right]_{1,h}$$

$$A_4 = \left[1+\rho_j+\sum_{i=1}^{k}\sigma_j^{(i)}.\alpha_i \ ;\ C_j^1\ldots\ldots C_j^{(r)},0\ldots\ldots 1^j\ldots\ldots.0\right]_{1,h}$$

$$A_5 = [1-a-b-\sum_{i=1}^{k}\mu_i.\alpha_i \ ;\ \zeta_i\ldots\ldots\zeta_r\,,1\ldots\ldots.1$$

$$L_1 = \begin{cases} Z_1(n-m)^{\zeta_1}\Big/\prod_{j=1}^{h}(p_j m+q_j)^{c_j^1} \\ \vdots \\ Z_r(n-m)^{\zeta_r}\Big/\prod_{j=1}^{h}(p_j m+q_j)^{c_j^{(r)}} \end{cases}$$

$$L_2 = \begin{cases} \dfrac{(n-m)p_1}{p_1 m+q_1} \\ \vdots \\ \dfrac{(n-m)p_h}{p_h m+q_h} \end{cases}$$

2. For $\zeta_1 = 0 = \cdots = \zeta_r$ *and* $\mu_1 = 0 = \cdots \mu_k$, the integral (8) reduces to

$$\int_m^n (t-m)^{a-1}(n-t)^{b-1}.\left\{\prod_{j=1}^{h}\left(p_j+q_j\right)^{\rho_j}\right\}$$

$$S_{N_1 N_k}^{M_1 M_k}\begin{bmatrix} x_1(t-m)^{\lambda_1}\ \prod_{j=1}^{h}\left(p_j t+q_j\right)^{\sigma_j^1} \\ \vdots \\ x_k(t-m)^{\lambda_k}\ \prod_{j=1}^{h}\left(p_j t+q_j\right)^{\sigma_j^k} \end{bmatrix}.I\begin{bmatrix} z_1(t-m)^{\gamma_1}\ \prod_{j=1}^{h}\left(p_j t+q_j\right)^{-c_j^1} \\ \vdots \\ z_r(t-m)^{\gamma_k}\ \prod_{j=1}^{h}\left(p_j t+q_j\right)^{-c_j^r} \end{bmatrix} = \Gamma(b)F_1\sum_{\alpha_1=0}^{\left[\frac{N_1}{M_1}\right]}\ldots\ldots$$

$$\sum_{\alpha_k=0}^{\left[\frac{N_k}{M_k}\right]}\frac{(-N_1)_{M_1\alpha_1}}{!\alpha_1}\ldots\ldots\frac{(-N_k)_{M_k\alpha_k}}{!\alpha_k}.B[N_1,\alpha_1,\ldots,N_k,\alpha_k].x_1^{\alpha_1}\ldots x_k^{\alpha_k}.F_2\ \times$$

$$I^{0,n_2:0,n_3\ldots 0,n_r+h+1:(m^1,n^1)\ldots(m^{(r)},n^{(r)}),(1,0)\ldots(1,0)}_{p_2,q_2:p_3,q_3\ldots p_r+h+1,q_r+h+1:(p^1,q^1)\ldots(p^{(r)},q^{(r)}),(0,1)\ldots(0,1)}$$

$$\left[\begin{matrix} B_1,B_2,\left(a_{2j}\alpha'_{2j},\alpha''_{2j}\right)_{1,p_2},\left(a_{3j}\alpha'_{3j},\alpha''_{3j},\alpha'''_{3j}\right)_{1,p_3},\ldots\left(a_{rj},\alpha'_{rj}\ldots\ldots,\alpha^{(r)}_{rj},0,\ldots\ldots0\right)_{1,p_r},(a'_j,\alpha'_j)_{1,p^1}\ldots\left(a^{(r)}_j,\alpha^{(r)}_j\right);--;--;--Q_1 \\ \left(b_{2j}\beta'_{2j},\beta''_{2j}\right)_{1,q_2},\left(b_{3j}\beta'_{3j},\beta''_{3j},\beta'''_{3j}\right)_{1,q_3},\ldots\left(b_{rj},\beta'_{rj}\ldots,\beta^{(r)}_{rj},0,\ldots0\right)_{1,q_r}X_4,X_5,(b'_j,\beta'_j)_{1,q^1}\ldots\left(b^{(r)}_j,\beta^{(r)}_j\right),(0,1)\ldots(0,1)\;\;Q_2\end{matrix}\right]$$

