IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN:2319-765X. Volume 9, Issue 4 (Jan. 2014), PP 07-12
www.iosrjournals.org

On the Banach Algebra Norm for the Functions of Bounded $k\varphi - Variation$ in the Sense of Riesz – Korenblum

DR. A. K. Choudhary[1], S. M. Nengem[1]
[1]*Department of Mathematical Sciences, Adamawa State University, P.M.B. 25, Mubi, Nigeria*

Abstract: *In this paper, we present the space of functions of bounded $k\varphi$ - variation in the sense of Riesz – Korenblum, denoted by $kV_\varphi^R[a,b]$, which is a combinations of the notions of bounded φ – variation in the sense of Riesz and of bounded k – variation in the sense of Korenblum. In the light of this, we prove that the space generated by this class of functions is Banach algebra with respect to a given norm and we give a brief characterization of the composition (Nemystkii) operator on the space $kV_\varphi^R[a,b]$.*

Keywords: *Banach algebra norm, Bounded $k\varphi - Variation$, composition (Nemystkii) operator.*

I. Introduction

When working with numbers such as real numbers $x \in \boldsymbol{R}$ or complex numbers $z \in \boldsymbol{C}$, there are unambiguous notion of a *magnitude* $|x|$ *or* $|z|$ of a number, with which to measure which numbers are larger and which are small. One can also use this notion of magnitude to define a distance $|x-y|$ *or* $|z-w|$ between two real numbers $x, y \in \boldsymbol{R}$, or between two complex numbers $z, w \in \boldsymbol{C}$, thus giving a quantitative measure of which pairs of numbers are close and which ones are far apart. This situation becomes more complicated however when dealing with objects with more degrees of freedom. Consider for instance the problem of determining the *magnitude* of a three-dimensional rectangular box. There are several candidates for such a magnitude: length, width, height, volume, surface area, diameter (i.e length of the diagonal), eccentricity, and so forth. Unfortunately, these magnitudes do not give equivalent comparisons: box A may be longer and have more volume than box B, but box B may be wider and have more surface area, and so forth. Because of this one abandon the idea that there should only be one notion of *magnitude* for boxes, and instead accept that there are instead multiplicities of such notions, all of which have some utility. Thus for some applications one may wish to distinguish the large volume boxes from the small volume boxes, while in others one may want to distinguish the eccentric boxes from the round boxes. Of course, there are several relationships between the different notions of magnitude (e.g. the isoperimetric inequality allows one to obtain the upper bound for the volume in terms of the surface area), so the situation is not as disorganized as it may first appear [5].

Now we turn to *functions* with a fixed domain and range (e.g. functions $f : [1,1] \rightarrow \mathbf{R}$ from the interval [1,1] to the real line **R**). These objects have infinitely many degrees of freedom, and so it should not be surprising that there are now infinitely many distinct notions of *magnitude*, all of which provide a different answer to the question "*how large is a given function f*?", or to the closely related question *"how close together are two functions f,g?"*, in some cases, certain functions may have infinite magnitude by one such measure, and finite magnitude by the another; similarly, a pair of functions may be very close by one measure and very far apart by another. Again this situation may seem chaotic, but it simply reflects the facts that functions have many distinct characteristics – some are tall, some are broad, some are smooth, some are oscillatory, and so forth – depending on the application at hand. One may want to give more weight to one of these characteristics than to others. In analysis, this is embodied in the variety of standard *function spaces*, and their associated *norms*, which are available to describe functions both qualitatively and quantitatively. While these spaces and norms are mostly distinct from each other, they are certainly interrelated, for instance, through such basic facts of analysis such as approximability by test functions (or in some cases by polynomials), by embedding such as sobolev embedding, and by interpolation theorems [2].

