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Common Fixed Point Theorems For Weakly Compatible Mappings In Generalisation Of Symmetric Spaces.

T.R.Vijayan,
Department of Mathematics, Pavai College of Technology, Namakkal - 637018, India

Abstract. *The main purpose of this paper is to obtain common fixed point theorem for weakly compatible mappings in generalisation symmetric spaces and a Property (E.A) introduced in [M. Aamri, D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270 (2002) 181-188]. Our theorem generalizes theorems Duran turkoglu and ishak altun, a common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007.*

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I. Introduction

It is well known that the Banach contraction principle is a fundamental result in fixed point theory, which has been used and extended in many different directions. Hicks [5] established some common fixed point theorems in symmetric spaces and proved that very general probabilistic structures admit a compatible symmetric or semi-metric. Recall that a symmetric on a set X is a nonnegative real valued function d on X × X such that (i) d(x, y) = 0 if, and only if, x = y, and (ii) d(x, y) = d(y, x).Let d be a symmetric on a set X and for r > 0 and any x ∈ X, let B(x, r) ={y ∈ X: d(x, y) < r}. A topology τ_d on X is given by U ∈ τ_d if, and only if, for each x ∈ U, B(x, r) ⊂ U for some r > 0. A symmetric d is a semi-metric if for each x ∈ X and each r > 0, B(x, r) is a neighbourhood of x in the topology τ_d .Note that $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology τ_d.

II. Preliminaries

Before proving our results, we need the following definitions and known results in this sequel.

Definition 2.1([4]) let (X, d) be a symmetric space. (W.3) Given $\{x_n\}$, x and y in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply x = y. (W.4) Given $\{x_n\}$, $\{y_n\}$ and x in X $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ imply that $\lim_{n\to\infty} d(y_n, x) = 0$.

Definition 2.2 ([12]) Two self mappings A and B of a metric space (X, d) are said to be weakly commuting if d (ABx, BAx) ≤ d (Ax, Bx), ∀x ∈ X.

Definition 2.3([6]) Let A and B be two self mappings of a metric space (X, d). A and B are said to be compatible if $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$, whenever (x_n) is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some t ∈ X.

Remark 2.4. Two weakly commuting mappings are compatibles but the converse is not true as is shown in [6].

Definition 2.5 ([7]) Two self mapping T and S of a metric space X are said to be weakly Compatible if they commute at there coincidence points, i.e., if Tu = Su for some u ∈ X, then TSu = STu.

Note 2.6. Two compatible maps are weakly compatible. M. Aamri [2] introduced the concept property (E.A) in the following way.

Definition 2.7 ([2]). Let S and T be two self mappings of a metric space (X, d). We say that T and S satisfy the property (E.A) if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some t ∈ X.

Definition 2.8 ([2]). Two self mappings S and T of a metric space (X, d) will be non-compatible if there exists at least one sequence $\{x_n\}$ in X such that if $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either nonzero or non-existent.

Remark 2.9. Two noncompatible self mappings of a metric space (X, d) satisfy the property (E.A).In the sequel, we need a function $\phi : R^+ \to R^+$ satisfying the condition $0 < \phi(t) < t$ for each t > 0.

Definition 2.10. Let A and B be two self mappings of a symmetric space (X, d). A and B are said to be compatible if $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$ whenever (x_n) is a sequence in X such that $\lim_{n\to\infty} d(Ax_n, t) = \lim_{n\to\infty} d(Bx_n, t) = 0$ for some t ∈ X.

Definition 2.11. Two self mappings A and B of a symmetric space (X, d) are said to be weakly compatible if they commute at their coincidence points.

Definition 2.12. Let A and B be two self mappings of a symmetric space (X, d).We say that A and B satisfy the property (E.A) if there exists a sequence (x_n) such that $\lim_{n\to\infty} d(Ax_n, t) = \lim_{n\to\infty} d(Bx_n, t) = 0$ for some t ∈ X.

Remark 2.13. It is clear from the above Definition 2.10, that two self mappings S and T of a symmetric space (X, d) will be noncompatible if there exists at least one sequence (x_n) in X such that $\lim_{n\to\infty} d(Sx_n, t)$ $=\lim_{n\to\infty} d(Tx_n, t) = 0$ for some t ∈ X. but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either non-zero or does not exist. Therefore, two noncompatible self mappings of a symmetric space (X, d) satisfy the property (E.A).

