On A Survey Of the Convergence and Stability of the Finite Difference Solutions of Partial Differential Equations

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Abstract: *Finite difference solution of partial differential equations must satisfy the requirement of convergence and stability if they are to be reasonably accurate. In this paper, we examined convergence and stability criterion of finite difference scheme for solving a partial differential equation. A case which is possible when the cumulative effect of all rounding errors is negligible. Consequently, we investigated the propagation of these errors as t increases by applying the Fourier series method.*

I.   Introduction

Several works has been carried out by many researchers on the stability of solutions of partial differential equation. Von Neumann [1,2] during the world war II developed a scheme using Fourier series methods to investigate the growth of errors arising from the arithmetic operation needed to solve a finite difference equation. This we shall use in the paper to investigate the stability of some explicit scheme for solving partial differential equation.

II.   Basic equation and results

Let $u(x,t)$ be the analytical solution of the partial differential equation and $W_{p,q}$ the solution of its finite difference approximation. Define the error function

$$E_{P,q} = u(x_p, t_q) - W_{p,q}, \qquad \begin{matrix} P = 0,1 \dots \dots, N \\ q = 0,1 \dots \dots, M \end{matrix}$$

Then $E_{p,q}$ satisfies the same difference equation as $W_{p,q}$

Let $E_p P = 0,1, \dots N$ denote the errors at mesh point $(x_p, 0)$ along the initial line $t = 0$.

To investigate the propagation of these errors as t increases we need to find a solution of the finite difference equation in $E_{p,q}$ that reduces to E_p when $t = 0$.

To apply the Fourier series method [3], we first express the errors in terms of finite Fourier series. Recall that the finite Fourier series of a function in the an interval $[-L, L]$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n cos\left(\frac{n\pi}{l}\right)x + \sum_{n=1}^{\infty} b_n sin\left(\frac{n\pi}{l}\right)x$$

The complex exponential form [4] of this series is

$$f(x) = \sum_{n=1}^{\infty} A_n e^{i\frac{n\pi}{\ell}x} \quad (2.2)$$

In the above expansions of $f(x), a_0, a_n, b_n \; and \; A_n$ are constants which have to be determined.

Expressing E_p in terms of complex Fourier series in an interval of length ℓ, we have

$$E_p = \sum_{n=1}^{\infty} A_n e^{i\frac{n\pi}{\ell}x_p} \qquad (i = \sqrt{-1})$$

In the domain $D = \{(x,t) \backslash 0 \le x \le \ell, 0 \le t \le T\}$, the mesh points (x_p, t_p) assume the form (ph, qk), where h and k are the mesh sizes.

$$\therefore E_p = \sum_{n=1}^{\infty} A_n e^{i \; \alpha_n ph} \qquad (2.3)$$

Where

$$\alpha_0 = \frac{n\pi}{\ell} = \frac{n\pi}{Nh} (since \ell = Nh), \quad i = \sqrt{-1}$$

Let us consider the propagation of the error at just one point $(x_p, 0)$ as t increases, such an error has a form $e^{i\beta ph}$.

Assuming that the corresponding error at $t > 0$ has the form

$$e^{i\beta ph}e^{\propto qk}, \quad \text{where} \propto \text{ is a complex constant, then we may write}$$

$$E_{P,q} = e^{i\beta ph}\xi^2 \quad (2.4)$$

Where $\xi = e^{\propto k}$

It is obvious that this form of $E_{P,q}$ ensures that $E_{P,q}$ reduce to $E_P = e^{i\beta ph}$ when $q = 0$

From equation (2.4), we deduce that the error will not increase as $t = qk$ increases provided $|\xi| \leq 1$

Example 2.1

Investigate the stability of the explicit finite differential equation

$$W_{i,j+1} = \lambda W_{i-1,j} + (1 - 2\lambda)W_{i,j} + \lambda W_{i+1,j}$$

Where $\lambda = \dfrac{k}{h^2}$

Solution

To avoid confusion over notation, let us $W_{p,q}$ for the solution of the given difference scheme, and $E_{P,q}$ for the error, since $E_{P,q}$ satisfies the same difference equations as

$W_{p,q}$, we have

$$E_{p,q+1} = \lambda E_{p-1,q} + (1 - 2\lambda)E_{P,q} + \lambda E_{P+1,q} \quad (2.5)$$

$$(p = 1, \dots\dots, N - 1, \quad q = 1, \dots\dots, m)$$

Writing $E_{P,q} = e^{i\beta ph}\xi^2$

Equation (2.5) becomes

$$e^{i\beta ph}\xi^{q+1} = \left[\lambda e^{i\beta(p-1)h} + (1 - 2\lambda)e^{i\beta ph} + \lambda e^{i\beta(p+1)h}\right]\xi^q$$

$$\therefore \xi = \lambda e^{-i\beta h} + (1 - 2\lambda) + \lambda e^{i\beta h}$$