More formally, a function space is a class X of functions together with a *norm* which assigns a non-negative number $\|f\|_X$ to every function f in X; this function is the function space's way of measuring how large a function is. It is common (though not universal) for the class X of functions to consist precisely of those functions for which the definition of the norm $\|f\|_X$ makes sense and is finite; thus the mere fact that a function f has membership in a function space X conveys some qualitative information about that function (it may imply some regularity, some decay, some boundedness, or some integrability on the function f), while the norm $\|f\|_X$ supplements this qualitative information with a more quantitative measurement of the function (e.g. how regular is *f?* how much decay does *f* have? by which constant is *f* bounded? what is the integral of *f*?). Typically, we assumed that the function space X and its associated norm $\|\cdot\|_X$ obey a certain number of axioms; for instance, a rather standard set of axioms is that X is a real or complex vector space, that the norm is non-degenerate

($\|f\|_X > 0$, for non-zero *f*), homogeneous of degree 1, and obeys the triangle inequality $\|f+g\|_X \leq \|f\|_X + \|g\|_X$; furthermore, the space X when viewed using the metric *d(f,g)* = $\|f-g\|_X$ is a complete metric space [10]. Spaces satisfying all of these axioms are known as *Banach spaces*, and enjoy a number of good properties [4]. A majority (but certainly not all) of the standard function spaces are Banach spaces. More also, a Banach space which is also closed under multiplication is refers to as a Banach algebra.

The concept of functions of bounded variation has been well-known since Jordan gave the complete characterization of functions of bounded variation as a difference of two increasing functions in 1881. This class of functions immediately proved to be important in connection with the rectification of curves and with the Dirichlet's theorem on convergence of Fourier series. Functions of bounded variation exhibit so many interesting properties that make them a suitable class of functions in a variety of contexts with wide applications in pure and applied mathematics [9].

Riesz in 1910 generalized the notion of Jordan and introduced the notion of bounded $p - variation(1 \leq p < \infty)$ and showed that, for $1 < p < \infty$, this class coincides with the class of functions absolutely continuous with the derivative in space L_p. On the other hand, this notion of bounded $p - variation$ was generalized by Medvedev in 1953 who introduce the concept of bounded $\varphi - variation$ in the sense of Riesz and also showed a Riesz's Lemma for this class of functions [7].

Korenblum in 1975 introduce the nation of bounded $k - variation$ variation. This concept differs from others due to the fact that it introduces a distribution function k that measures intervals in the domain of the function and not in the range [1]. In 1985, Cyphert and Kelingos showed that a function u is of bounded $k - variation$ if it can be written as the difference of two k-decreasing functions. In 2010, Park introduced the notion of functions of $k\phi - bounded\ variation$ on a compact interval $[a,b] \subset R$ which is a combination of concept of bounded $k - variation$ and bounded $\varphi - variation$ in the sense of Schramm [8], and in 2010 Aziz et al. showed that the space of bounded $k - variation$ satisfies Matkowski's weak condition [16].

Recently, Castillo et al. introduce the notion of bounded $k - variation$ in the sense of Riesz-Korenblum, which is a combination of the notions of bounded $p - variation$ in the sense of Riesz and bounded $k - variation$ in sense of Korenblum. More also, Castillo et al. in 2013 introduce the concept of bounded $k\phi - variation$ in the sense of Riesz-Korenblum, which is a combination of the notions of bounded $\varphi - variation$ in the sense of Riesz and bounded $k - variation$ in the sense of Korenblum, and proved some properties of this class of functions and it relation with the functions of bounded $k - variation$ and bounded $\varphi - variation$ in the sense of Riesz. The same study proved that the space generated by this class of functions is a Banach space with a given norm and that the uniformly bounded composition operator satisfies Matkowski's weak condition in this space [9].

II. Preliminary

In this section we present some definitions and preliminary results related to the notion of functions of bounded $k\varphi$ - variation in the sense of Riesz – Korenblum.

Definition 1. *A partition of an interval* [a, b]: A partition say P of an interval $[a, b]$ is a set of points $\{x_1, x_2, ..., x_n\}$ such that P={a=$x_1, x_2, ..., x_n$ = b} [14].