Definition 2.14. Let (X, d) be a symmetric space. We say that (X, d) satisfies the property (H_E) if given {x_n}, {y_n} and x in X, and $\lim_{n\to\infty} d(x_n, x) = 0$ *and* $\lim_{n\to\infty} d(y_n, x) = 0$ imply $\lim_{n\to\infty} d(y_n, x_n) = 0$.

Note that (X, d) is not a metric space.

III. Implicit Relation

Implicit relations on metric spaces have been used in many articles. (See [4], [10], [13].

Let R_+ denote the non-negative real numbers and let F be the set of all Continuous functions F: $R_+ \to R_+$ satisfying the following conditions:

F_1: there exists an upper semi-continuous and non-decreasing function f: $R_+ \to R_+$, f (0) = 0, f (t) < t for t > 0, such that for u ≥ 0, F (u, v, v, 0) ≤ 0 or F (u, v, 0, v) ≤ 0 or F (u, 0, v, v) ≤ 0 implies u ≤ f (v).

***F_2*:** F (u, 0, 0, 0) > 0 and F (u, u, u, 0) > 0, ∀ u > 0.

Example (3.1). F (t_1, t_2, t_3, t_4) = t_1 -α max {t_2, t_3, t_4}, where 0 < α < 1.

F_1 : Let u > 0 and F (u, v, v, 0) = u-αv≤ 0, then u ≤ αv. Similarly, let u > 0 and F (u, v, 0, v) ≤ 0, then u ≤ αv and again let u > 0 and F (u, 0, v, v) ≤ 0, then u ≤ αv. If u = 0 then u ≤αv. Thus F_1 is satisfied with f (t) = αt.

F_2: F (u, 0, 0, 0) = u > 0, ∀ u > 0 and F (u, u, u, 0) = u(1 − α) > 0 , ∀ u > 0.

Thus F ∈ $\mathcal{F}$

Example (3.2). F (t_1, t_2, t_3, t_4*)* = $t_1 - \psi$ (max {t_2, t_3, t_4}), where: $R_+ \to R_+$ is upper semi-continuous, non-decreasing and ψ (0) = 0, ψ (t) < t for t > 0.

F_1 : Let u > 0 and F (u, v, v, 0) = u − ψ (v) ≤ 0, then u ≤ψ (v). Similarly, let u > 0 and F(u, v, 0, v) ≤ 0, then u ≤ ψ (v) and again let u > 0 and F(u, 0, v, v)≤0, then *u* ≤ ψ *(v). If u = 0 then u*≤ ψ *(v). Thus* F_1 is satisfied with f = ψ.

F_2 : F(u, 0, 0, 0) = u > 0, ∀ u > 0 and F(u, u, u, 0) = u − ψ (u) > 0 , ∀ u > 0.

Thus F ∈ $\mathcal{F}$

IV.Main Result

Theorem 4.1: Let d be a symmetric for X that satisfies (W.3), (W.4) and (H_E).Let {A_i}, {A_j} (i≠j)and S be self mappings of (X,d) such that

(1)$F\Big(\int_0^{d(A_ix,A_jy)} \phi(t)dt, \int_0^{d(Sx,Sy)} \phi(t)dt, \int_0^{d(Sx,A_jy)} \phi(t)dt, \int_0^{d(Sy,A_jy)} \phi(t)dt,\Big) \le 0.$

for all (x,y) ∈ X^2, (i≠j) where F∈ $\mathcal{F}$ and ϕ : $R_+ \to R_+$ is a Lebesque-integrable mapping which is summable, non-negative and such that (2) $\int_0^{\varepsilon} \phi(t)dt > 0$ for all ε>0.

Suppose that $A_iX \subset SX$ and $A_jX \subset SX$, (i≠j) (A_i, S) and (A_j,S) (i≠j) are weakly compatible and (A_i, S) or (A_j, S) (i≠j) satisfies property (E.A). If the range of one ofthe mappings {A_i}, {A_j} or S (i≠j) is a closed subspace of X, then {A_i}, {A_j} and S (i≠j) have a unique common fixed point in X.

