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KummerDirichlet Distributions of Matrix Variate in the Complex Case

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Abstract: *The aim of this paper is to investigate matrix variate generalizations of multivariate Kummer-Beta and multivariate Kummer-Gamma families of distributions in the complex case. The multivatiateKummer-Beta and multivariate Kummer-Gamma families of distributions have been proposed and studied recently by Ng and Kotz. These distributions are extensions of Kummer-Beta and Kummer-Gamma distributions. Many known or new results have been made with the help of multivatiateKummer-Beta and multivariate Kummer-Gamma families of distributions.*

I. Introduction

The Kummer-Beta and kummer-Gamma families of distributions are defined by the density functions involving hermition positive definite matrix

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\ \{{}_1F_1(\alpha;\ \alpha+\beta;\ -\lambda)\}^{-1}\mathrm{ex}(-\lambda u)u^{\alpha-1}(1-u)\beta\text{-}1,\ 0<u<1, \qquad (1.1)$$

$$\{\Gamma(\alpha)\psi(\alpha,\alpha-\gamma+;\ \varepsilon)\}^{-1}\exp(-\varepsilon v)v^{\alpha-1}(1+v)^{-\gamma},\ v>0, \qquad (1.2)$$

respectively, where $\alpha > 0$, $\beta > 0$, $\varepsilon > 0$, $-\infty<\gamma$, $\lambda<\infty$, ${}_1F_1$, and ψ are confluent hypergeometric functions. These distributions are extensions of Gamma and Beta distributions, and for $\alpha< 1$ (and certain values of $\lambda$and γ) yield bimodal distributions on finite and infinite ranges, respectively. These distributions are used (i) in the3 Bayesian analysis of queueing system where posterior distribution of certain basic parameters in M / M / ∞queueing system is Kummer-Gamma and (ii) in common value auctions where the posterior distribution of "value of a single good" is Kummer-Beta. For properties and applications of these distributions the reader is referred to NG and Kotz [7], Armero and Bayarri [1], and Gordy [2].

As the corresponding multivariate generalization of these distributions, we have the following *n*-dimensional densities:

$$\frac{\tilde{\Gamma}\left(\sum_{i=1}^{n}\alpha_i+\beta\right)}{\prod_{i=1}^{n}\tilde{\Gamma}(\alpha_i)\tilde{\Gamma}(\beta)}\left\{{}_1F_1\left(\sum_{i=1}^{n}\alpha_i;\sum_{i=1}^{n}\alpha_i+\beta;-\lambda\right)\right\}^{-1}\exp\left(-\lambda\sum_{i=1}^{n}u_i\right)$$

$$\times\prod_{i=1}^{n}u_i^{\alpha_i-1}\left(1-\sum_{i=1}^{n}u_i\right)^{\beta-1},\ 0<u_i<1,\ \sum_{i=1}^{n}u_i<1, \qquad (1.3)$$

Where $\alpha_i > 0$, $i = 1, \ldots, n$, $\beta > 0$, $-\infty<\lambda<\infty$, and

$$\left\{\tilde{\Gamma}\left(\sum_{i=1}^{n}\alpha_i\right)\psi\left(\sum_{i=1}^{n}\alpha_i,\sum_{i=1}^{n}\alpha_i-\lambda+1;\varepsilon\right)\right\}^{-1}\exp\left(-\varepsilon\sum_{i=1}^{n}v_i\right) \qquad (1.4)$$

$$\times\prod_{i=1}^{n}v_i^{a_i-1}\left(1+\sum_{i=1}^{n}v_i\right)^{-\gamma},\ v_i>0,$$

Where $\alpha_i>$ o, $i = 1, \ldots, n$, $\varepsilon > 0$, $-\infty<\gamma<\infty$, respectively. These distributions have been considered by Ng and Kotz [7] who refer to (1.3) and (1.4) as multivariate Kummer-Beta and multivariate Kummer-Gamma distributions, respectively. For $\lambda = 0$, (1.1) and (1.3) reduce to Beta and Dirichlet distributions with probability density functions

$$\frac{\tilde{\Gamma}(\alpha+\beta)}{\tilde{\Gamma}(\alpha)\tilde{\Gamma}(\beta)}u^{\alpha-1}(1-u)^{\beta-1},\ 0<u<1,$$

$$\frac{\tilde{\Gamma}\left(\sum_{i=1}^{n}\alpha_i+\beta\right)}{\prod_{i=1}^{n}\tilde{\Gamma}(\alpha_i)\tilde{\Gamma}(\beta)}\prod_{i=1}^{n}u_i^{\alpha_i-1}\left(1-\sum_{i=1}^{n}u_i\right)^{\beta-1},\ 0<u_i<1,\ \sum_{i=1}^{n}u_i<1 \qquad (1.5)$$

respectively. Since (1.3) is an extension of Dirichlet distribution and a multivariate generalization of Kummer-Beta distribution, anppropriate nomenclature for this distribution would be Kummer-Dirichlet distribution. In the same vein, we may call (1.4) a Kummer-Dirichlet distribution. Further, in order to distinguish between these two distributions ((1.3) and (1.4)), we call them Kummer-Dirichlet type I and Kummer-Dirichlet type II distributions.