Definition 2. *Variation of a function f:* Let $f : [a, b] \to$ R be a function and let [*c*,*d*] be any closed interval of [*a*, *b*] if the set S = $\{|f(x_i) - f(x_{i-1})|\}$ such that $\{x_i : 1 \leq i \leq n\}$ *is a partition of* $[c, d]$, is bounded then, the variation of f on [*c*, *d*] is defined to be V(f,[*c*, *d*]) = sup S. If S is unbounded then, the variation of f is said to be ∞. A function f is of bounded variation on [*c*, *d*], if V(f,[*c*, *d*]) is finite [13], [14].

Definition 3. Let $u : [a, b] \to \mathbb{R}$ be a function. For each partition $\pi : a = t_0 < t_1 < t_2 < \cdots < t_n = b$ of the interval $[a, b]$, we define

$$V(u; [a, b]) = \sup_{\pi} \sum_{i=1}^{n} |u(t_i) - u(t_{i-1})|,$$

Where the supremum is taken over all partitions π of the interval $[a, b]$ [9]. If $V(u; [a, b]) < \infty$, we say that u has bounded variation. Denoted by $BV[a, b]$ the collection of all functions of bounded variation on $[a, b]$.

The following are some well – known properties of the space of functions of bounded $BV[a, b]$.

(1) If the function u is monotone, then, $V(u; [a, b]) = |u(b) - u(a)|$.

(2) If $u \in BV[a, b]$, then u is bounded on $[a, b]$.

(3) A function u is of bounded variation of an interval $[a, b]$ if and only if it can be decomposed as a difference of increasing functions.

(4) Every function of bounded variation has left- and right- hand limits at each point of its domain.

(5) $BV[a, b]$ is a Banach space endowed with the norm

$$\|u\|_{BV} = |u(a)| + V(u; [a, b]), \qquad u \in BV[a, b]$$

Definition 4. A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be a φ - function if it satisfies the following properties.

(a) φ is continuous on $[0,\infty)$.
(b) $\varphi(t) = 0$ if and only if $t = 0$.
(c) φ is strictly increasing.
(d) $\lim_{t\to\infty} \varphi(t) = \infty$.
Definition 5. (conditions ∞_1 and Δ_2). Let φ be a convex φ – function, then
(a) φ satisfies the condition ∞_1 if $\lim_{t\to\infty}(\varphi(t)/t) = \infty$.
(b) φ satisfies the condition $\Delta_2(\infty)$ if there is $C > 0$, $x_0 > 0$ such that

$$\varphi(2t) \le C\varphi(t), \ \ t \ge x_0.$$

Definition 6. A *normed algebra* A is a norm space which is an algebra such that for all $x, y \in A$

$$\|xy\| \le \|x\|\|y\|$$

Definition 7. Let φ be a φ – function and $u : [a,b] \to \mathbb{R}$ be a function. For each partition $\pi : a = t_0 < t_1 < t_2 < \cdots < t_n = b$ of the interval $[a, b]$, we define

$$V_\varphi^R(u) = V_\varphi^R(u;[a,b]) = \sup_\pi \sum_{i=1}^n \varphi\left(\frac{|u(t_i)-u(t_{i-1})|}{|t_i-t_{i-1}|}\right)|t_i - t_{i-1}|,$$

where the supremum is taken over all partitions π of the interval $[a,b]$ [9]. If $V_\varphi^R(u;[a,b]) < \infty$, we say that u has bounded φ – variation in the sense of Riesz. Denoted by $V_\varphi^R[a,b]$ the collection of all functions of bounded φ – variation in the sense of Riesz on $[a,b]$. This set of functions share similar properties with $BV[a,b]$. In fact if φ is convex then, $V_\varphi^R[a,b] \subset BV[a,b]$ and if $\lim_{t\to\infty}(\varphi(t)/) = \gamma < \infty$, then $V_\varphi^R[a,b] = BV[a,b]$.
Definition 8. A function $k: [0,1] \to [0,1]$ is said to be a $k - function$ if it satisfies the following properties:
(a) k is continuous with $k(0) = 0 \ and \ k(1) = 1$.
(b) k is concave (down), increasing, and
(c) $\lim_{t\to 0}(k(t)/t) = \infty$. (Castillo, 203)
The set of all $k - function$ will be denoted by $\mathcal{K}$. Note that every $k - function$ is sub-additive; that is,