Proof: Suppose that {A_j} and *T*, ∀ j satisfy property (*E.A*). Then, there exists a sequence {x_n} in *X* such that that $\lim_{n\to\infty} d(A_jx_n, z) = \lim_{n\to\infty} d(Sx_n, z) = 0$ for some z ∈ X. ∀ j

Therefore, by (H_E) we have $\lim_{n\to\infty} d(A_jx_n, Sx_n) = 0$. ∀ j

Since $A_j(X) \subset S(X)$ ∀ j, there exists in *X* a sequence {y_n} such that $A_jx_n = Sy_n$. ∀ j

Hence, $\lim_{n\to\infty} d(Sy_n, z) = 0$.

Let us show that $\lim_{n\to\infty} d(A_iy_n, z) = 0$. ∀ i.

Suppose that $\text{im}_{n\to\infty} d(A_iy_n, A_jx_n) > 0$. Then, using (1), we have

$F\Big(\int_0^{d(A_iy_n,A_jx_n)} \phi(t)dt, \int_0^{d(Sy_n,Sx_n)} \phi(t)dt, \int_0^{d(Sy_n,A_jx_n)} \phi(t)dt, \int_0^{d(Sx_n,A_jx_n)} \phi(t)dt\Big) \le 0.$(i≠j)

We have,

$F\Big(\lim_{n\to\infty} \int_0^{d(A_iy_n,A_jx_n)} \phi(t)dt, \lim_{n\to\infty} \int_0^{d(A_jx_n,Sx_n)} \phi(t)dt, \lim_{n\to\infty} \int_0^{d(A_jx_n,Sx_n)} \phi(t)dt,\Big) \le 0.$

(i≠j). From F_1, there exists an upper semi-continuous and non-decreasing function f : $R_+ \to R_+$ $f(0) = 0$, $f(t) < t$ for $t > 0$

such that $\lim_{n\to\infty} \int_0^{d(A_iy_n,A_jx_n)} \phi(t)dt \le f\Big(\lim_{n\to\infty} \int_0^{d(A_jx_n,Sx_n)} \phi(t)dt\Big) < \lim_{n\to\infty} \int_0^{d(A_jx_n,Sx_n)} \phi(t)dt$,(i≠j).

Therefore $\lim_{n\to\infty} \int_0^{d(A_jx_n,Sx_n)} \phi(t)dt, > 0$which is a contradiction. Then we have

that $\lim_{n\to\infty} \int_0^{d(A_iy_n,A_jx_n)} \phi(t)dt = 0$. By (*W.*4), we deduce that $\lim_{n\to\infty} d(A_iy_n, z) = 0$. ∀ i.