In this article we propose and study matrix variate generalizations of (1.3) and (1.4), respectively.

II. Matrix variateKummer-Dirichlet distributions in the complex case

We begin with a brief review of some definitions and notations. We adhere to standard notations (cf. Gupta and Nagar [3]). Let $A = (a_{ij})$ be a p ×p matrix.

Then, A′ denotes the transpose of A; tr(A) = a_{11} ... + a_{pp}; etr(A) = exp (tr(A));
det (A) = determinant of A; A > 0 means that A is hermition square root of A > 0.

The multivariate gamma function $\tilde{\Gamma}_p(m)$ is defined as

$$\tilde{\Gamma}_p(m) = \pi^{p(p-1)/4}\prod_{j=1}^{p}\tilde{\Gamma}\left(m-\frac{j-1}{2}\right),\ \mathrm{Re}(m) > (p-1) \qquad (2.1)$$

where Re(.) denotes the real part of (.). It is straightforward to show that

$$\tilde{\Gamma}_p(m) = \int_{R>0} \det(\bar{R})^{m-p}\,\mathrm{etr}(\bar{R})\, d\bar{R},\ \mathrm{Re}(m) > (p-1) \qquad (2.2)$$

Where the integral has been evaluated over the space of the p ×phermition positive definite matrices. The integral representation of the confluent hypergeometric function $_1F_1$ is given by

$$1F1\ (a; b; \bar{X}) = \frac{\tilde{\Gamma}_p(b)}{\tilde{\Gamma}_p(a)\tilde{\Gamma}_p(b-a)}\times\int_{<R<I_p}\det(\bar{R})^{a-p}\det(I_p-\bar{R})^{b-a-p}\,\mathrm{etr}(X\bar{R})\,d\bar{R}, \qquad (2.3)$$

where Re(a) > (p-1) and Re (b-a) > (p-1). The confluent hypergeometric function ψ of a p ×p hermition matrix X is defined by

$$\psi(a, c; X) = \frac{1}{\tilde{\Gamma}_p(a)}\times\int_{R>0}\mathrm{etr}(-X\bar{R})\det(\bar{R})^{a-p}\det(I_p+\bar{R})^{c-a-p}\,d\bar{R}, \qquad (2.4)$$

where Re($\bar{X}$) > 0 and Re(a) > (p - 1)

Now we define the corresponding matrix variate generalizations of (1.3) and (1.4) as follows

.

Definition 2.1 The p ×p Hermition positive definite random matrices u_1 ,..., u_n are said to have the matrix variate Kummer-Drichlet type I distribution with parameters α_1, ..., α_n, β and Λ, denoted by (u_1, ..., u_n) ~ K D_p^I (α_1, ..., α_n, β, Λ), if their joint probability density function (pdf) is given by

$$K_1(\alpha_1, ..., \alpha_n, \beta, \Lambda)\ \mathrm{etr}\left(-\Lambda\sum_{i=1}^{n}\bar{u}_i\right)$$

$$\times\prod_{i=1}^{n}\det(\bar{u}_i)^{\alpha_i-p}\det\left(I_p-\sum_{i=1}^{n}\bar{u}_i\right)^{\beta-p} \qquad (2.5)$$

$$0<\bar{u}_i<I_p,\ 0<\sum_{i=1}^{n}\bar{u}_i<I_p,$$

where α_i> (p - 1) i = 1, ..., n, β> (p - 1) Λ (p × p) is Hermition K_1 (α_1, ..., α_n, β, Λ) is the normalizing constant.

Definition 2.2. The p ×pHermiotion positive definite random matrices V_1, ..., V_n are said to have the matrix variateKummer-Dirichlet type II distribution with parameters α_1,...., α_n, γ and Ξ, denoted by (V_1, ..., V_n) ~ K D_p^{II} (α_1,...., α_n, γ, Ξ), if their joint pdf is given by

$$K_2(\alpha_1, ...,\alpha_n, \gamma, \Xi)\ \mathrm{etr}\left(-\Xi\sum_{i=1}^{n}\overline{V}_i\right)$$

$$\times\prod_{i=1}^{n}\det(\overline{V}_i)^{\alpha_i-p}\det\left(I_p-\sum_{i=1}^{n}\overline{V}_i\right)^{-\gamma},\ V_i>0, \qquad (2.6)$$

where $\alpha_i>(p-1)$, $i=1, ..., n$, $-\infty<\gamma<\infty$, $\Xi\ (p\times p)>0$, and $k_2(\alpha_1,...., \alpha_n, \gamma, \Xi)$ IS The normalizing constant.