$$k(t_1 + t_2) \le k(t_1) + k(t_2), \qquad t_1, t_2 \in [0,1]$$

Then, for all partitions $\pi: a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$, we have

$$1 = k(1) = k\left(\sum_{i=1}^n \frac{t_i - t_{i-1}}{b-a}\right) \le \sum_{i=1}^n k\left(\frac{t_i-t_{i-1}}{b-a}\right).$$

Korenblum introduces the definition of bounded $k - variation$ as follows:
Definition 9. A real value function u on $[a,b]$ is said to be of bounded k – variation, if

$$kV(u) = kV(u;[a,b]) = \sup_\pi \frac{\sum_{i=1}^n |u(t_i)-u(t_{i-1})|}{\sum_{i=1}^n k((t_i-t_{i-1})/(b-a)} < \infty,$$

Where the supremum is taken over all partitions π of the interval $[a,b]$. We denote by $kV[a,b]$ the collection of all functions of bounded k – variation on $[a,b]$.
Some properties of the space $kBV[a,b]$ were exposed by [3].
(1) if the function u is monotone, then $kBV(u;[a,b] = |u(b) - u(a)|$.
(2) if $u \in kBV[a,b]$, then u is bounded on $[a,b]$.
(3) if $u \in BV[a,b]$, then if $u \in kBV[a,b]$; that is if $BV[a,b] \subset kBV[a,b]$.
(4) A function u has bounded $k - variation$ in an interval $[a,b]$ if and only if it can be decomposed as a difference of $k - decreasing$ functions.
(5) Every function of bounded $k - variation$ has left- and right-hand limits at each point of its domain.
(6) $kBV[a,b]$ is Banach space endowed with the norm

$$\|u\|_k = |u(a)| + kV(u;[a,b]), \qquad u \in kBV[a,b], [11].$$

Definition 10. Let φ be a φ – function, $k \in \mathcal{K}$, and $u : [a,b] \to \mathbb{R}$ be a function. For each partition $\pi : a = t_0 < t_1 < t_2 < \cdots < t_n = b$ of the interval $[a, b]$, we define

$$kV_\varphi^R(u) = kV_\varphi^R(u;[a,b]) = \sup_\pi \frac{\sum_{i=1}^n \varphi(|u(t_i)-u(t_{i-1})|/|t_i-t_{i-1}|)|t_i-t_{i-1}|}{\sum_{i=1}^n k((t_i-t_{i-1})/(b-a)},$$

Where the supremum is taken over all partitions π of the interval $[a,b]$. If $kV_\varphi^R(u;[a,b]) < \infty$, we say that u has bounded $k\varphi$ – variation in the sense of Riesz – Korenblum. Denoted by $V_\varphi^R[a,b]$ the collection of all functions of bounded $k\varphi$ – variation in the sense of Riesz - Korenblum on $[a,b]$.
Proposition 11. Let φ be a φ – function, $k \in \mathcal{K}$, and $u : [a,b] \to \mathbb{R}$ be a function, Then
(a) The function $kV_\varphi^R(\cdot) : kV_\varphi^R[a,b] \to \mathbb{R}$ is an even function, that is, $kV_\varphi^R(u) = kV_\varphi^R(-u)$.
(b) $kV_\varphi^R(u) = 0$ if and only if u is a constant.

III. Main Results

In this section we present the principal results of this paper. Next, let φ be a convex function such that φ satisfies definition $5(a)$, that is, $\lim_{t\to\infty}(\varphi(t)/t) = \infty$.