Where, $F_1 = (n-m)^{a+b-1}\left\{\prod_{j=1}^{h}(p_j m + q_j)^{\rho_j}\right\}$

$$F_2 = (n-m)^{\sum_{i=1}^{k}\lambda_i\alpha_i}\,.\left\{\prod_{j=1}^{h}\left(p_j m+q_j\right)^{\sum_{i=1}^{k}\sigma_j^{(i)}.\alpha_i}\right\}$$

$$B_1 = \left[1-a-\sum_{i=1}^{k}\lambda_i.\alpha_i\;;\gamma_i\ldots\ldots\gamma_r\,,1\ldots\ldots1\right]$$

$$B_2 = \left[1+\rho_j+\sum_{i=1}^{k}\sigma_j^{(i)}.\alpha_i\;;C_j^1\ldots\ldots C_j^{(r)},0\ldots\ldots1\ldots\ldots.0\right]_{1,h}$$

$$B_3 = \left[1+\rho_j+\sum_{i=1}^{k}\sigma_j^{(i)}.\alpha_i\;;C_j^1\ldots\ldots C_j^{(r)},0\ldots\ldots\ldots\ldots.0\right]_{1,h}$$

$$B_4 = [1-a-b-\sum_{i=1}^{k}\lambda_i\alpha_i\;;\gamma_i\ldots\ldots\gamma_r\,,1\ldots\ldots.1$$

$$Q_1 = \begin{cases} Z_1(n-m)^{\gamma_1}\Big/\prod_{j=1}^{h}(p_j m+q_j)^{c_j^1} \\ \vdots \\ Z_r(n-m)^{\gamma_r}\Big/\prod_{j=1}^{h}(p_j m+q_j)^{c_j^{(r)}}\end{cases}$$

$$Q_2 = \begin{cases}\dfrac{(n-m)p_1}{p_1 m+q_1} \\ \vdots \\ \dfrac{(n-m)p_h}{p_h m+q_h}\end{cases}$$

3. When $\zeta_1 = 0 = \cdots = \zeta_r = 0 = \gamma_1 = \cdots\gamma_r$ *and* $\lambda_i = 0 = \mu_i, (i = 1\ldots\ldots k)$, then integral (8) reduces to

$$\int_m^n (t-m)^{a-1}(n-t)^{b-1}.\left\{\prod_{j=1}^{h}\left(p_j+q_j\right)^{\rho_j}\right\}$$

$$S_{N_1\ldots\ldots N_k}^{M_1\ldots\ldots M_k}\begin{bmatrix} x_1\;\prod_{j=1}^{h}\left(p_j t+q_j\right)^{\sigma_j^1} \\ \vdots \\ x_k\;\prod_{j=1}^{h}\left(p_j t+q_j\right)^{\sigma_j^k}\end{bmatrix}.I\begin{bmatrix} z_1\;\prod_{j=1}^{h}\left(p_j t+q_j\right)^{-c_j^1} \\ \vdots \\ z_r\;\prod_{j=1}^{h}\left(p_j t+q_j\right)^{-c_j^r}\end{bmatrix} = \Gamma(b)G_1\sum_{\alpha_1=0}^{\left[\frac{N_1}{M_1}\right]}\ldots\ldots$$

$$\sum_{\alpha_k=0}^{\left[\frac{N_k}{M_k}\right]}\frac{(-N_1)_{M_1\alpha_1}}{!\alpha_1}\ldots\ldots\frac{(-N_k)_{M_k\alpha_k}}{!\alpha_k}.B[N_1,\alpha_1,\ldots,N_k,\alpha_k].{x_1}^{\alpha_1}\ldots{x_k}^{\alpha_k}.G_2\;\times$$

$$I^{0,n_2:0,n_3\ldots0,n_r+h+1:(m^1,n^1)\ldots(m^{(r)},n^{(r)}),(1,0)\ldots(1,0)}_{p_2,q_2:p_3,q_3\ldots p_r+h+1,q_r+h+1:(p^1,q^1)\ldots(p^{(r)},q^{(r)}),(0,1)\ldots(0,1)}$$