The normalizing constants in (2.5) and (2.6) are given as

$\{K_1(\alpha_1, ...,\alpha_n, \beta, \Lambda)\}^{-1}$

$$=\int_{\substack{0<\sum_{i=1}^{n} u_i<I_p\\ u_i>0}}\cdots\int \mathrm{etr}\left(-\Lambda\sum_{i=1}^{n}\overline{u}_i\right)$$

$$\times\prod_{i=1}^{n}\det(\overline{u}_i)^{\alpha_i-p}\det\left(I_p-\sum_{i=1}^{n}\overline{u}_i\right)^{\beta-p}\prod_{i=1}^{n}d\,\overline{u}_i$$

$$=\frac{\prod_{i=1}^{n}\tilde{\Gamma}_p(\alpha_i)}{\tilde{\Gamma}_p\left(\sum_{i=1}^{n}\alpha_i\right)}\int_{0<u<I_p}\mathrm{etr}(-\Lambda\overline{u})\det(\overline{u})^{\sum_{i=1}^{n}\alpha_i-p}$$

$$\times\det(I_p-\overline{u})^{\beta-p}\,d\overline{u}$$

$$=\frac{\prod_{i=1}^{n}\tilde{\Gamma}_p(\alpha_i)\tilde{\Gamma}_p(\beta_i)}{\tilde{\Gamma}_p\left(\sum_{i=1}^{n}\alpha_i+\beta\right)}\ 1F1\left(\sum_{i=1}^{n}\alpha_i;\sum_{i=1}^{n}\alpha_i+\beta;-\Lambda\right), \qquad (2.7)$$

$\{K_2(\alpha_1,...,\alpha_n, \gamma, \Xi)\}^{-1}$

$$=\int_{V_1>0}\cdots\int_{V_1>0}\mathrm{etr}\left(-\Xi\sum_{i=1}^{n}\overline{V}_i\right)$$

$$\times\prod_{i=1}^{n}\det(\overline{V}_i)^{\alpha_i-p}\det\left(I_p-\sum_{i=1}^{n}\overline{V}_i\right)^{-\gamma}\prod_{i=1}^{n}d\,\overline{V}_i$$

$$=\frac{\prod_{i=1}^{n}\tilde{\Gamma}_p(\alpha_i)}{\tilde{\Gamma}_p\left(\sum_{i=1}^{n}\alpha_i\right)}\int_{V>0}\mathrm{etr}(-\Xi\overline{V})\det(\overline{V})^{\sum_{i=1}^{n}\alpha_i-p}\det(I_p+\overline{V})^{-\gamma}\,d\overline{V}$$

$$=\prod_{i=1}^{n}\tilde{\Gamma}_p(\alpha_i)\psi\left(\sum_{i=1}^{n}\alpha_i;\sum_{i=1}^{n}\alpha_i+\gamma+p;\Xi\right), \qquad (2.8)$$

respectively, where $_1F_1$ and ψ are confluent hypergeometric functions of matrix argument.

For $\Lambda = 0$, the matrix variateKummer-Dirichlet type I distribution collapses to an ordinary matrix variateDirichlet type I distribution with pdf

$$\frac{\tilde{\Gamma}_p\left(\sum_{i=1}^{n}\alpha_i+\beta\right)}{\prod_{i=1}^{n}\tilde{\Gamma}_p(\alpha_i)\tilde{\Gamma}_p(\beta_i)}\prod_{i=1}^{n}\det(\overline{u}_i)^{\alpha_i-p}\det\left(I_p-\sum_{i=1}^{n}\overline{u}_i\right)^{\beta-p}$$

$$0<\overline{u}_i<I_p,\ 0<\sum_{i=1}^{n}\overline{u}_i<I_p, \qquad (2.9)$$

where$\alpha_i>(p-1)$ i = 1, ..., n, and $\beta>(p-1)/2$. A common notation to designate that $(u_1, ..., u_n)$ has this density is $(u_1, ... u_n)\sim D_p^I(\alpha_1, ..., \alpha_n;\beta)$. For $\gamma = 0$, the matrix variateKummer-Dirichlet type II density simplifies to the product of n matrix variate Gamma densities.

For p = 1, the densities in (2.5) and (2.6) simplify to Kummer-Dirichlet type I (multivariate Kummer Beta) and Kummer – Dirichlet type II distributions reduce to the matrix variateKummer-Beta and matrix variateKummer-Gamma distributions, respectively. These two distributions have been studied by Nagar and Gupta [6] and Nagar and Cardeno [5]. Substituting n = 1 in (2.5) and (2.6), the matrix variateKummer-Beta and matrix variateKummer-Gamma densities are obtained as

K_1 (α, β, Λ) etr(-Λ $\bar{u}$) det($\bar{u}$)$^{\alpha\text{-P}}$
×det(I_p - $\bar{u}$)$^{\beta\text{-p}}$, 0 < $\bar{u}$ <I_p, (2.10)
K_2 (α, γ, Ξ) etr (-Ξ $\bar{V}$) det ($\bar{V}$)$^{\alpha\text{-p}}$det (I_p + $\bar{V}$)$^{-\gamma}$, V > 0,

respectively, where α> (p-1), β> (p - 1), -∞, Λ = Λ′, and Ξ (p × p) > 0. These two distributions are designated by u ~ KB_p (α, β, Λ) and V ~ KG_p (α, γ, Ξ). It may be noted that the matrix variateKummer-Dirichlet distributions are special cases of the matrix variateLiouville distribution.

Using certain transformations, generalized matrix variateKummer-Dirichlet distributions are generated as given in the next two theorems.