Remark 1. From proposition $11(a)$ and (b), it follows that $kV_\varphi^R(u;[a,b])$ is a symmetric and convex subset of the linear space X consisting of all functions $u : [a,b] \to \mathbb{R}$. Then the linear space $\langle kV_\varphi^R(u;[a,b])\rangle$ generated by $kV_\varphi^R(u;[a,b])$ may be written in the form

$\langle kV_\varphi^R(u;[a,b])\rangle = \{u \in X : there\ is\ \lambda > 0\ such\ that\ \lambda u \in kV_\varphi^R(u;[a,b])\}$ denoted by $kV_\varphi^R[a,b]$.

Moreover, the set $\wedge = \{u : [a,b] \to \mathbb{R} : kV_\varphi^R\left(\frac{u}{\lambda}\right) \le 1, for\ some\ \lambda > 0\}$ is absorbent and balance, so the Minkowski's functional associated to the set $\wedge$ is a semi-norm.

Remark 2. Since the set $\left\{\varepsilon > 0 :\ kV_\varphi^R\left(\frac{u}{\varepsilon}\right) \le 1\right\}$ is nonempty therefore, the following definition has sense

Definition 3. Let φ be a convex φ – function, and $\|\cdot\|_{k\varphi} : kRV_\varphi \to \mathbb{R}$, be defined by the formula. $\|u\|_{k\varphi} = inf\left\{\varepsilon > 0 :\ kV_\varphi^R\left(\frac{u}{\varepsilon}\right) \le 1\right\}$, with $kV_{\varphi,o}^R[a,b] = \{f \in kV_\varphi^R[a,b] :\ f(a) = 0\}$. Then, $kV_\varphi^R[a,b]$ has Banach space structure with respect to the norm

$$\|u\|_{k\varphi}^R = |u(a)| + inf\left\{\varepsilon > 0 :\ kV_\varphi^R\left(\frac{u}{\varepsilon}\right) \le 1\right\}\ for\ f \in kV_\varphi^R[a,b], [11].$$

Theorem 4. Let φ be a convex φ – function, then, $(kV_\varphi^R[a,b], \|\cdot\|_{k\varphi})$, where the functional $\|\cdot\|_{k\varphi} : kV_\varphi^R[a,b] \to \mathbb{R}$ defined by $\|u\|_{k\varphi} = |u(a)| + \mu_\Lambda(u)$, $u \in kV_\varphi^R[a,b]$ is a normed space, where μ_Λ is the Minkowski functional associated to the set Λ. Proof (see [9]).

Theorem 5. Let φ be a convex φ – function, such that φ - function such that φ satisfies the condition ∞_1, then, the space $(kV_\varphi^R[a,b], \|\cdot\|_{k\varphi})$ is Banach space. Proof (see [9]).

IV. The Banach Algebra $kV_\varphi^R[a,b]$

The techniques and approach used in this section are similar to those used by [16] and [15]

The first main result of this paper is contained in the following theorem.

Theorem 1. The space $(kV_\varphi^R([a,b];\mathbb{R}), \|\cdot\|_{k\varphi})$ is Banach algebra. In addition, $\|uv\|_{k\varphi} \le \|u\|_{k\varphi} \cdot \|v\|_{k\varphi}\ for\ u,v \in kV_\varphi^R([a,b];\mathbb{R})$.

