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On Co Distributive Pair and Dually Co Distributive Pair in a Lattice

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***Abstract**: In this paper we have obtained some interesting results on Co distributive pair in lattices. We also obtained some results on dually Co distributive pair in general lattice.*

***Key Words**: Co distributive pair, Dually co distributive, d-prime element, Dually d-prime element, d-prime Ideals, Dually d-prime element.*

I. Introduction

In this paper we have defined some definitions like Co distributive pair, d-meet irreducible element, d-prime element of a lattice 'L', d-prime is transformed to dually co distributive pair, dually d-prime ideals, dually d-prime element.

Using above definitions we have achieved some theorems [like,(4) If (x,y) is dually co distributive then for any $a\in L$, $(x\wedge a)$, $(y\wedge a)$ is also dually co distributive.(6) Relation between dually d-prime ideal with

(1) distributive (2) Standard (3) Neutral] and result(s), If 'a' is dually d-prime element $\Leftrightarrow$ [a) is dually d-prime and $(x\wedge a)$, $(y\wedge a)$ is dually co distributive pair. Also we have obtained some of the most important theorems,(8) If (a,b) and (b,c) are dually codistributive, then $(a\wedge c,b)$ is also dually co distributive and (9) Suppouse I is a sublattice of L and m_a, $a\in I$,and m_a is an ideal of I,minimal w.r.to the property of containing 'a', then there is a d-prime ideal 'p' of L ∍ $P\cap I= m_a$ which is followed by lemma, If 'L' is any lattice, then every dually d-meet irreducible element is dually d-prime.
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II. In the first part of this paper we start with the following preliminaries

Def (1):- Co distributive pair : Let 'L' be a lattice, $x,y\in L$, then (x,y) is said to be codistributive, if $(x\wedge y)\vee z = (x\vee z)\wedge(y\vee z)\ \forall\ z\in L$.

Def (2):- d-meet irreducible element : An element 'a' of lattice 'L' is called d-meet irreducible $\Leftrightarrow$ $a= x\wedge y$ and (x,y) is codistributive $\Rightarrow$ either a=x or a=y.

Def (3):- d-prime element of a lattice L : An element 'a' of a lattice is d-prime $\Leftrightarrow$ $a \geq x\wedge y$ and (x,y) is codistributive $\Rightarrow$ either $a\geq x$ or $a\geq y$.

Def (4):- d-prime Ideals : An Ideal 'I' of a lattice 'L' is called a d-prime Ideal if for any codistributive pair $(a,b)\in L^2$ with $a\wedge b \in I$ then $a\in I$ or $b\in I$.

Theorem (1) :- Connection between d-meet irreducible element of a lattice 'L' with either distributive/Standard/Neutral.

Proof :- Let 'a' be a d-meet irreducible element, Also let (x,y) be co distributive with $a= x\wedge y$.

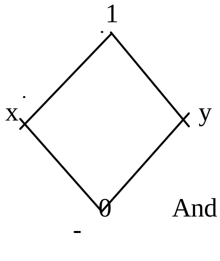
Claim:- (i) 'a' is distributive, i.e, $a\vee(x\wedge y) = (a\vee x)\wedge(a\vee y)$

Consider, $a\vee(x\wedge y) = a\vee(a) = a$.

Also, $(a\vee x)\wedge(a\vee y) = ((x\wedge y)\vee x)\wedge((x\wedge y)\vee y) = x\wedge y = a$.

Converse :- If 'a' is distributive then 'a' is a d-meet irreducible element.

Consider,

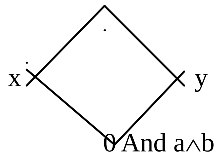


$a\wedge b = 0$, $a\vee b = 1$.
Since, $a\vee(x\wedge y) = (a\vee x)\wedge(a\vee y)$,
Let x = 1 , y = b , then $a\vee(1\wedge b) = a\vee b = 1$.
Also, $(a\vee 1)\wedge(a\vee b)= 1\wedge 1=1$.

And $a\wedge b = 0 \Rightarrow$ (a,b) is co distributive $\Rightarrow$ either a=0 or b = 0.

