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# Integral Collocation Approximation Methods for the Numerical Solution of Linear Integro Differential Equations.

[1]Taiwo, O. A and [2] Abubakar, A
[1]*Department of Mathematics, University of Ilorin, Nigeria.*
[2]*Department of Mathematics and Computer Science, Ibrahim Badamasi Babangida University, Lapai, Nigeria.*

***Abstract:*** *In this paper, we employed Standard and Perturbed Integral Collocation methods to find numerical solution of ordinary differential equations. Power series form of approximation is used as basis function and Chebyshev Polynomial is used as perturbation term in the case of perturbed Integral Collocation Method. Numerical Computations are carried out to illustrate the application of the methods and also the results obtained by the methods are compared in terms of accuracy and computations involved in the two methods. Two examples each of first and second orders linear integro differential equations are solved to demonstrate the methods.*

## I. Introduction

Integro differential equation (IDE) is an important branch of modern mathematics and a arise frequently in many applied areas which include engineering , mechanics, physics ,astronomy, biology ,economics, potential theory and electrostatic[2].

This paper concerns the development of integral collocation approximation methods by power series as our basis function for the solution of first order integro differential equations .

Many different method have been used to obtain the solution of linear and nonlinear integro differential equations ,such methods include the Successive Approximation Method, Homotopy Perturbation Method [6], Adomian Decomposition Method (ADM [13], Wavelet Methods (See [4],[5] ) and so on , to mention just a few . [1] used the integrated formulation of the Tau method and Error Estimation for over determined differential equations which actually motivated the beginning of this work.

For the purpose of our discussion, we consider the general nth order integro differential equation of the form :

$$Ly(x) \cong \sum_{i=0}^{n}\left(p_i x^i \frac{d^i}{dx^i}\right)y(x) + \int_a^b k(x,t)y(t)dt = f(x) \quad (1)$$

Subject to the conditions

$$Ly(x) \cong \sum_{i=0}^{N} a_i y^i(x_i) = \alpha_k \quad (2)$$

Where $p_i$ , are given real numbers, $x_i$ are point belonging to the internal $a \le x \le b$ at which the conditions are satisfied , $f(x)$ can be polynomial or transcendental or hyperbolic functions and or $(\gamma \ge o)$ are constant to be determined.

Here, we let

$$\iiint ..k...\int g(x)dx$$

Denote the indefinite integration applied to the $g(x)$ $k$ times and denote $I_L = \int\int\int .....n....\int L(.)dx$

## II. Numerical solution Techniques

**Method1. Standard Integral Collocation Method**

This method used the idea of [1] as applied to solve non over determined differential equations.

In [1], $f(x)$ in (1) is a polynomial of degree less or equal to the order of or the differential equation considered .these methods handle non polynomial or transcendental function which is an added advantage over [1] (See[11], [12]).

The method does not require truncation or approximation of non polynomial functions.

Also the problem of over determination does not arise in these methods. Without loss of generality, we integrated both sides of (1), we have

$$I_L \quad = \int \int \int ...............n.................\int [f(x) - \int_a^b k(x,t)dt]dx \qquad (4)$$

This implies,

$$\int \int \int ..........n.........\int \sum_{i=o}^{n} (p_i x^i \frac{d^i}{dx^i})dx = \int \int \int .........n.......\int [f(x) - \int_a^b k(n,t)y(t)dt]dx \qquad (5)$$

We assumed an approximation solution of the form

$$y(x) = y_N(x) = \sum_{r=o}^{N} a_r x^i \qquad (6)$$

Thus, (6) is substituted into (5), we have

(7)

Hence, we collocate (7) at point to have

$$\int \int \int ........n.........\int \ (\sum_{i=o}^{n} p_i x^i \frac{d^i}{dx^i} y_N(x_K)dx_K$$
$$= \int \int \int .......n.......\int [f(x_K) - \int_a^b k(x_K,t)y_N(t)dt]dx_K \qquad (8)$$

Where,

$$x_K = a + \frac{(b+a)k}{N+1}; \quad k = 1,2.................n+1 \qquad (9)$$

Thus (8) gives $(N+1)$ algebraic linear system of equations in $(N+1)$ unknown constants $a_r \quad (r \geq o)$ . These $(N+1)$ algebraic linear equations are then solved by Gaussian Elimination method to obtain the unknown constants $a_r \quad (r \geq o)$ which are then substituted back into (6) to obtain the approximate solution for the value of N.

Method 2: **Perturbed Integral Collocation Method**.