Theorem 1. Let ($\bar{u}_1$, ..., $\bar{u}_n$) ~ KD_p^I (α_1,..., α_n, β, Λ) and ψ_1, ..., ψ_n, Ω be hermitionmatricies such that Ω> 0 and Ω - $\sum_{i=1}^n$ ψ_i> 0. Define

$$Zi = \left(\Omega-\sum_{i=1}^{n}\psi_i\right)^{1/2} u_i\left(\Omega-\sum_{i=1}^{n}\psi_i\right)^{1/2} + \psi_i,\ i = 1, ..., n. \qquad (2.11)$$

The (Z_1, ..., Z_n) have the generalized matrix variateKummer-Dirichlet type I distribution with pdf

$$\frac{K_1(\alpha_1,...,\alpha_n,\beta,\Lambda)}{\text{etr}(\Omega-\sum_{i=1}^n\psi_i)^{\sum_{i=1}^n\alpha_i+\beta-p_i}}$$
$$\times\frac{\prod_{i=1}^n\det(\bar{Z}_i-\psi_i)^{\alpha_i-p}\det(\Omega-\sum_{i=1}^n\bar{Z}_i)^{\beta-p}}{etr\{(\Omega-\sum_{i=1}^n\psi_i)^{-1/2}\Lambda(\Omega-\sum_{i=1}^n\psi_i)^{-1/2}\sum_{i=1}^n(Z_i-\psi_i)\}}$$
$$\psi_i< \bar{Z}_i <\Omega,\ i = 1, ..., n,\ \sum_{i=1}^{n}\bar{Z}_i <\Omega. \qquad (2.12)$$

Proof. Making the transformation $\bar{u}_i$ = (Ω - $\sum_{i=1}^n\psi_i$)$^{-1/2}$ (Z_i - ψ_i) (Ω - $\sum_{i=1}^n\psi_i$)$^{-1/2}$, *i* = 1, ..., *n*, with Jacobin J(u_1, ..., u_n→ Z_1, ..., Z_n) = det (Ω - $\sum_{i=1}^n\psi_i$)$^{-np}$in (2.5), we get (2.12).

If (Z_1, ..., Z_n) has the pdf (2.12), then we write (Z_1, ..., Z_n) ~ GKD_P^I (α_1, ..., α_n, β, Λ; Ω; ψ_1, ..., ψ_n). Note that GKD_P^I (α_1, ..., α_n, β, Λ; I_p; 0, ..., 0) ≡ KD_P^I (α_1, ..., α_n, β, Λ).

Theorem 2. Let (V1, ..., Vn) ~ KD_P^{II} (α_1, ..., α_n, γ, Ξ) and ψ_1, ... , ψ_n, Ω be hermition matrices such that Ω> 0 and Ω + $\sum_{i=1}^n\psi_1$ > 0. Define

$$Y_i = \left(\Omega-\sum_{i=1}^{n}\psi_i\right)^{1/2} V_i\left(\Omega-\sum_{i=1}^{n}\psi_i\right)^{1/2} + \psi_i,\ i = 1, ..., n. \qquad (2.13)$$

Then, (Y_1, ...,Y_n) have the generalized matrix variaeKummer-Dirichlet type II distribution with pdf

$$\frac{K_2(\alpha_1,...,\alpha_n,\gamma,\Xi)}{\det(\Omega-\sum_{i=1}^n\psi_i)^{\sum_{i=1}^n\alpha_i-\gamma}}$$
$$\times\frac{\prod_{i=1}^n\det(\bar{Y}_i-\psi_i)^{\alpha_i-p}\det(\Omega-\sum_{i=1}^n\bar{Y}_i)^{-\gamma}}{etr\{(\Omega-\sum_{i=1}^n\psi_i)^{-1/2}\Xi(\Omega-\sum_{i=1}^n\psi_i)^{-1/2}\sum_{i=1}^n(Y_i-\psi_i)\}}$$
$$Y_i>\psi_i,\ i = 1, ..., n. \qquad (2.14)$$

Proof. Making the transformation V_i = (Ω + $\sum_{i=1}^n\psi_i)^{-1/2}$) ($Y_i$ - ψ_i) (Ω + $\sum_{i=1}^n\psi_i)^{-1/2}$, *i* = 1, ..., n, with the Jacobian J (V_1, ..., V_n→ Y_1,... , Y_n) = det (Ω + $\sum_{i=1}^n\psi_i)^{-np}$ in (2.6), we get (2.14).

If (Y_1, ..., Y_n) has pdf (2.14), then we write (Y_1, ..., Y_n) ~ GKD_P^{II} (α_1, ..., α_n, γ, Ξ; Ωψ_1, ... , ψ_n). In this case GKD_P^{II} (α_1, ..., α_n, γ, I_p; 0, ..., 0) ≡ KD_P^{II} (α_1, ..., α_n; γ, Ξ).