Or

$$\xi = 1 - 2\lambda\left[1 - \cos(Bh)\right]$$

$$= 1 - 2\lambda\left[2\sin^2\left(\frac{Bh}{2}\right)\right]$$

$$= 1 - 4\lambda\sin^2\left(\frac{Bh}{2}\right)$$

For stability,

$$|\xi| \leq 1$$

$$\Rightarrow \quad \left|1 - 4\lambda\sin^2\left(\frac{Bh}{2}\right)\right| \leq 1$$

$$\therefore \quad -2 \leq -4\lambda\sin^2\left(\frac{Bh}{2}\right) < 0$$

$$1 \geq 2\lambda\sin^2\left(\frac{Bh}{2}\right) \geq 0$$

i.e.

$$0 \leq 2\lambda\sin^2\left(\frac{Bh}{2}\right) \leq 1$$

$$0 \leq \lambda\sin^2\left(\frac{Bh}{2}\right) \leq \frac{1}{2}$$

Since

$$0 \leq \sin^2\left(\frac{Bh}{2}\right) \leq 1$$

We must have,

$$0 < \lambda \leq \frac{1}{2}.$$

Example 2.2

The hyperbolic equation $U_{tt} = U_{xx}$ is approximated by the explicit scheme

$$\frac{W_{p,q+1} - 2W_{p,q} + W_{p,q-1}}{k^2} = \frac{W_{p+1,q} - 2W_{p,q} + W_{p-1,q}}{h^2}$$

Investigate its stability?

Solution

Let $E_{P,q} = e^{i\beta ph}$ be the error at $t = 0$ and $E_{P,q} = e^{i\beta ph}\xi^q$ be the error at any time t.

$E_{P,q} = U(x_p, t_q) - W_{p,q}$ satisfies the same difference equation as W so we have

$$E_{p,q+1} = \sigma^2 E_{p-1,q} + 2(1-\sigma^2)E_{P,q} + \sigma^2 E_{p+1,q} - E_{p,q-1}$$

$$e^{i\beta ph}\xi^{q+1} = \{[\sigma^2 e^{i\beta(p-1)h} + 2(1-\sigma^2)e^{i\beta ph} + \sigma^2 e^{i\beta(p+1)h}]\xi^q - e^{i\beta ph}\xi^{q-1}\}$$

$$\therefore \quad \xi = \sigma^2 e^{-i\beta h} + 2(1-\sigma^2) + \sigma^2 e^{i\beta h} - \xi^{-1}$$

$$\xi + \xi^{-1} = 2 + \sigma^2(e^{i\beta h} + e^{i\beta h} - 2)$$

$$= 2 + 2\sigma^2(cosph - 1)$$

$$\xi + \xi^{-1} = 2 - 4\sigma^2 \sin^2\left(\frac{Bh}{2}\right)$$

$$= 2\left[1 - 2\sigma^2 \sin^2\left(\frac{Bh}{2}\right)\right]$$

$$\therefore \quad \xi^2 - 2A + 1 = 0$$

Where $A = 1 - 2\sigma^2 \sin^2\left(\frac{Bh}{2}\right)$, and $\sigma = \frac{k}{h}$

$$\xi = \frac{2A \pm \sqrt{4A^2 - 4}}{2} = A \pm \sqrt{A^2 - 1}$$

Note

If $A < -1$, $\quad \sqrt{A^2 - 1} > 0$

If $A > -1$, $\quad \sqrt{A^2 - 1} > 0$

For stability, $|\xi| \leq 1$

$$\therefore \quad |\xi| = \left|A + \sqrt{A^2 - 1}\right| \leq 1$$

and $|\xi_2| = \left|A - \sqrt{A^2 - 1}\right| \leq 1$

if $A < -1$, $|\xi_2| > 1$, giving instability and

if $A > 1, |\xi_1| \leq 1$ and $|\xi_2| \leq 1$

i.e. for

$$\left|1 - 2\sigma^2 \sin^2\left(\frac{Bh}{2}\right)\right| \leq 1$$

$$-1 \leq 1 - 2\sigma^2 \sin^2\left(\frac{Bh}{2}\right) \leq 1$$

$$-2 \leq -2\sigma^2 \sin^2\left(\frac{Bh}{2}\right) \leq 0$$

$$\Rightarrow \qquad 0 \leq \sigma^2 \sin^2\left(\frac{Bh}{2}\right) \leq 1$$

$$\Rightarrow \qquad 0 \leq \sigma \leq 1$$

Thus, the given explicit scheme is stable for $0 \leq \sigma = \frac{k}{h} \leq 1$ and unstable otherwise

1. **Convergence Criterion**

If the solution $u(x,t)$ of the parabolic equation

$u_t = u_{xx}(0 < x < 1, \qquad 0 < t)$

hascontinuous and finite partial derivative of sufficiently high order in x and t, and if $\sigma t \leq \frac{(\sigma x)^2}{2}$

