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Generalized Single Integral Involving Multivariable Kampé De Fériet Function

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***Abstract**: The object of this paper is to obtain twenty five Eulerian type single integrals in the form of a general single integral involving multivariable extension of the Kampé de Fériet function [1]. The results are derived with the help of the generalized classical Watson's theorem obtained earlier by Lavoie et. Al. [2]. A few interesting special cases of our main result have also been discussed.*
***Key Words**: Multivariable Kampé de Fériet function, Generalized Watson's theorem.*

I. Introduction

We make use of following abbreviation,

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(k)} = a(a+1)\ldots(a+k-1);$$

and in what follows for the sake of brevity and elegance we recall the definition of multivariabe generalization of Kampé de Fériet function [1] in the notations of Burchnall and Chaundy [5]:

$$F^{p:q_1;\cdots;q_n}_{l:m_1;\cdots;m_n}\left[\begin{matrix}(a_p):(b'_{q_1});\cdots;(b^{(n)}_{q_n});\\(\alpha_l):(\beta'_{m_1});\cdots;(\beta^{(n)}_{m_n});\end{matrix}\; z_1;\cdots;z_n\right] = \sum_{s_1;\cdots;s_n=0}^{\infty} \Lambda(s_1;\cdots;s_n)\frac{{z_1}^{s_1}}{s_1!}\cdots\frac{{z_n}^{s_n}}{s_n!}, \quad (1.1)$$

where

$$\Lambda(s_1;\cdots;s_n) = \frac{\prod_{j=1}^{p}(a_j)_{s_1+\cdots+s_n}\prod_{j=1}^{q_1}(b'_j)_{s_1}\cdots\prod_{j=1}^{q_n}(b_j^{(n)})_{s_n}}{\prod_{j=1}^{l}(\alpha_j)_{s_1+\cdots+s_n}\prod_{j=1}^{m_1}(\beta'_j)_{s_1}\cdots\prod_{j=1}^{m_n}(\beta_j^{(n)})_{s_n}}, \quad (1.2)$$

and, for convergence of the multiple hypergeometric series in (1.1)

$$1 + l + m_k - p - q_k \geq 0, \;\; k = 1,2,\cdots n; \quad (1.3)$$

The equality holds when, in addition, either

$$|z_1|^{1/(p-l)} + \cdots + |z_n|^{1/(p-l)} < 1, \qquad \text{if } p > l \quad (1.4)$$

or

$$\max\{|z_1|,\cdots|z_n|\} < 1, \qquad \text{if } p \leq l. \quad (1.5)$$

Although the multiple hypergeometric series defined by (1.1) reduces to the Kampé de Fériet function [3] in the special case: $q_1 = \cdots = q_n$ and $m_1 = \cdots = m_n$.

The Kampé de Fériet function defined in (1.1) can be specialized to be expressed in terms of generalized hypergeometric series, among other things, as following instance:

$$F^{p:0;\cdots;0}_{l:0;\cdots;0}\left[\begin{matrix}\alpha_1,\cdots,\alpha_p\\ \beta_1,\cdots,\beta_l\end{matrix}; z_1;\cdots;z_n\right] = {}_pF_l\left[\begin{matrix}\alpha_1,\cdots,\alpha_p\\ \beta_1,\cdots,\beta_l\end{matrix}; z_1+\cdots+z_n\right]. \quad (1.6)$$

$$F^{0:q_1;\cdots;q_n}_{0:m_1;\cdots;m_n}\left[\begin{matrix}-:(b'_{q_1});\cdots;(b^{(n)}_{q_n});\\ -:(\beta'_{m_1});\cdots;(\beta^{(n)}_{m_n});\end{matrix}\; z_1;\cdots;z_n\right]$$

$$= {}_{q_1}F_{m_1}\left[\begin{matrix}b'_1,\cdots,b'_{q_1}\\ \beta'_1,\cdots,\beta'_{m_1}\end{matrix}; z_1\right]\cdots {}_{q_n}F_{m_n}\left[\begin{matrix}b^{(n)}_1,\cdots,b^{(n)}_{q_n}\\ \beta^{(n)}_1,\cdots,\beta^{(n)}_{m_n}\end{matrix}; z_n\right]. \quad (1.7)$$

For more details, see Karlsson [4, pp. 28-32].