III. Properties

In this section, we study certain properties of matrix variateKummer-Dirichlet type I and II invariant. That is, for any fixed orthogonal matrix $\Gamma(p \times p)$, the distribution of $(\Gamma u_1\Gamma', ..., \Gamma u_n\Gamma')$ is the same as the distribution of $(u_1, ..., u_n)$. Our next two results give marginal and conditional distributions.distributions. It may be noted that for $\Lambda = \lambda I_p$, $\Xi = \varepsilon I_p$ densities (2.5) and (2.6) are othogonaily

Theorem 3. If $(u_1, ..., u_n) \sim K D_p^I(\alpha_1, ..., \alpha_n, \beta, \Lambda)$, then the joint marginal pdf of $u_1, ..., u_m$, $m \leq n$, is given by

$$K_1\left(\alpha_1,...,\alpha_m, \sum_{i=m+1}^{n}\alpha_i+\beta,\Lambda\right)\operatorname{etr}\left(-\Lambda\sum_{i=1}^{m}\bar{u}_i\right)$$
$$\times\prod_{i=1}^{m}\det(\bar{u}_i)^{\alpha_i-p}\det\left(I_p-\sum_{i=1}^{m}\bar{u}_i\right)^{\sum_{i=m+1}^{n}\alpha_i+\beta-p}$$
$$\times {}_1F_1\left(\sum_{i=m+1}^{n}\alpha_i;\sum_{i=m+1}^{n}\alpha_i+\beta;-\Lambda\left(I_p-\sum_{i=1}^{m}\bar{u}_i\right)\right)$$
$$0<\bar{u}_i<I_p,\ 0<\sum_{i=1}^{m}\bar{u}_i<I_p, \tag{3.1}$$

and the conditional density of $(u_m + 1, ..., u_n) \mid (u_1, ..., u_m)$ is given by

$$\frac{K_1(\alpha_1,...,a_n,\beta,\Lambda)}{K_1(\alpha_1,...,a_m,\sum_{i=m+1}^{n}\alpha_i+\beta,\Lambda)}.$$
$$\times\frac{etr(-\Lambda\sum_{i=m+1}^{n}\bar{u}_i)}{\det(I_p-\sum_{i=1}^{m}\bar{u}_i)^{\sum_{i=m+1}^{n}\alpha_i+\beta-p}}$$
$$\times\frac{\prod_{i=m+1}^{n}\det(\bar{u}_i)^{\alpha_i-p}\det(I_p-\sum_{i=1}^{m}\bar{u}_i-\sum_{i=m+1}^{n}\bar{u}_i)^{\beta-p}}{{}_1F_1(\sum_{i=m+1}^{n}\alpha_i;\sum_{i=m+1}^{n}\alpha_i+\beta;-\Lambda(I_p\sum_{i=m+1}^{n}\bar{u}_i))} \tag{3.2}$$
$$0<\bar{u}_i\ I_p-\sum_{i=1}^{m}\bar{u}_i,\ i=m+1,...,n,\ \sum_{i=m+1}^{m}\bar{u}_i<I_p-\sum_{i=1}^{m}\bar{u}_i$$

Proof. First we find the marginal density of $u_1, ..., u_{n-1}$ by integrating out un from the joint density of $u_1, ..., u_n$ as

$$K_1(\alpha_1,...,\alpha_n,\beta,\Lambda)\int_{0<u_n<I_p-\sum_{i=1}^{n-1}}\operatorname{etr}\left(-\sum_{i=1}^{m}\bar{u}_i\right)$$
$$\times\prod_{i=1}^{n}\det(\bar{u}_i)^{\alpha_i-p}\det\left(I_p-\sum_{i=1}^{m}\bar{u}_i\right)^{\beta-p}d\bar{u}_n. \tag{3.3}$$

Now, substituting $Z_n = (I_p - \sum_{i=1}^{n-1}\bar{u}_i)^{-1/2} u_n (I_p - \sum_{i=1}^{n-1}\bar{u}_i)^{-1/2}$ with Jacobian $J(\bar{u}_n \to Z_n) = \det(I_p - \sum_{i=1}^{n-1}\bar{u}_i)^p$ in (3.2), we get

$$K_1(\alpha_1,...,\alpha_n,\beta,\Lambda)\operatorname{etr}\left(-\Lambda\sum_{i=1}^{n-1}\bar{u}_i\right)$$
$$\times\prod_{i=1}^{n-1}\det(\bar{u}_i)^{\alpha_i-p}\det\left(I_p-\sum_{i=1}^{n-1}\bar{u}_i\right)^{\alpha_n+\beta-p}$$
$$\times\int_{0<Z_n<I_p}\operatorname{etr}\left[-\left(-I_p-\sum_{i=1}^{n-1}\bar{u}_i\right)^{1/2}\Lambda\left(-I_p-\sum_{i=1}^{n-1}\bar{u}_i\right)^{1/2}\bar{Z}_n\right]$$

$$\times \det(\overline{Z}_n)^{\alpha_n - p} \det(I_p - \overline{Z}_n)^{\beta - p} d\overline{Z}_n . \tag{3.4}$$