Then

I.   The error e_{ij} due to discretization satisfies the inequality

$$|e_{ij}| \leq T\left\{\frac{k}{2}|u_{tt}|_{max} + \frac{h^2}{12}|u_{xxxx}|_{max}\right\}$$

$$where \; 0 < t \leq T, \qquad h = \sigma x \; and \; x = \sigma t$$

II.   The numerical solution converge to the analytical solution at a rate that is $\quad 0(k + h^2)$

Proof

Let the analytic and the numerical solution at a point (x_i, t_j) in the domain of the parabolic partial differential equation

$$u_t = u_{xx}, \quad 0 \leq x \leq \ell, \quad t > 0 \tag{3.1}$$

be denoted by $u(x_i, t_j)$ and $W_{i,j,}$ respectively thus the error due to discretization is

$$e_{i,j,} = u(x_1, t_j) - W_{i,j} \tag{3.2}$$

The simplest explicit finite difference approximation to the parabolic equation (3.1) is

$$\frac{W_{i,j+1} - W_{i,j}}{k} - \frac{W_{1-1,j} - 2W_{i,j} + W_{1+1,j}}{h^2}$$

Where $h = \sigma x$ and $k = \sigma t$

Using equation (3.2), we get

$$W_{i,j} = u(x_1, t_j) - e_{i,j}$$

and substituting this into equation (3.3), we get

$$\frac{u(x_1, t_{j+1}) - u(x_1, t_j)}{k} - \frac{u(x_{1-1}, t_j) - 2u(x_1, t_j) + u(x_{1+1}, t_j)}{h^2}$$
$$= \frac{e_{i,j+1} - e_{i,j}}{k} - \frac{e_{i,-1\,j} - 2e_{i,j} + e_{i+1,j}}{h^2}$$

Taylor's expansion of $u(x, t)$ in the variable t gives

$$u(x + h, t) \cong u(x, t) + h u_x(x, t) + \frac{h^2}{2!} u_{xx}(\eta, t),$$
$$(x < \eta_1 < x + h)$$

$$u(x - h, t) \cong u(x, t) + h u_x(x, t) + \frac{h^2}{2!} u_{xx}(\eta_2, t),$$
$$(x - 1, \eta_2 < x)$$

i.e. we may write

$$u_t(x_i, \xi_j) - u_{xx}(\eta_i, t_j) = \frac{1}{k}(e_{i,j+1} - e_{i,j}) - \frac{1}{h^2}(e_{i-1,j} - 2e_{i,j} + e_{i+1,j})$$
$$\therefore\ e_{i,j+1} = \lambda e_{i-1,j} + (1 - 2\lambda)e_{i,j} + \lambda e_{i+1,j} + k[u_t(x_i, \xi_j) - u_{xx}(\eta_i, t_j)] \quad (3.4)$$

Where $\lambda = \frac{k}{h^2}$, $\eta_i = x_i + \propto h$, $\xi_j = t_j + \beta k$

$$(-1 < \propto < 1), \qquad (0 < \beta < k)$$

Let $\left|u_t(x_i, \xi_j) - u_{xx}(\eta_i, t_j)\right|_{max} = M$

and

$$\left|e_{i,j}\right| \leq \lambda\left|e_{i-1,j}\right| + |(1 - 2\lambda)|\left|e_{i,j}\right| + \lambda\left|e_{i+1,j}\right| + k\left|u_t(x_i, \xi_j) - u_{xx}(\eta_i, t_j)\right|$$

i.e.

$$E_{j+1} \leq \{\lambda + (1 - 2\lambda) + \lambda\}E_j + km = E_j + km$$
$$\therefore\quad E_{j+1} \leq E_j + km \quad (3.5)$$

i.e.

$$E_j \leq E_{j-1} + km \leq E_{j-2} + 2km \leq E_{j-3} + 3km \leq E_0 + Jkm$$

But $E_0 = 0$ and $t_j = jk$

$$\therefore\quad E_j \leq t_j m$$
$$\text{As } h \to 0 \text{ and } k \to 0$$
$$M \to [u_t(x_i, t_j) - u_{xx}(x_i, t_j)] = 0$$
$$E_j \to 0 \text{ as } \sigma x \to 0 \text{ and } \sigma t \to 0$$

Since $\left|e_{i,j}\right| = \left|u_{i,j} - W_{i,j}\right| \leq E_j \forall j$ it follows that

$$W \to u \text{ as } \sigma x \text{ and } \sigma t \text{ both tend to zero provided } \lambda \leq \frac{1}{2}$$

III.   Conclusion

The above proof shows that, for the finite difference solution to be convergent, u_t and u_{xx} must be uniformly continuous and bounded throughout the solution domain.

Besides, in example (2.1) and (2.2), we saw that for the stability of a finite difference scheme for solving P.D.E to be possible, the cumulative effect of all rounding errors must be negligible.

References

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