Proof: Let $u,v \in kV_\varphi^R([a,b];\mathbb{R})$ be given, for any natural number n and any collection of non-overlapping subintervals $\pi : a = t_1, t_2, \dots, t_n = b$, by definition we have

$$\begin{aligned}
kV_\varphi^R(uv) = kV_\varphi^R(uv;[a,b]) &= \sup_\pi \frac{\sum_{i=1}^n \varphi(|uv(t_i) - uv(t_{i-1})|/|t_i - t_{i-1}|)|t_i - t_{i-1}|}{\sum_{i=1}^n k((t_i - t_{i-1})/(b-a))}\\
&= \sup_\pi \frac{\sum_{i=1}^n \varphi(|u(t_i)v(t_i) - u(t_{i-1})v(t_{i-1})|/|t_i - t_{i-1}|)|t_i - t_{i-1}|}{\sum_{i=1}^n k((t_i - t_{i-1})/(b-a))}\\
&= \sup_\pi \frac{\sum_{i=1}^n \varphi(|u(t_i)v(t_i) - u(t_i)v(t_{i-1}) + u(t_i)v(t_{i-1}) - u(t_{i-1})v(t_{i-1})|/|t_i - t_{i-1}|)|t_i - t_{i-1}|}{\sum_{i=1}^n k((t_i - t_{i-1})/(b-a))}\\
&= \sup_\pi \frac{\sum_{i=1}^n \varphi(|u(t_i)(v(t_i) - v(t_{i-1})) + v(t_{i-1})(u(t_i) - u(t_{i-1}))|/|t_i - t_{i-1}|)|t_i - t_{i-1}|}{\sum_{i=1}^n k((t_i - t_{i-1})/(b-a))}\\
&\le \sup_\pi \frac{\sum_{i=1}^n \varphi(|u(t_i)(v(t_i) - v(t_{i-1}))|/|t_i - t_{i-1}|)|t_i - t_{i-1}|}{\sum_{i=1}^n k((t_i - t_{i-1})/(b-a))} +\\
&\qquad \frac{\sum_{i=1}^n \varphi(|v(t_{i-1})(u(t_i) - u(t_{i-1}))|/|t_i - t_{i-1}|)|t_i - t_{i-1}|}{\sum_{i=1}^n k((t_i - t_{i-1})/(b-a))}\\
&= \|u\|_\infty kV_\varphi^R(v;[a,b]) + \|v\|_\infty kV_\varphi^R(u;[a,b])
\end{aligned}$$

Where $\|u\|_\infty\ is\ the\ supremum\ norm\ of\ the\ functional\ u$. Hence, by definition 9(6) we have

$$\begin{aligned}
\|uv\|_{k\varphi}^R &= \|uv\|_\infty + kV_\varphi^R(uv)\\
&\le \|u\|_\infty\|v\|_\infty + \|u\|_\infty kV_\varphi^R(v;[a,b]) + \|v\|_\infty kV_\varphi^R(u;[a,b])\\
&\le \|u\|_\infty\|v\|_\infty + \|u\|_\infty kV_\varphi^R(v;[a,b]) + \|v\|_\infty kV_\varphi^R(u;[a,b]) + kV_\varphi^R(u)kV_\varphi^R(v)\\
&= (\|u\|_\infty + kV_\varphi^R(u))(\|v\|_\infty kV_\varphi^R(v))\\
&= \|u\|_{k\varphi}^R\|v\|_{k\varphi}^R.
\end{aligned}$$

The proof is complete.

V. The Composition Operator On $kV_\varphi^R[a,b]$

The main objective of this section is to give a brief characterization of the composition (Nemystkii) operator on the space $kV_\varphi^R[a,b]$ of functions of bounded $k\varphi$ - variation in the sense of Riesz-Korenblum.

Theorem 1. Let $\varphi \in \Phi$ be a convex function, satisfying the condition ∞_1 and let $H : \mathbb{R}^{[a,b]} \to \mathbb{R}^{[a,b]}$ be the composition operator generated by the function $h: [a,b] \times \mathbb{R} \to \mathbb{R}$ and defined by the formula $(Hu)(t_i) =$

$h(t_i, u(t_i))$ for $u \in \mathbb{R}^{[a,b]}, t_i \in [a,b]$. If H map $kV_\varphi^R([a,b];\mathbb{R})$ into itself and is globally Lipschitzian, then, the following condition is satisfied.