But

$K_1(\alpha_1, ..., \alpha_n, \beta, \Lambda)$

$$\times \int_{0 < Z_r < I_p} \operatorname{etr}\left[-\left(- I_p - \sum_{i=1}^{n-1} \overline{u}_i \right)^{1/2} \Lambda \left(- I_p - \sum_{i=1}^{n-1} \overline{u}_i \right)^{1/2} \overline{Z}_n \right]$$

$$\times \det(\overline{Z}_n)^{\alpha_n - p} \det(I_p - \overline{Z}_n)^{\beta - p} d\overline{Z}_n .$$

$$K_1(\alpha_1, ..., \alpha_n, \beta, \Lambda) \frac{\tilde{\Gamma}_p(\alpha_n)\tilde{\Gamma}_p(\beta)}{\tilde{\Gamma}_p(\alpha_n + \beta)} {}_1F_1\left(\alpha_n ; \alpha_n + \beta ; -\Lambda\left(I_p - \sum_{i=1}^{n-1} \overline{u}_i \right)\right)$$

$$= K_1(\alpha_1, ..., \alpha_n, -1, \alpha_n + \beta, \Lambda) \left(\alpha_n ; \alpha_n + \beta ; -\Lambda\left(I_p - \sum_{i=1}^{n-1} \overline{u}_i \right)\right) \tag{3.5}$$

Hence, we get the joint density of $(u_1, ..., u_{n-1})$ as

$$K_1(\alpha_1, ..., \alpha_n, -1, \alpha_n + \beta, \Lambda) \operatorname{etr}\left(I_p - \sum_{i=1}^{n-1} \overline{u}_i \right) \prod_{i=1}^{n-1} \det(\overline{u}_i)^{\alpha_i - p}$$

$$\times \det\left(I_p - \sum_{i=1}^{n-1} \overline{u}_i \right)^{\alpha_n + \beta - p} {}_1F_1\left(\alpha_n ; \alpha_n + \beta ; -\Lambda\left(I_p - \sum_{i=1}^{n-1} \overline{u}_i \right)\right) \tag{3.6}$$

Repeating this procedure n – m times gives the marginal density of $(u_1, ..., u_m)$ as

$$K1 \left(\alpha_1, ..., \alpha_m \sum_{i=m+1}^{n} \alpha_i + \beta, \Lambda \right) \operatorname{etr}\left(-\Lambda \sum_{i=1}^{n-1} \overline{u}_i \right)$$

$$\times \prod_{i=1}^{m} \det(\overline{u}_i)^{\alpha_i - p} \det\left(I_p - \sum_{i=1}^{n-1} \overline{u}_i \right)^{\sum_{i=m}^{n} \alpha_n + \beta - p}$$

$$\times {}_1F_1\left(\sum_{i=m+1}^{n} \alpha_i ; \sum_{i=m+1}^{n} \alpha_i + \beta ; -\Lambda\left(I_p - \sum_{i=1}^{m} \overline{u}_i \right)\right). \tag{3.7}$$

Now the second part of the theorem follows immediately.

Corollary 3.1 – If $(u_1, ... u_n) \sim K\, D_p^{I} (\alpha_1 ,... \alpha_n, \beta, \Lambda)$, then the marginal pdf of u_i, $i = 1, ..., n$ is given by

$$K_1\left(\alpha_i , \sum_{j=1(\neq i)}^{n} \alpha_j + \beta ; -\Lambda \right) \operatorname{etr}(-\Lambda \overline{u}_i) \det(\overline{u}_i) \alpha_i - p$$

$$\times \det(I_p - \overline{u}_i)^{\sum_{j=1(\neq i)}^{n} \alpha_j + \beta - p}$$

$$\times {}_1F_1\left(\alpha_i , \sum_{j=1(\neq i)}^{n} \alpha_j ; + \sum_{j=1(\neq i)}^{n} \alpha_j + \beta - \Lambda(I_p - \overline{u}_i) \right), 0 < \overline{u}_i < I_p. \tag{3.8}$$

It is interesting to note that the marginal density of u_i does not belong to the Kummer-Beta family and differs by an additional factor containing confluent hypergeometric function ${}_1F_1$.

In Theorem 4, we give results on marginal and conditional distributions for Kummer-Dirchlet type II distribution. Before doing so, we need to give an integral that will be used in the derivation of marginal distribution. From (2.6) and, (2.8) we have

$$\int_{x>0} \int_{Y>0} \operatorname{etr}[-\Xi] [(\overline{X} + \overline{Y})] \operatorname{Det}(\overline{Y})^{a_1 - p}$$

$$\times \det(\overline{X})^{a_2 - p} \det(I_p + \overline{X} + \overline{Y})^{-b} d\overline{X} d\overline{Y} \tag{3.9}$$

$$= \tilde{\Gamma}_p(a_1) \tilde{\Gamma}_p(a_2) \psi(a_1 + a_2, a_1 + a_2 - b + p; \Xi),$$

where Re $(a_1) > (p-1)$, Re $(a_2) > (p-1)$ and Re $\Xi > 0$. Substituting

$W = (I_p + X)^{-1/2} Y (I_p + X)\text{-}1/2$ with the Jacobian $J(Y \to W) = \operatorname{DET}(I_p + \overline{X})^p$ in (3.9) and integrating W, we obtain

$$\int_{x>0} \operatorname{etr}(-\Xi \overline{X})\det(\overline{X})^{a_2-p} \det(I_p + \overline{X})^{a_1-b}$$

$$\times\psi(a_1, a_1 - b + p; \Xi(I_p + \overline{X}) \, d\overline{X} \tag{3.10}$$

$$= \Gamma_p(a_2)\, \psi(a_1, a_1 - b + p).$$

Now we turn to our problem of finding the marginal and conditional distributions.