Suppose that *SX* is a closed subspace of *X*. Then z = Su for some u$\in$ X. Consequently, we have $\lim_{n\to\infty} d(A_i y_n, A_j x_n)$=$\lim_{n\to\infty} d(A_j x_n, Su)$ = $\lim_{n\to\infty} d(Sx_n, \text{Su})$=$\lim_{n\to\infty} d(Sy_n, \text{Su})$=0.
We claim that Au = *S*u. Using (1),
$F\left(\int_0^{d(A_i u, A_j x_n)} \phi(t)dt, \int_0^{d(Su, Sx_n)} \phi(t)dt, \int_0^{d(Su,, A_j x_n)} \phi(t)dt, \int_0^{d(Sx_n A_j x_n)} \phi(t)dt,\right) \leq 0.$
and letting $n \to \infty$, we have $F\left(\int_0^{d(A_i u, A_j x_n)} \emptyset(t)dt, 0,0,0\right) \leq 0$. $\forall$ i,j(i$\neq$j).
which is a contradiction with F_2, if $\lim_{n\to\infty} \int_0^{d(A_i u, A_j x_n)} \phi$ (t)dt>0
Thus we obtain $\lim_{n\to\infty} \int_0^{d(A_i u, A_j x_n)} \phi$ (t)dt=0 and (2) implies that
$\lim_{n\to\infty} d(A_i u, A_j x_n)$=0 (i$\neq$j).
By (W.3) we have z = A_iu = Su. $\forall$ i The weak compatibility of {A_i} and S $\forall$ i implies that
A_iSu = SA_iu $\forall$ i ; i.e., A_iz = Sz. $\forall$ i On the other hand, since A_iX $\subseteq$ SX, $\forall$ i there exists
v $\in$ X such that A_iu = Sv. $\forall$ i We claim that A_jv = Sv. $\forall$ j If not, condition (1) gives
$F\left(\int_0^{d(A_i u, A_j v)} \phi(t)dt, \int_0^{d(Su, Sv)} \phi(t)dt, \int_0^{d(Su, A_j v)} \phi(t)dt, \int_0^{d(Su, A_j v)} \phi(t)dt,\right) \leq 0$. (i$\neq$j).
And we have, $F\left(\int_0^{d(A_i u, A_j v)} \phi(t)dt, \int_0^{d(Sv, A_j v)} \phi(t)dt, \int_0^{d(Sv, A_j v)} \phi(t)dt,\right) \leq 0$. (i$\neq$j).
From F_2, $\int_0^{d(Su, A_j v)} \phi(t)dt = \int_0^{d(A_i u, A_j v)} \phi(t)dt \leq f\left(\int_0^{d(Sv, A_j v)} \phi(t)dt\right)$(i$\neq$j).
Which is a contradiction since $\int_0^{d(Sv, A_j v)} \phi(t)dt > 0$, by (2)
Hence, z = A_iu =Su = A_jv = Sv. (i$\neq$j), the weak compatibility of {A_j} and S $\forall$ j implies that A_jSv =SA_jv
i.e., A_jz = Sz.
Let us show that *z* is a common fixed point of {A_i}, {A_j}, and S (i$\neq$j).
If z$\neq$ A_iz, $\forall$ i using (1), we get
$F\left(\int_0^{d(A_i, A_j v)} \phi(t)dt, \int_0^{d(Sz, Sv)} \phi(t)dt, \int_0^{d(Sz, A_j v)} \phi(t)dt, \int_0^{d(Sz, A_j v)} \phi(t)dt,\right) \leq 0$. (i$\neq$j)
And we have, $F\left(\int_0^{d(A_i z, z)} \phi(t)dt, \int_0^{d(A_i z, z)} \phi(t)dt, \int_0^{d(A_i z, z)} \phi(t)dt, 0\right) \leq 0$. (i$\neq$j)
Which is a contradiction with F_2, since $\int_0^{d(A_i z, z)} \phi$ (t) dt >0 by (2)
Thus z=A_iz=Sz $\forall$ i
If z$\neq A_j$z using (1) we get
$F\left(\int_0^{d(A_i z, A_j z)} \phi(t)dt, \int_0^{d(Sz, Sz)} \phi(t)dt, \int_0^{d(Sz, A_j z)} \phi(t)dt, \int_0^{d(Sz, A_j z)} \phi(t)dt,\right) \leq 0$. (i$\neq$j)
And we have $F\left(\int_0^{d(z, A_j z)} \phi(t)dt, 0, \int_0^{d(z, A_j z)} \phi(t)dt, \int_0^{d(z, A_j z)} \phi(t)dt,\right) \leq 0$.(i$\neq$j)
which is a contradiction with F_2 since $\int_0^{d(z, A_j Z)} \phi(t)dt > 0$ by (2).
Thus z =A_jz = Sz=A_iz.
The cases in which A_iX or A_jX is a closed subspace of *X* are similar to the cases in which *SX* is closed since A_iX$\subset$SX and A_jX$\subset$SX.(i$\neq$j)

Uniqueness.

For the uniqueness of *z*, suppose that *w*$\neq$ z is another common fixed point of {A_i}, {A_j} and S. (i$\neq$j)
Using (1), we obtain, $F\left(\int_0^{d(A_i z, A_j w)} \phi(t)dt, \int_0^{d(Sz, Sw)} \phi(t)dt, \int_0^{d(Sz, A_j w)} \phi(t)dt, \int_0^{d(Sw, A_j w)} \phi(t)dt,\right) \leq 0$. (i$\neq$j)
And we have, $F\left(\int_0^{d(z,w)} \phi(t)dt, \int_0^{d(z,w)} \phi(t)dt, \int_0^{d(z,w)} \phi(t)dt, 0\right) \leq 0$. (i$\neq$j)
which is a contradiction with F_2 since $\int_0^{d(z,w)} \phi(t)dt > 0$ by (2). Thus *z* = *w*, and the common fixed point is unique. This completes the proof of the theorem.