$$|h(x,y_1) - h(x,y_2)| \le \delta|u_1 - u_2|, for\ x \in [a,b] and\ all\ y_1, y_2 \in \mathbb{R}.$$

Proof; Notice that the approach used here is similar to that from [16]. Suppose for any arbitrarily fixed $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, let us put

$$\eta_{\alpha,\beta} = \begin{cases} 0 & for\ t \le \alpha \\ \frac{t-\alpha}{\beta-\alpha} & for\ \alpha \le t \le \beta \\ 1 & for\ t \ge \beta \end{cases}$$

Observe that $\eta_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ and is obviously Lipschitzian.

Now, since $H :\ kV_\varphi^R([a,b];\mathbb{R}) \to kV_\varphi^R([a,b];\mathbb{R})$ is Lipschitzian, there exist a constant $\mu > 0$ such that

$$\|Hu_1 - Hu_2\|_\varphi^R \le \mu\|u_1 - u_2\|_{kV_\varphi^R}\ for\ u_1, u_2 \in kV_\varphi^R([a,b];\mathbb{R}).$$

The definition of norm $\|\cdot\|_{kV_\varphi^R}$ implies that $\mathcal{P}_\varphi(Hu_1 - Hu_2) \le \mu\|u_1 - u_2\|_{kV_\varphi^R}$, now from lemma 4.1 of [16] we infer that if $\|u_1 - u_2\|_{kV_\varphi^R} > 0$, then the last inequality can b equivalently written as

$$kV_\varphi^R\left(\frac{Hu_1 - Hu_2}{\mu\|u_1 - u_2\|_{kV_\varphi^R}}\right) \le 1$$

From the definitions of the operators kV_φ^R and H we deduce that for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R},\ x_1 < y_1$, $x_2 < x_2$ we have that

$$\varphi\left(\frac{\|(Hu_1 - Hu_2)(x_1,x_2) - (Hu_1 - Hu_2)(y_1,y_2)\|}{\mu\|u_1 - u_2\|_{kV_\varphi^R}\|y - x\|}\right)\|y - x\| \le 1$$

Hence, by taking the inverse function we get

$$\left|h\big(x,u_1(x)\big) - h\big(x,u_2(x)\big) - h(y,u_1(y) + h(y,u_2(y))\right| \\ \le \mu\|u_1 - u_2\|_{kV_\varphi^R}\|y - x\|\varphi^{-1}(\|y - x\|)$$

By critical examination of the four possible cases $(i)\, a_1 < x_1 \le b_1\ and\ a_2 < x_2 \le b_2,\ (ii)\, a_1 < x_1 \le b_1\ and\ x_2 = a_2,\ (iii)\, a_2 < x_2 \le b_2\ and\ x_1 = a_1, (iv)\, x_1 = a_1\ and\ x_2 = a_2$ by [9] gives
$\|u_1 - u_2\|_{kV_\varphi^R} = \mathcal{P}_\varphi(u_1 - u_2) = r = \frac{|u_1 - u_2|}{|y-x|\varphi^{-1}(\frac{1}{|y-x|})}$, hence the result follows from the last inequality above.

VI. Conclusion

At the beginning of the paper we followed the laid down steps by Castillo [9] and other authors in the references to introduced the notion of bounded $k\varphi$ - variation in the sense of Riesz-Korenblum. This is a combination of the notions of bounded φ - variation in the sense of Riesz and bounded k - variation in the sense of Korenblum. We further introduced a norm such that the space generated by this class of functions is Banach algebra. The problems are readily side-stepped by restricting attention to functions of bounded $k\varphi$ – variation in the sense of Riesz-Korenblum. The submultiplicativity of the norm is obtained and characterization of the composition operator briefly demonstrated. However, other interesting aspects such as inclusion of the space $kV_\varphi^R([a,b];\mathbb{R})$ into $BV[a,b]$ were not touched because of time space. Thus, that may be an interesting task for our readers.

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