Theorem 4. If $(V_1, ..., V_n) \sim K\, D_p^{II}$ $(\alpha_1, ..., \alpha_n, \gamma, \Xi)$, the joint marginal pdf of $V_1, ..., V_m$, $m<n$, is given by

$$\tilde{\Gamma}_p\left(\sum_{i=m+1}^{n}\alpha_i\right) K2\left(\alpha_1 ..., \alpha_m, \sum_{i=m+1}^{n}\alpha_i, \gamma, \Xi\right) \operatorname{etr}\left(-\Xi\sum_{i=1}^{m}\overline{V}_i\right)$$

$$\times\prod_{i=1}^{m}\det(\overline{V}_i)^{\alpha_i - p}\det\left(I_p + \sum_{i=1}^{n}\overline{V}_i\right)^{-\gamma+\sum_{i=m+1}^{n}\alpha_i}$$

$$\times\psi\left(\sum_{i=m+1}^{n}\alpha_i, \sum_{i=m+1}^{n}\alpha_i - \gamma + p; \Xi\left(I_p + \sum_{i=1}^{m}\overline{V}_j\right)\right), \tag{3.11}$$

$V_j > 0$, j = 1, ..., m

and the conditional density of $(V_{m+1}, ..., V_n) \mid (V_1, ... V_m)$ is given by

$$\frac{K_2(\alpha_1,...,\alpha_n,\gamma,\Xi)}{\tilde{\Gamma}_p(\sum_{i=m+1}^{n}\alpha_i)K_2(\alpha_1,...,\alpha_m,\sum_{i=m+1}^{n}\alpha_i,\gamma,\Xi)}$$

$$\times\frac{\operatorname{etr}(-\Xi\sum_{i=m+1}^{n}\overline{V}_i)}{\det(I_p + \sum_{i=1}^{m}\overline{V}_i)^{-\gamma\sum_{i=m+1}^{n}\alpha_i}}$$

$$\times\frac{\prod_{i=m+1}^{n}\det(\overline{V}_i)^{\alpha_i-p}\det(I_p + \sum_{i=1}^{m}\overline{V}_i + \sum_{i=1}^{n}\overline{V}_i)^{-\gamma}}{\psi(\sum_{i=m+1}^{m}\alpha_i, \sum_{i=m+1}^{n}\alpha_i - \gamma + p\Xi(I_p + \sum_{j=1}^{m}\overline{V}_j))} s$$

$V_i > 0$, i = m + 1, ..., n.

(3.12)

Proof. In this case, to obtain the marginal density of V1, ..., Vn-1, we substitute $W_n = (I_p + \sum_{i=1}^{n-1}\overline{V}_i)^{-1/2} V_n (I_p + \sum_{i=1}^{n-1}\overline{V}_i)^{-1/2}$ with the jacobian $J(V_n \to W_n) = \det(I_p + \sum_{i=1}^{n-1}\overline{V}_i)^p$.

Thus, the joint density of $V_1, ..., V_{n-1}$ is obtained as

$$K2(\alpha_1,...,\alpha_n,\gamma,\Xi)\operatorname{etr}\left(-\Xi\sum_{i=1}^{n-1}\overline{V}_i\right)$$

$$\times\prod_{i=1}^{n-1}\det(\overline{V}_i)^{\alpha_i-p}\det\left(I_p + \sum_{i=1}^{n-1}\overline{V}_i\right)^{-\gamma+\alpha_n}$$

$$\times\int_{w_n>0}\operatorname{etr}\left[-\left(I_p + \sum_{i=1}^{n-1}\overline{V}_i\right)^{1/2}\Xi\left(I_p + \sum_{i=1}^{n-1}\overline{V}_i\right)^{1/2}\overline{W}_n\right]$$

$$\times\det(\overline{W}_n)^{a_n-p}\det(I_p + \overline{W}_n)^{-\gamma} d\overline{W}_n \tag{3.13}$$

$$= \Gamma_p(\alpha_n)\, K_2(\alpha_1,..., \alpha_n, \gamma, \Xi)\operatorname{etr}\left(-\Xi\sum_{i=1}^{n-1}\overline{V}_i\right)$$

$$\times\prod_{i=1}^{n-1}\det(\overline{V}_i)^{\alpha_i-p}\det\left(I_p + \sum_{i=1}^{n-1}\overline{V}_i\right)^{-\gamma+\alpha_n}$$

$$\times\psi\left(\alpha_n, \alpha_n - \gamma + p; \Xi\left(I_p + \sum_{i=1}^{n-1}\overline{V}_i\right)\right)$$