II. Results required

The following results will be required in our present investigations.

$$\int_0^1 x^{c-1}(1-x)^{c+j-1}\;{}_2F_1\left[\begin{matrix}a,\; b\\ \frac{1}{2}(a+b+i+1)\end{matrix}; x\right]dx$$

$$= \frac{\Gamma(c)\Gamma(c+j)}{\Gamma(2c+j)}\ {}_3F_2\left[\begin{matrix} a,\ b,\ \ c \\ \frac{1}{2}(a+b+i+1), 2c+j \end{matrix}; 1\right], \tag{2.1}$$

provided $\Re(c) > 0, \Re(c+j) > 0$, and $\Re(2c-a-b+i+1) > 0$, for $i, j = 0, \pm 1, \pm 2$. The result (2.1) is a special case of a general double integral given in Erdelyi et. Al. [6, pp. 399, Eq. (5)].

Lavoie et. Al. [2] have given the generalization of the Watson's theorem on the sum of a ${}_3F_2$ and obtained the following twenty five results in the form of a single result:

$$\begin{aligned} &{}_3F_2\left[\begin{matrix} a,\ b,\ \ c \\ \frac{1}{2}(a+b+i+1), 2c+j \end{matrix}; 1\right] \\ &= A_{i,j}\frac{2^{a+b+i-2}\Gamma\left(\frac{a+b+i+1}{2}\right)2^{a+b+i-2}\Gamma\left(c+\left[\frac{j}{2}\right]+\frac{1}{2}\right)2^{a+b+i-2}\Gamma\left(c-\frac{a+b+|i+j|-j-1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(a)\Gamma(b)} \\ &\times\left\{B_{i,j}\frac{\Gamma\left(\frac{a}{2}+\frac{1-(-1)^i}{4}\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(c-\frac{a}{2}+\left[\frac{j}{2}\right]+\frac{1}{2}-\frac{(-1)^j(1-(-1)^i)}{4}\right)\Gamma\left(c-\frac{b}{2}+\left[\frac{j}{2}\right]+\frac{1}{2}\right)}\right. \\ &\left.+C_{i,j}\frac{\Gamma\left(\frac{a}{2}+\frac{1-(-1)^i}{4}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)}{\Gamma\left(c-\frac{a}{2}+\left[\frac{j+1}{2}\right]+\frac{(-1)^j(1-(-1)^i)}{4}\right)\Gamma\left(c-\frac{b}{2}+\left[\frac{j+1}{2}\right]\right)}\right\}, \end{aligned} \tag{2.2}$$

provided $\Re(2c-a-b) > -1-i-2j$, for $i, j = 0, \pm 1, \pm 2$. Here, $[x]$ is the greatest integer less than or equal to x and $|x|$ denotes the usual absolute value of x. The coefficients $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ are given respectively in [2].

III. Main Results

The following results for reducibility of multivariable Kampé de Fériet function will be established in this section.