(ii) 'a' is standard, i.e, $x\wedge(a\vee y) = (x\wedge a)\vee(x\wedge y)$.
Consider, $x\wedge(a\vee y) = x\wedge((x\wedge y)\vee y) = x\wedge y = a$.
Also, $(x\wedge a)\vee(x\wedge y) = (x\wedge a)\vee a = a$.
Converse :- If 'a' is standard, then 'a' is not d-meet irreducible element, because of the following

example, 1



Since, $x\wedge(a\vee y) = (x\wedge a)\vee(x\wedge y)$,
Let $x = 1$, $y = b$, then $1\wedge(a\vee b) = 1\wedge 1 = 1$.
Also, $(x\wedge a)\vee(x\wedge y)= (1\wedge a)\vee(1\wedge b)= a\vee b = 1$.
And $a\wedge b = 0 \Rightarrow (a,b)$ is co distributive $\Rightarrow$ either a=0 or b = 0.

Theorem (2) :- Connection between d-prime element of a lattice 'L' with either distributive / Standard / Neutral.

Proof :- Let 'a' is d-prime element, also let (x,y) be codistributive with $a\geq x\wedge y$.

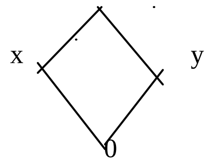
Claim:- (i) 'a' is distributive, i.e, $a\vee(x\wedge y) = (a\vee x)\wedge(a\vee y)$.

Since, $a\geq x\wedge y \Rightarrow (x\wedge y)\vee a = a. \Rightarrow (x\vee a)\wedge(y\vee a) = a.$, as (x,y) is co distributive.

$\Rightarrow$ 'a' is distributive.

Converse of this need not be true, because of the following example,

1



Since, $a\vee(x\wedge y) = (a\vee x)\wedge(a\vee y)$,
$0\vee(x\wedge y) = (0\vee x)\wedge(0\vee y)$,
$0 \vee 0 \quad = x\wedge y$,
$0 = 0$.

$\therefore$ {0} is distributive.

Since, $0 \geq x\wedge y$ and (x,y) is codistributive, but, 0□ x and 0□ y.

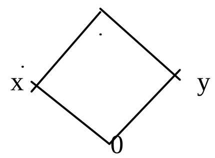
$\therefore$ Any distributive element need not be d-prime element.

(ii) 'a' is standard, i.e, $x\wedge(a\vee y) = (x\wedge a)\vee(x\wedge y)$.

Consider, $x\wedge(a\vee y) = (x\vee(x\wedge y))\wedge(a\vee y) = (x\vee a)\wedge(y\vee a) = (x\wedge y)\wedge a = a\wedge a = a$.

Also, $(x\wedge a)\vee(x\wedge y) = (x\wedge a)\vee a = a$.

Converse of this need not be true, because of the following example, 1



Since, $x\wedge(a\vee y) = (x\wedge a)\vee(x\wedge y)$,
$x\wedge(0\vee y) = (x\wedge 0)\vee(x\wedge y)$,
$x\wedge y = 0 \vee 0$,
$0 = 0$.

$\therefore$ {0} is standard.

Since, $0 \geq x\wedge y$ and (x,y) is co distributive, but, 0□x and 0□ y.

$\therefore$ Any Standard element need not be d-prime element.

III. In the second part of the paper we start with the following preliminaries

Def (1):- Dually co distributive : Let 'L' be a lattice and $(x,y)\in L^2$, then the pair (x,y) is said to be dually co distributive, if $(x\vee y)\wedge z = (x\wedge z)\vee(y\wedge z)\ \forall\ z\in L$.

Def (2):- Dually d-prime Ideal : An Ideal P of L is called a dually d-prime Ideal if $(x,y)\in L^2$ with $(x\vee y)\in P$. $\Rightarrow x\in P$ and $y\in P$ for any codistributive pair (x,y) in L^2.

Def (3):- Dually d-prime element : An element 'a' of a lattice 'L' is dually d-prime $\Leftrightarrow a \leq x\vee y$ and (x,y) is codistributive $\Rightarrow$ either $a\leq x$ and $a\leq y$.