Further, substituting $W_{n-1} = (I_p + \sum_{i=1}^{n-2} \overline{V}_i)^{-1/2} V_{n-1} (I_p + \sum_{i=1}^{n-2} \overline{V}_i)$-1/2 with the Jacobian j ($V_{n-1} \to W_{n-1}$) = det ($I_p + \sum_{i=1}^{n-2} \overline{V}_i)^p$ in (3.13) and integrating Wn-1 using (3.10), we get the joint marginal density of $V_1, \ldots, V_{n-2}$ as

$$\tilde{\Gamma}_p(\alpha_n)\, K_2(\alpha_1, \ldots, \alpha_n, \gamma, \Xi)\, \mathrm{etr}\left(-\Xi \sum_{i=1}^{n-2} \overline{V}_i\right)$$

$$\times \prod_{i=1}^{n-2} \det(\overline{V}_i)^{\alpha_i - p} \det\left(I_p + \sum_{i=1}^{n-2} \overline{V}_i\right)^{-\gamma+\alpha_n+\alpha_{n-1}}$$

$$\times \int_{w_n>0} \mathrm{etr}\left[-\left(I_p + \sum_{i=1}^{n-2} \overline{V}_i\right)^{1/2} \Xi \left(I_p + \sum_{i=1}^{n-2} \overline{V}_i\right)^{1/2} \overline{W}_{n-1}\right]$$

$$\times \det(\overline{W}_{n-1})^{a_n - p} \det(I_p + \overline{W}_{n-1})^{-\gamma+\alpha n}$$

$$\times \psi\left(\alpha_n, \alpha_n - \gamma + p; \left(I_p + \sum_{i=1}^{n-2} \overline{V}_i\right)^{1/2}\right.$$

$$\left.\times \Xi \left(I_p + \sum_{i=1}^{n-2} \overline{V}_i\right)^{1/2} W_{n-1}\right) d\overline{W}_{n-1}$$

$$= \Gamma_p(\alpha_n)\, \Gamma_p(\alpha_{n-1})\, K_2(\alpha_1, \ldots, \alpha_n, \gamma, \Xi)\, \mathrm{etr} \left(-\Xi \sum_{i=1}^{n-2} \overline{V}_i\right)$$

$$\times \prod_{i=1}^{n-2} \det(\overline{V}_i)^{\alpha_i - p} \det\left(I_p + \sum_{i=1}^{n-2} \overline{V}_i\right)^{-\gamma+\alpha_n+\alpha_{n-1}}$$

$$\times \psi\left(\alpha_n, \alpha_{n-1}, \alpha_n + \alpha_{n-1} - \gamma + p; \left(I_p + \sum_{i=1}^{n-2} \overline{V}_i\right)\right) \tag{3.14}$$

Integrating out $V_{n-2}, \ldots, V_{m+1}$ similarly, we get marginal density of $V_1, \ldots, V_m$ as

$$\times \prod_{i=m+1}^{n} \tilde{\Gamma}_p(\alpha_1)\, K_2(\alpha_1, \ldots, \alpha_n, \gamma, \Xi)\, \mathrm{etr}\left(-\Xi \sum_{i=1}^{n-2} \overline{V}_i\right)\Bigg)$$

$$\times \prod_{i=1}^{m} \det(\overline{V}_i)^{\alpha_i - p} \det\left(I_p + \sum_{i=1}^{m} \overline{V}_i\right)^{-\gamma + \sum_{i=m+i}^{n} + \alpha_1} \tag{3.5}$$

$$\times \psi\left(\alpha_n, \alpha_{n-1}, \alpha_n + \alpha_{n-1} - \gamma + p; \left(I_p + \sum_{i=1}^{n-2} \overline{V}_i\right)\right)$$

$$\times \prod_{i=m+1}^{n} \tilde{\Gamma}_p(\alpha_1)\, K_2(\alpha_1, \ldots, \alpha_n, \gamma, \Xi)$$

$$= \tilde{\Gamma}_p\left(\sum_{i=m+1}^{n} \alpha_i\right) K_2\left(\alpha_1, \ldots, \alpha_m, \sum_{i=m+1}^{n} \alpha_1, \gamma, \Xi\right). \tag{3.16}$$

The derivation of the conditional density is now straightforward.

Corollary 3.2. If $(V_1, \ldots, V_n) \sim K\, D_p^{II}(\alpha_1, \ldots, \alpha_n, \gamma, \Xi)$, then density of V_i, $i = 1, \ldots, n$ is given by

$$\tilde{\Gamma}_p\left(\sum_{j=1(\neq i)}^{n} \alpha_i\right) K_2\left(\alpha_1, \sum_{j=1(\neq i)}^{n} \alpha_1, \gamma, \Xi\right) \mathrm{etr}(-\Xi\, \overline{V}_i)$$

$$\times \det(\overline{V}_i)^{\alpha_i - p} \det(I_p + \overline{V}_i)^{-\gamma + \sum_{j=1(\neq i)}^{n} \alpha_j}$$

$$\times\psi\left(\sum_{j=1(\neq i)}^{n}\alpha_1, \sum_{j=1(\neq i)}^{n}\alpha_j - \gamma + p, \Xi(I_p + \overline{V}_i)\right) V_i > 0 \qquad (3.17)$$

Note that the marginal density of V_i differs from the Kummer-Gamma density. It is a pdf with an additional factor containing confluent hypergeometricfunction ψ.

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