$$\begin{aligned} &\int_0^1 x^{c-1}(1-x)^{c+j-1}\ {}_2F_1\left[\begin{matrix} a,\ b \\ \frac{1}{2}(a+b+i+1) \end{matrix}; x\right] \\ &F^{p:\rho;\cdots;\rho}_{l:\sigma;\cdots;\sigma}\left[\begin{matrix}(a_p):(b'_\rho);\cdots;(b^{(n)}_\rho); \\ (\alpha_l):(\beta'_\sigma);\cdots;(\beta^{(n)}_\sigma);\end{matrix}\ z_1x(1-x);\cdots;z_nx(1-x)\right]dx \\ &= \sum_{s_1;\cdots;s_n=0}^{\infty}\Omega(s_1;\cdots;s_n)\frac{\Gamma(c+\sum_{i=1}^n s_i)\Gamma(c+j+\sum_{i=1}^n s_i)}{\Gamma(2c+j+\sum_{i=1}^n 2s_i)} \\ &A_{i,j}\frac{2^{a+b+i-2}\Gamma\left(\frac{a+b+i+1}{2}\right)2^{a+b+i-2}\Gamma\left(c+\sum_{i=1}^n s_i+\left[\frac{j}{2}\right]+\frac{1}{2}\right)2^{a+b+i-2}\Gamma\left(c+\sum_{i=1}^n s_i-\frac{a+b+|i+j|-j-1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(a)\Gamma(b)} \\ &\times\left\{B_{i,j}\frac{\Gamma\left(\frac{a}{2}+\frac{1-(-1)^i}{4}\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(c+\sum_{i=1}^n s_i-\frac{a}{2}+\left[\frac{j}{2}\right]+\frac{1}{2}-\frac{(-1)^j(1-(-1)^i)}{4}\right)\Gamma\left(c+\sum_{i=1}^n s_i-\frac{b}{2}+\left[\frac{j}{2}\right]+\frac{1}{2}\right)}\right. \\ &\left.+C_{i,j}\frac{\Gamma\left(\frac{a}{2}+\frac{1-(-1)^i}{4}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)}{\Gamma\left(c+\sum_{i=1}^n s_i-\frac{a}{2}+\left[\frac{j+1}{2}\right]+\frac{(-1)^j(1-(-1)^i)}{4}\right)\Gamma\left(c+\sum_{i=1}^n s_i-\frac{b}{2}+\left[\frac{j+1}{2}\right]\right)}\right\}, \end{aligned} \tag{3.1}$$

provided $\Re(c) > 0$, for $j = -1, -2$; $\Re(c+j) > 0$ for $j = 0,1,2$. Also,

(i) $1 + l + \sigma - p - \rho \geq 0$,

(ii) $|z_1|^{1/(p-l)} + \cdots + |z_n|^{1/(p-l)} < 1$, if $p > l$ and $\max\{|z_1|, \cdots |z_n|\} < 1$, if $p \leq l$.

Also, the coefficients $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ can be obtain easily from the tables given in [2] by replacing c by $c+s_1+\cdots+s_n$, and $\Omega(s_1;\cdots;s_n)$ is defined as

$$\Omega(s_1;\cdots;s_n)=\frac{\prod_{j=1}^{p}(a_j)_{s_1+\cdots+s_n}\prod_{j=1}^{\rho}(b_j')_{s_1}\cdots\prod_{j=1}^{\rho}(b_j^{(n)})_{s_n}}{\prod_{j=1}^{l}(\alpha_j)_{s_1+\cdots+s_n}\prod_{j=1}^{\sigma}(\beta_j')_{s_1}\cdots\prod_{j=1}^{\sigma}(\beta_j^{(n)})_{s_n}}. \tag{3.2}$$

IV. Proof of (3.1)

To prove (3.1), we proceed as follows : Let

$$I=\int_0^1 x^{c-1}(1-x)^{c+j-1}\,{}_2F_1\left[\begin{matrix}a,\ b\\ \frac{1}{2}(a+b+i+1)\end{matrix};x\right] F^{p:\rho;\cdots;\rho}_{l:\sigma;\cdots;\sigma}\left[\begin{matrix}(a_p):(b_\rho');\cdots;(b_\rho^{(n)});\\ (\alpha_l):(\beta_\sigma');\cdots;(\beta_\sigma^{(n)});\end{matrix}\ z_1x(1-x);\cdots;z_nx(1-x)\right]dx. \tag{4.1}$$

Expressing multivariable Kampé de Fériet function in series form as defined in (1.1), we have

$$I=\int_0^1 x^{c-1}(1-x)^{c+j-1}\,{}_2F_1\left[\begin{matrix}a,\ b\\ \frac{1}{2}(a+b+i+1)\end{matrix};x\right] \sum_{s_1;\cdots;s_n=0}^{\infty}\Omega(s_1;\cdots;s_n)[x(1-x)]^{s_1}\cdots[x(1-x)]^{s_1}\frac{z_1^{s_1}}{s_1!}\cdots\frac{z_n^{s_n}}{s_n!}\,dx, \tag{4.2}$$