Corollary4.2: Let d be a symmetric for X that satisfies (W.3),(W.4) and (H_E).Let A,B and S be self mappings of (X,d) such that (1)$F\left(\int_0^{d(Ax, By)} \phi(t)dt, \int_0^{d(Sx, Sy)} \phi(t)dt, \int_0^{d(Sx, By)} \phi(t)dt, \int_0^{d(Sy, By)} \phi(t)dt,\right) \leq 0$.
for all (x,y)$\in X^2$, where F$\in \mathcal{F}$ and ϕ: $R_+ \to R_+$ is a Lebesque-integrable mapping which is summable, non-negative and such that (2) $\int_0^{\varepsilon} \phi(t)dt$ >0 for all ε>0.
Suppose that AX$\subset$SX and BX$\subset$SX, (A, S) and (B, S) are weakly compatible and (A, S) or (B, S) satisfies property (E.A). If the range of one of the mappings A, B or S is a closed subspace of X, then A, B and S have a unique common fixed point in X.
Proof. The proof of Corollary 4.2 follows from theorem 4.1 by putting A_i=A; A_j=B. (i$\neq$j).

Theorem4.3. Let d be a symmetric for X that satisfies (W.3),(W.4) and (H_E).Let $\{A_i\}$,B $\forall$ i be self mappings of (X,d) such that (1)$F\left(\int_0^{d(A_ix,A_iy)} \phi(t)dt, \int_0^{d(Bx,By)} \phi(t)dt, \int_0^{d(Bx,A_iy)} \phi(t)dt, \int_0^{d(A_i y,By)} \phi(t)dt,\right) \leq 0$. $\forall$ i for all (x,y)$\in X^2$, where F$\in \mathcal{F}$ and ϕ: $R_+ \to R_+$ is a Lebesque-integrable mapping which is summable, non-negative and such that (2) $\int_0^{\varepsilon} \phi(t)dt > 0$ for all $\varepsilon > 0$. Suppose that $A_iX \subset BX$ $\forall$ i, (A_i,B) $\forall$ i is weakly compatible and (A_i,B) $\forall$ i satisfy the property (E.A). If the range of one of the mappings $\{A_i\}$, or B is a closed subspace of X, then $\{A_i\}$ and B $\forall$ i have a unique common fixed point in X.

Proof. Suppose that A_j and B $\forall_j$ satisfy property (E.A). Then there exists a sequence $\{x_n\}$ in X such that that $\lim_{n\to\infty} d(A_ix_n, z) = \lim_{n\to\infty} d(Bx_n, z) = 0$ for some z $\in$ X.
Therefore, by (H_E) we have $\lim_{n\to\infty} d(A_ix_n, Bx_n) = 0$. $\forall$ i.
Since $A_iX \subset BX$ $\forall$ i, there exists in X a sequence $\{y_n\}$ such that $A_ix_n = By_n$.
Hence, $\lim_{n\to\infty} d(By_n, z) = 0$.
Let us show that $\lim_{n\to\infty} d(A_iy_n, z) = 0$. $\forall$ i.
Suppose that $\lim_{n\to\infty} d(A_iy_n, A_ix_n) > 0$. Then, using (1), we have
$F\left(\int_0^{d(A_iy_n,A_ix_n)} \phi(t)dt, \int_0^{d(By_n,Bx_n)} \phi(t)dt, \int_0^{d(By_n,A_ix_n)} \phi(t)dt, \int_0^{d(A_ix_n,Bx_n)} \phi(t)dt,\right) \leq 0$. $\forall$ i
And we have, $F\left(\lim_{n\to\infty} \int_0^{d(A_iy_n,A_ix_n)} \phi(t)dt, \lim_{n\to\infty} \int_0^{d(A_ix_n,Bx_n)} \phi(t)dt, 0, \lim_{n\to\infty} \int_0^{d(A_ix_n,Bx_n)} \phi(t)dt,\right) \leq 0$. $\forall$ i From F_1, there exists an upper semi-continuous and non-decreasing function
f: $R_+ \to R_+$ $f(0) = 0$, $f(t) < t$ for $t > 0$ such that $\lim_{n\to\infty} \int_0^{d(A_iy_n,A_ix_n)} \phi(t)dt \leq$
$f\left(\lim_{n\to\infty} \int_0^{d(A_ix_n,Bx_n)} \phi(t)dt\right) < \lim_{n\to\infty} \int_0^{d(A_ix_n,Bx_n)} \phi(t)dt$,(i≠j).
Therefore $\lim_{n\to\infty} \int_0^{d(A_ix_n,Bx_n)} \phi(t)dt, > 0$ which is a contradiction. Then we have
that $\lim_{n\to\infty} \int_0^{d(A_iy_n,A_ix_n)} \phi(t)dt = 0$. (2) implies that$\lim_{n\to\infty} d(A_iy_n, A_ix_n)$=0
By (W.4), we deduce that $\lim_{n\to\infty} d(A_iy_n, z) = 0$. $\forall$ i.
Suppose that BX is a closed subspace of X. Then z = Bu for some u$\in$ X. Consequently, we have $\lim_{n\to\infty} d(A_iy_n, A_ix_n) = \lim_{n\to\infty} d(Bx_n, Bu) = \lim_{n\to\infty} d(By_n, Bu) = \lim_{n\to\infty} d(A_ix_n, Bu) = 0$. We claim that Au = Bu. Using (1), $F\left(\int_0^{d(A_iu,Bx_n)} \phi(t)dt, \int_0^{d(Bu,Bx_n)} \phi(t)dt, \int_0^{d(Bu,A_ix_n)} \phi(t)dt, \int_0^{d(A_ix_n,Bx_n)} \phi(t)dt,\right) \leq 0$. $\forall$ i
and letting $n \to \infty$, we have $F\left(\int_0^{d(A_iu,Bx_n)} \phi(t)dt, 0,0,0\right) \leq 0$. $\forall$ i.
which is a contradiction with F_2, if $\lim_{n\to\infty} \int_0^{d(A_iu,Bx_n)} \phi(t)dt > 0$. Thus we obtain $\lim_{n\to\infty} \int_0^{d(A_iu,Bx_n)} \phi(t)dt = 0$ and (2) implies that $\lim_{n\to\infty} d(A_iu, Bx_n)$=0 $\forall$ i.
By (W.3) we have $z = A_iu = B_u$. $\forall$ i The weak compatibility of $\{A_i\}$and B $\forall$ i, implies that
$A_iB_u = BA_iu$ $\forall$ i; i.e., A_iz = Bz. $\forall$ i
The proof is similar when A_iX $\forall$ i is assumed to be a closed subspace of X, since,$A_iX \subset BX$ $\forall$ i