$$\left[\begin{matrix}R_1\\R_2\end{matrix}\left|\begin{matrix} D_1,D_2,\left(a_{2j}\alpha'_{2j},\alpha''_{2j}\right)_{1,p_2},\left(a_{3j}\alpha'_{3j},\alpha''_{3j},\alpha'''_{3j}\right)_{1,p_3},\ldots\left(a_{rj},\alpha'_{rj}\ldots\ldots,\alpha^{(r)}_{rj},0,\ldots\ldots0\right)_{1,p_r},(a'_j,\alpha'_j)_{1,p^1}\ldots\left(a^{(r)}_j,\alpha^{(r)}_j\right)_{1,p^{(r)}};--;--;-- \\ \left(b_{2j}\beta'_{2j},\beta''_{2j}\right)_{1,q_2},\left(b_{3j}\beta'_{3j},\beta''_{3j},\beta'''_{3j}\right)_{1,q_3},\ldots\left(b_{rj},\beta'_{rj}\ldots,\beta^{(r)}_{rj},0,\ldots0\right)_{1,q_r}X_3,X_4,(b'_j,\beta'_j)\ldots\left(b^{(r)}_j,\beta^{(r)}_j\right)_{1,q^{(r)}},(0,1)\ldots(0,1)\end{matrix}\right.\right]$$

Where, $G_1 = (n-m)^{a+b-1}\left\{\prod_{j=1}^{h}(p_j m+q_j)^{\rho_j}\right\}$

$$G_2 = \left\{\prod_{j=1}^{h}(p_j m+q_j)^{\sum_{i=1}^{k}\sigma_j^{(i)}.\alpha_i}\right\}$$

$$D_1 = \left[1-a;\overbrace{0\ldots\ldots0}^{r},1\ldots\ldots.1\right]$$

$$D_2 = \left[1+\rho_j+\sum_{i=1}^{k}\sigma_j^{(i)}.\alpha_i\ ;\ C_j^1\ldots\ldots C_j^{(r)},0\ldots\ldots1\ldots\ldots.0\right]_{1,h}$$

$$D_3 = \left[1+\rho_j+\sum_{i=1}^{k}\sigma_j^{(i)}.\alpha_i\ ;\ C_j^1\ldots\ldots C_j^{(r)},0\ldots\ldots\ldots\ldots.0\right]_{1,h}$$

$$D_4 = [1-a-b;\overbrace{0\ldots\ldots0}^{r},1\ldots\ldots.1$$

$$R_1 = \begin{cases} Z_1.\prod_{j=1}^{h}(p_j m+q_j)^{c_j^1} \\ \vdots \\ Z_r.\prod_{j=1}^{h}(p_j m+q_j)^{c_j^{(r)}} \end{cases}$$

$$R_2 = \begin{cases} \frac{(n-m)p_1}{p_1 m+q_1} \\ \vdots \\ \frac{(n-m)p_h}{p_h m+q_h} \end{cases}$$

V. MAIN_THEOREM

Let $f(t) = (t-m)^{a-1}\left\{\prod_{j=1}^{h}\left(p_j t+q_j\right)^{\rho_j}\right\}$.

$$S_{N_1 N_k}^{M_1 M_k}\begin{bmatrix} x_1(t-m)^{\lambda_1}\prod_{j=1}^{h}\left(p_j t+q_j\right)^{\sigma_j^1} \\ \vdots \\ x_k(t-m)^{\lambda_k}\prod_{j=1}^{h}\left(p_j t+q_j\right)^{\sigma_j^k} \end{bmatrix}.I\begin{bmatrix} z_1(t-m)^{\gamma_1}\prod_{j=1}^{h}\left(p_j t+q_j\right)^{-c_j^1} \\ \vdots \\ z_r(t-m)^{\gamma_k}\prod_{j=1}^{h}\left(p_j t+q_j\right)^{-c_j^r} \end{bmatrix}$$

Then, ${}_mD_y^{-b}\{f(y)\} = \frac{1}{\Gamma(\mathrm{b})}.\int_m^y (y-t)^{b-1}.f(t)dt$

$${}_mD_y^{-b}[f(y)].\frac{1}{\Gamma(\mathrm{b})}.\int_m^y (t-m)^{a-1}(y-t)^{b-1}.\left\{\prod_{j=1}^{h}\left(p_j t+q_j\right)^{\rho_j}\right\}$$

$$S_{N_1\ldots N_k}^{M_1\ldots M_k}\begin{bmatrix} x_1(t-m)^{\lambda_1}\prod_{j=1}^{h}\left(p_j t+q_j\right)^{\sigma_j^1} \\ \vdots \\ x_k(t-m)^{\lambda_k}\prod_{j=1}^{h}\left(p_j t+q_j\right)^{\sigma_j^k} \end{bmatrix}.I\begin{bmatrix} z_1(t-m)^{\gamma_1}\prod_{j=1}^{h}\left(p_j t+q_j\right)^{-c_j^1} \\ \vdots \\ z_r(t-m)^{\gamma_k}\prod_{j=1}^{h}\left(p_j t+q_j\right)^{-c_j^r} \end{bmatrix}$$