where $\Omega(s_1;\cdots;s_n)$ is given with (3.2).
Changing the order of integration and summation which is justified due to uniformly convergence of the series, we obtain

$$I=\sum_{s_1;\cdots;s_n=0}^{\infty}\Omega(s_1;\cdots;s_n)\frac{z_1^{s_1}}{s_1!}\cdots\frac{z_n^{s_n}}{s_n!} \times\left\{\int_0^1 x^{c+\sum_{i=1}^{n}s_i-1}(1-x)^{c+\sum_{i=1}^{n}s_i+j-1}\,{}_2F_1\left[\begin{matrix}a,\ b\\ \frac{1}{2}(a+b+i+1)\end{matrix};x\right]dx\right\}, \tag{4.3}$$

which, upon using (2.1), becomes

$$I=\sum_{s_1;\cdots;s_n=0}^{\infty}\Omega(s_1;\cdots;s_n)\frac{z_1^{s_1}}{s_1!}\cdots\frac{z_n^{s_n}}{s_n!}\frac{\Gamma(c+\sum_{i=1}^{n}s_i)\Gamma(c+j+\sum_{i=1}^{n}s_i)}{\Gamma(2c+j+\sum_{i=1}^{n}2s_i)} \times\ {}_3F_2\left[\begin{matrix}a,\ b,\ \ c+\sum_{i=1}^{n}s_i\\ \frac{1}{2}(a+b+i+1),2c+\sum_{i=1}^{n}2s_i+j\end{matrix};1\right].$$

By making use of (2.2) and replacing c by $c+s_1+\cdots+s_n$, we finally arrive at the right-hand side of (3.1). This completes the proof of (3.1).

V. Special cases

In this section, we shall mention some of the interesting special cases of our main result (3.1).

(i) If we take $i=j=0$ in (3.1), then we have, after a little simplification, the following transformation formula:

$$\int_0^1 x^{c-1}(1-x)^{c-1}\,{}_2F_1\left[\begin{matrix}a,\ b\\ \frac{1}{2}(a+b+1)\end{matrix};x\right] F^{p:\rho;\cdots;\rho}_{l:\sigma;\cdots;\sigma}\left[\begin{matrix}(a_p):(b_\rho');\cdots;(b_\rho^{(n)});\\ (\alpha_l):(\beta_\sigma');\cdots;(\beta_\sigma^{(n)});\end{matrix}\ z_1x(1-x);\cdots;z_nx(1-x)\right]dx$$
$$=\frac{2^{a+b+2c-1}\,\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)\Gamma(c)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma(a)\,\Gamma(b)\,\Gamma\left(c-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{b}{2}+\frac{1}{2}\right)} \times F^{p+2:\rho;\cdots;\rho}_{l+2:\sigma;\cdots;\sigma}\left[\begin{matrix}(a_p),c,c+\frac{1-a-b}{2}:(b_\rho');\cdots;(b_\rho^{(n)});\\ (\alpha_l),c+\frac{1-a}{2},c+\frac{1-b}{2}:(\beta_\sigma');\cdots;(\beta_\sigma^{(n)});\end{matrix}\ \frac{z_1}{4};\cdots;\frac{z_n}{4}\right], \tag{5.1}$$

provided that the conditions easily obtainable from (3.1) are satisfied.

(ii) In (3.1), if we take $i = 0; j = -1$ then we have, after a little simplification, the following transformation formula:

$$\int_0^1 x^{c-1}(1-x)^{c-2}\ {}_2F_1\left[\begin{matrix} a,\ b \\ \frac{1}{2}(a+b+1) \end{matrix};x\right]$$

$$F^{p:\rho;\cdots;\rho}_{l:\sigma;\cdots;\sigma}\left[\begin{matrix}(a_p):(b'_\rho);\cdots;(b^{(n)}_\rho); \\ (\alpha_l):(\beta'_\sigma);\cdots;(\beta^{(n)}_\sigma);\end{matrix}\ z_1x(1-x);\cdots;z_nx(1-x)\right]dx$$