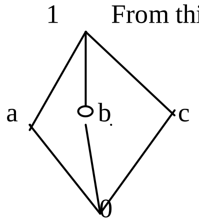
Theorem (4):- If (x,y) is dually codistibutive, then for any $a\in L$, $(x\wedge a)$, $(y\wedge a)$ is also dually co distributive.

Proof:- It is clear.

Theorem (5):- Relation between dually d-prime element with, (1) Distributive (2) Standard (3) Neutral.

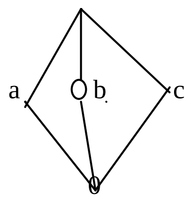
Proof:- Consider the following example for (1), i.e, dually d-prime element to distributive.

1 From this fig. put x =b, y =c



Then $a\vee(x\wedge y) = a\vee(b\wedge c) = a \vee 0 = a$.
Also $(a\vee x)\wedge(a\vee y) = (a\vee b)\wedge(a\vee c) = 1\wedge 1 = 1$.
$\therefore\ a \neq 1$.
Also $a \leq x\vee y$
$\leq b\vee c \ \leq\ 1$, and (x,y) is not codistributive, also a not$\leq$x and a not$\leq$y.

Also consider the following example for (2), i.e, dually d-prime element to standard.



From this fig. put x =b, y =c

Then $x\wedge(a\vee y) = b\wedge(a\vee c)= b\wedge 1 = b$.

Also $(x\wedge a)\vee(x\wedge y) = (b\wedge a)\vee(b\wedge c) = 0\vee 0 = 0$.

$\therefore\ b \neq 0$.

Also $a \leq x\vee y$

$\leq b\vee c \leq 1$, and (x,y) is not codistributive, also a not≤x and a not≤y.

Result (6) :- 'a' is d-prime element ⇔ (a] is d-prime Ideal.

Proof:- Let 'a' be d-prime element.

Claim:- (a] is d-prime Ideal.

Let (x,y) be a co distributive pair with $x\wedge y\in(a]$. ⇒ $x\wedge y\leq a$. ⇒ $x\leq a$ or $y\leq a$.

If $x\leq a$ ⇒ $x\in(a]$. If $y\leq a$ ⇒ $y\in(a]$. Hence (a] is d-prime.

Conversely, Let (a] be d-prime.

Claim:- 'a' is d-prime element.

i.e, $a \geq x\wedge y$ and (x,y) is co distributive, then $a\geq x$ or $a\geq y$.

Since $a\geq x\wedge y$ ⇒ $a =a\vee(x\wedge y) = (a\vee x)\wedge(a\vee y)$ ⇒ $(a\vee x)\wedge(a\vee y)\in(a]$.

Since, we know that (x,y) is co distributive then for any 'a'$(a\vee x,a\vee y)$ is also co distributive.

Hence $a\vee x \in(a]$ or $a\vee y\in (a]$.

If $a\vee x\in (a]$, then $a\vee x \leq a$. But $a\vee x\geq a$, hence $a\vee x=a$. Hence $x \leq a$.

If $a\vee y\in (a]$, then $a\vee y \leq a$. But $a\vee y\geq a$, hence $a\vee y=a$. Hence $y \leq a$.

Hence 'a' is d-prime element.

Result (7) :- 'a' is dually d-prime element ⇔ [a) is dually d-prime and for any (x,y) $(x\wedge a),(y\wedge a)$ is also dually co distributive.

Proof:- Let 'a' be dually d-prime element ⇒ $a \leq x\vee y$ and (x,y) is co distributive pair ⇒ $a\leq x$ and $a\leq y$.

Claim:- [a) is dually d-prime, where $[a) = \{x\in S/ x\geq a\}$.

Let $x\vee y \in[a)$

⇒ $a \leq x\vee y$ ⇒ $a\leq x$ and $a\leq y$.

If $a\leq x$ ⇒ $x\in[a)$ and $a\leq y$ ⇒ $y\in[a)$

Hence [a) is dually d-prime.

Conversely, let [a) is dually d-prime element.

Claim:- 'a' is dually d-prime element.

i.e, $a \leq x\vee y$ and (x,y) is dually co distributive ⇒ $a\leq x$ and $a\leq y$.

Since $a\leq x\vee y$ ⇒ $a = a\wedge(x\vee y) = (a\wedge x)\vee(a\wedge y)$.