$$= I \sum_{\alpha_1=0}^{\left[\frac{N_1}{M_1}\right]} \dots\dots$$

$$\sum_{\alpha_k=0}^{\left[\frac{N_k}{M_k}\right]} \frac{(-N_1)_{M_1\alpha_1}}{!\alpha_1} \dots\dots \frac{(-N_k)_{M_k\alpha_k}}{!\alpha_k} . B[N_1, \alpha_1, \dots, N_k, \alpha_k]. {x_1}^{\alpha_1} \dots {x_k}^{\alpha_k}. I_2 \times$$

$$I^{0,n_2:0,n_3\dots 0,n_r+h+1:(m^1,n^1)\dots(m^{(r)},n^{(r)}),(1,0)\dots(1,0)}_{p_2,q_2:p_3,q_3\dots p_r+h+1,q_r+h+1:(p^1,q^1)\dots(p^{(r)},q^{(r)}),(0,1)\dots(0,1)}$$

$$\left[\begin{matrix} K_1 \\ K_2 \end{matrix}\left| \begin{matrix} H_1, H_2, (a_{2j}\alpha'_{2j}, \alpha''_{2j}), (a_{3j}, \alpha'_{3j}, \alpha''_{3j}, \alpha'''_{3j})_{1,p_3}, \dots (a_{rj}, \alpha'_{rj} \dots\dots, \alpha^{(r)}_{rj}, 0, \dots\dots 0)_{1,p_r}, (a'_j, \alpha'_j)_{1,p^1} \dots (a^{(r)}_j, \alpha^{(r)}_j)_{1,p^{(r)}}; --; --; -- \\ (b_{2j}\beta'_{2j}, \beta''_{2j})_{1,q_2}, (b_{3j}, \beta'_{3j}, \beta''_{3j}, \beta'''_{3j})_{1,q_3}, \dots (b_{rj}, \beta'_{rj} \dots, \beta^{(r)}_{rj}, 0, \dots 0)_{1,q_r} H_3, H_4, (b'_j, \beta'_j)_{1,q^1} \dots (b^{(r)}_j, \beta^{(r)}_j)_{1,q^{(r)}}, (0,1) \dots (0,1) \end{matrix}\right.\right]$$

Where, $G_1 = (y-m)^{a+b-1} \left\{\prod_{j=1}^{h} (p_j m + q_j)^{\rho_j}\right\}$

$$G_2 = (y-m)^{\sum_{i=1}^{k} \lambda_i \alpha_i} \left\{\prod_{j=1}^{h} (p_j m + q_j)^{\sum_{i=1}^{k} \sigma_j^{(i)} . \alpha_i}\right\}$$

$$H_1 = \left[1 - a - \sum_{i=1}^{k} \lambda_i . \alpha_i \; ; \gamma_i \dots\dots \gamma_r, 1 \dots\dots 1\right]$$

$$H_2 = \left[1 + \rho_j + \sum_{i=1}^{k} \sigma_j^{(i)} . \alpha_i \; ; C_j^1 \dots\dots C_j^{(r)}, 0 \dots\dots 1 \dots\dots .0\right]_{1,h}$$

$$H_3 = \left[1 + \rho_j + \sum_{i=1}^{k} \sigma_j^{(i)} . \alpha_i \; ; C_j^1 \dots\dots C_j^{(r)}, 0 \dots\dots\dots\dots .0\right]_{1,h}$$

$$H_4 = [1 - a - b - \sum_{i=1}^{k} \lambda_i \alpha_i \; ; \gamma_1 \dots\dots \gamma_r, 1 \dots\dots .1$$

$$K_1 = \begin{cases} Z_1 (y-m)^{\gamma_1} \Big/ \prod_{j=1}^{h} (p_j m + q_j)^{c_j^1} \\ \vdots \\ Z_r (y-m)^{\gamma_r} \Big/ \prod_{j=1}^{h} (p_j m + q_j)^{c_j^{(r)}} \end{cases}$$

$$K_2 = \begin{cases} \frac{(y-m)p_1}{p_1 m + q_1} \\ \vdots \\ \frac{(y-m)p_h}{p_h m + q_h} \end{cases}$$

Conditions of validity of the theorem are the same as stated in Eulerian Integral.

Special case

By specializing the various parameters, we get some known and unknown results.

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