$$= \frac{2^{a+b+2c}\,\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)\Gamma(c-1)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}-\frac{1}{2}\right)}{\Gamma(a)\,\Gamma(b)\,\Gamma\left(c-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{b}{2}-\frac{1}{2}\right)}$$

$$\times F^{p+2:\rho;\cdots;\rho}_{l+2:\sigma;\cdots;\sigma}\left[\begin{matrix}(a_p), c-1, c-\frac{1+a+b}{2}:(b'_\rho);\cdots;(b^{(n)}_\rho); \\ (\alpha_l), c-\frac{1+a}{2}, c-\frac{1+b}{2}:(\beta'_\sigma);\cdots;(\beta^{(n)}_\sigma);\end{matrix}\ \frac{z_1}{4};\cdots;\frac{z_n}{4}\right]$$

$$+\frac{2^{a+b-2c}\,\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)\Gamma(c-1)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}-\frac{1}{2}\right)}{\Gamma(a)\,\Gamma(b)\,\Gamma\left(c-\frac{a}{2}\right)\Gamma\left(c-\frac{b}{2}\right)}$$

$$\times F^{p+2:\rho;\cdots;\rho}_{l+2:\sigma;\cdots;\sigma}\left[\begin{matrix}(a_p), c-1, c-\frac{1+a+b}{2}:(b'_\rho);\cdots;(b^{(n)}_\rho); \\ (\alpha_l), c-\frac{a}{2}, c-\frac{b}{2}:(\beta'_\sigma);\cdots;(\beta^{(n)}_\sigma);\end{matrix}\ \frac{z_1}{4};\cdots;\frac{z_n}{4}\right], \quad (5.2)$$

provided that the conditions easily obtainable from (3.1) are satisfied.

(iii) If we take $i = 0; j = 1$ in (3.1), then we have, after a little simplification, the following transformation formula:

$$\int_0^1 x^{c-1}(1-x)^{c-2}\ {}_2F_1\left[\begin{matrix} a,\ b \\ \frac{1}{2}(a+b+1) \end{matrix};x\right]$$

$$F^{p:\rho;\cdots;\rho}_{l:\sigma;\cdots;\sigma}\left[\begin{matrix}(a_p):(b'_\rho);\cdots;(b^{(n)}_\rho); \\ (\alpha_l):(\beta'_\sigma);\cdots;(\beta^{(n)}_\sigma);\end{matrix}\ z_1x(1-x);\cdots;z_nx(1-x)\right]dx$$

$$= \frac{2^{a+b-2c-2}\,\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)\Gamma(c-1)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma(a)\,\Gamma(b)\,\Gamma\left(c-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{b}{2}+\frac{1}{2}\right)}$$

$$\times F^{p+2:\rho;\cdots;\rho}_{l+2:\sigma;\cdots;\sigma}\left[\begin{matrix}(a_p), c, c+\frac{1-a-b}{2}:(b'_\rho);\cdots;(b^{(n)}_\rho); \\ (\alpha_l), c+\frac{1-a}{2}, c+\frac{1-b}{2}:(\beta'_\sigma);\cdots;(\beta^{(n)}_\sigma);\end{matrix}\ \frac{z_1}{4};\cdots;\frac{z_n}{4}\right]$$

$$-\frac{2^{a+b-2c-2}\,\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)\Gamma(c)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}-\frac{1}{2}\right)}{\Gamma(a)\,\Gamma(b)\,\Gamma\left(c-\frac{a+1}{2}\right)\Gamma\left(c-\frac{b}{2}+1\right)}$$

$$\times F^{p+2:\rho;\cdots;\rho}_{l+2:\sigma;\cdots;\sigma}\left[\begin{matrix}(a_p), c, c+\frac{1-a-b}{2}:(b'_\rho);\cdots;(b^{(n)}_\rho); \\ (\alpha_l), c-\frac{a}{2}+1, c-\frac{b}{2}+1:(\beta'_\sigma);\cdots;(\beta^{(n)}_\sigma);\end{matrix}\ \frac{z_1}{4};\cdots;\frac{z_n}{4}\right], \quad (5.3)$$

provided that the conditions easily obtainable from (3.1) are satisfied. Similarly other results can also be obtained.

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