⇒ $(a\wedge x)\vee(a\wedge y) \in [a)$.

[since (x,y) is dually co distributive and $(a\wedge x,a\wedge y)$ is also dually co distributive]

Hence $a\wedge x\in[a)$ and $a\wedge y\in[a)$.

If $a\wedge x\in[a)$, then $a\wedge x\geq a$, but $a\wedge x\leq a$, $\therefore$ $a\wedge x=a$, hence $x\geq a$.

Also if $a\wedge y\in[a)$, then $a\wedge y\geq a$, but $a\wedge y\leq a$, $\therefore$ $a\wedge y=a$, hence $y\geq a$.

Theorem (8):- If (a,b) and (b,c) are dually co distributive,then $(a\wedge c,b)$ is also dually co distributive.

Proof:- Since (a,b) is dually co distributive, for any element $x\in L$, we have $(a\vee b)\wedge x = (a\wedge x)\vee(b\wedge x)$.

Also since (b,c) is dually co distributive, for any element $x\in L$, we have $(b\vee c)\wedge x = (b\wedge x)\vee(c\wedge x)$.

To show that $(a\wedge c,b)$ is also dually co distributive.

Suppousing that, let $(a\wedge c,b)$ is not dually co distributive, then $[(a\wedge c)\vee b]\wedge x > [(a\wedge c)\wedge x]\vee[b\wedge x]$,for some $x\in L$, hence ∃ an ideal P which is minimal w.r.to the property of containing$[(a\wedge c)\wedge x]\vee[b\wedge x]$ but not containing $[(a\wedge c)\vee b]\wedge x$.

Thus $[(a\wedge c)\wedge x]\vee[b\wedge x]\in P$ and $[(a\wedge c)\vee b]\wedge x \notin P$.

Now $(a\wedge c)\vee b \notin P$ and $(a\vee b)\leq (a\wedge c)\vee b \notin P$. We have $(a\vee b) \notin P$.

Similarly, $(b\vee c) \leq (a\wedge c)\vee b$, and hence $(b\vee c) \notin P$. Since, $(b\wedge x)\in P$, this shows that $b\in P$, lly $c\in P$.

If $a\in P$, then since $(a\vee b) \notin P$, we have $b \notin P$ which is a contradiction. If $c\in P$, then since $(b\vee c) \notin P$, We have $b\notin P$, which is also a contradiction. Hence $(a\wedge c,b)$ is dually co distributive.

Theorem (9):- Suppouse I is a sublattice of L and m_a,$a\in I$ and m_a is an Ideal of I, minimal w.r.to the property of containing 'a', then there is a d-prime ideal P of L ∍ $P\cap I = m_a$.

Proof:- Let $x\in[\ m_a)\cap(a]$, then $x\in[\ m_a)$ and $x\in(a]$, so that $x\leq a$ for some $x\in[\ m_a)$ and hence $x\geq k$ for some $k\in m_a$. Thus $a\geq x\geq k \Rightarrow a\geq k$ for some $k\in m_a$ which is a contradiction.

Hence $[\ m_a)\cap(a] = \emptyset$. $\therefore \exists$ a d-prime ideal P of L $\ni [\ m_a)\subseteq P$, and $P\cap(a] = \emptyset$, hence 'P' is d-prime.

Claim:- $P\cap I = m_a$.

Now $m_a\subseteq P$, $m_a\subseteq I \Rightarrow m_a\subseteq P\cap I$ --------------(*)

Suppouse $x\in P\cap I$ so that $x\in P$ and $x\in I$,

If $x\notin m_a$ then $a\in m_a\wedge[x)$ and hence $a\geq k\wedge x$, for some $k\in m_a$.

$\therefore$ $k\in m_a \Rightarrow k\in P$. for some $x\in P$. since $k\in P$, $x\wedge k\in P$ so that, $a\notin P$ which is a contradiction.

Hence $x\in m_a$.

Thus $P\cap I\subseteq m_a$ ------------------(**)

From (*) and (**) , $P\cap I = m_a$.

Lemma (10):- If L is any lattice then every dually d-meet irreducible element is dually d-prime.

Proof:- Proof is clear.

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