Uniqueness.

If A_iu=Bu =u and A_i v=Bv=v $\forall$ i and u≠v then (1) given,
$F\left(\int_0^{d(A_iu,A_iv)} \phi(t)dt, \int_0^{d(Bu,Bv)} \phi(t)dt, \int_0^{d(Bu,A_iv)} \phi(t)dt, \int_0^{d(A_iv,Bv)} \phi(t)dt,\right) \leq 0$. $\forall$ i
And we have
$F\left(\int_0^{d(u,v)} \phi(t)dt, \int_0^{d(u,v)} \phi(t)dt, \int_0^{d(u,v)} \phi(t)dt, 0\right) \leq 0$. $\forall$ i.
which is a contradiction with F_2 since$\int_0^{d(u,v)} \phi(t)dt > 0$ by (2). Thus u=v and the common fixed point is unique. This completes the proof of the theorem.

Corollary4.4: Let d be a symmetric for X that satisfies (W.3),(W.4) and (H_E).Let A,B be self mappings of (X,d) such that (1)$F\left(\int_0^{d(Ax,Ay)} \phi(t)dt, \int_0^{d(Bx,By)} \phi(t)dt, \int_0^{d(Bx,Ay)} \phi(t)dt, \int_0^{d(Ay,By)} \phi(t)dt,\right) \leq 0$.
for all (x,y)$\in X^2$, where F$\in \mathcal{F}$ and ϕ: $R_+ \to R_+$ is a Lebesque-integrable mapping which is summable, non-negative and such that (2) $\int_0^{\varepsilon} \phi(t)dt > 0$ for all $\varepsilon > 0$.
Suppose that AX$\subset$BX, (A, B) is weakly compatibles and (A, B) satisfy the property (E.A). If the range of one of the mappings A or B is a closed subspace of X, then A, B have a unique common fixed point in X.

Proof. The proof of Corollary 4.4 follows from theorem 4.3 by putting A_i=A $\forall$ i.
If $\phi(t) = 1$, A_i = A, $\forall$ i and in Corollary (4.4), we obtain Theorem 2.1 of [1].

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