The perturbed integral collocation method is an attempt to improve the accuracy and efficiency of the standard integral collocation method .
In order to apply this method, we employed the ideas of [11, 12]and the approximation solution (6) is substituted into a slightly perturbed (7) to get

$$\int \int \int ........n.........\int \sum_{i=o}^{n} (p_i x^i \frac{d^i}{dx^i}) y_N(x)dx$$
$$= \int \int \int ......n.......\int [(x) - \int_a^b k(x_1 t)y_N(t)dt]dx + H_n(x) \qquad (10)$$

Together with the conditions

$$L^* y_N(x_K) = \alpha_K; k = (1)n \qquad (11)$$

Where $H_n(x) = \sum_{r=o}^{n} \tau_r T_r(x)$ (12)

And $T_r(x)$ are the Chebyshev Polynomials valid in $a \leq x \leq b$ and defined by

$$T_r(x) = \cos[n\cos^{-1}(\frac{2x-a-b}{b-a})]; \qquad a \leq x \leq b \qquad (13)$$

Substituting (12) into (10), we have

$$\int \int \int .......n........\int \sum_{i=o}^{n} (p_i x^i \frac{d^i}{dx^i}) y_N(x)dx$$
$$\int \int \int .......n........\int \sum_{i=o}^{n} (p_i x^i \frac{d^i}{dx_k{}^i}) y_N(x)dx \qquad (14)$$

$$=\int\int\int .......n.......\int [f(x)-\int_a^b k(x,t)y_N(t)dt]dx+\tau_1T_1(x)+.......+\tau_NT_N(x)$$

Thus, (14) is collocated at the point $x = x_K$ ; we have

$$\int\int\int .......n........\int \sum_{i=o}^{n} (p_i x_k{}^i \frac{d^i}{dx^i}) y_N(x_k)dx$$

$$\int\int\int .......n........\int \sum_{i=o}^{n} (p_i x_k{}^i \frac{d^i}{dx_k{}^i}) y_N(x_k)dx$$

$$=\int\int\int .......n.......\int [f(x_k)-\int_a^b k(x_k,t)y_N(t)dt]dx+\tau_1T_1(x_k)+.......+\tau_NT_N(x_k)$$

Where,

$$x_K = \quad a+\frac{(b-a)k}{N+n+1} \qquad ;k=1,2,..........N+n+1 \tag{16}$$

Hence, we have $(N+n+1)$ algebraic system of linear equations in $(N+n+1)$ unknown constants $(a_r(r \geq o),\ \tau_1,\ \tau_{2,.,\ .........,\ ,}\tau_N)$. The linear equations are then solved by Gaussian Elimination method to obtain $(N+n=1)$ unknown constants which are then substituted back into (6) to obtain the approximate solution for value of N.

**Demonstration with first order integro differentiation equation**

We consider here case n=1 in equation (1). Thus, we have

$$p_0 y(x) \quad + \ p_1 xy\,'(x) + \int_a^b k(x,t)\,y(t)dt \,=\, f(x) \tag{17}$$

We integrated the terms in (17) to have

$$\int_o^x p_o y(t)dt + \int_o^x p_1 t y'(t)dt + \int_o^x \int_a^b k(z,t)y(t)dtdz = \int_o^x f(z)dz$$

$$p_o \int_o^x y(t)dt + p_1 \int_o^x ty(t)dt + \int_o^x \int_o^b k(z,t)y(t)dtdz = \int_o^x f(z)dz \tag{18}$$

Hence, evaluated the integrals in (18), we have

$$p_o \int_0^x y(t)dt + p_1[[ty(t)]_o^x - \int_o^x y(t)dt] + \int_o^x \int_a^b k(z,t)y(t)dtdz = \int_o^x f(z)dz$$

This implies,

$$p_o \int_o^x y(t)dt + p_1[xy(x) - \int_o^x y(t)dt] + \int_o^x \int_o^b k(z,t)y(t)dtdz = \int_o^x f(z)dz \tag{19}$$

Substituting the approximation solution (6) into (19), we have

$$p_o \int_o^x \sum_{r=o}^{N} a_r t^r dt + p_1[x\sum_{r=o}^{N} a_r x^r - \int_o^x \sum_{r=o}^{N} a_r t^r dt] + \int_o^x \int_a^b k(z,t)\sum_{r=o}^{N} a_r t^r dtdz = \int_o^x f(z)dz$$

This implies,

$$p_o \sum_{r=o}^{N} a_r \frac{x^{r+1}}{r+1} + p_1[\sum_{r=o}^{N} a_r x^{r+1} - \sum_{r=o}^{N} a_r \frac{x^{r+1}}{r+1}] + \int_o^x \int_o^b k(z,t)\sum_{r=o}^{N} a_r t^r dtdz = \int_o^x f(z)dz$$

Further simplification gives

$$\sum_{r=o}^{N} (p_o + rp_1)a_r \frac{x^{r+1}}{r+1} + \int_o^x \int_a^b k(z,t)\sum_{r=o}^{N} a_r t^r dtdz = \int_o^x f(z)dz$$

This implies

$$\sum_{r=o}^{N} (p_o + rp_1)a_r \frac{x^{r+1}}{r+1} + G(a,x) = F(x) \tag{20}$$

Where,

$$G(a,x)=\int_o^x \int_a^b k(z,t)\sum_{r=o}^{N} a_r t^r \, dtdz$$

And

$$F(x)=\int_o^x f(z)dz$$

Thus, from (20), we have.

$$(p_o+4p_1)a_4\frac{x^5}{5}+.........+(p_o+Np_1)a_N\frac{x^{N+1}}{N+1}+G(a,x)=f(x) \qquad (21)$$

Hence, we collocated (21) at the point $x=x_{K,}$ to have

$$p_o a_o x_K+(p_o+p_1)a_1\frac{x_k{}^2}{2}+(p_o+2p_1)a_2\frac{x_k{}^3}{3}+(p_o+3\text{p}_1)a_3\frac{x_k{}^4}{4}+ \qquad (22)$$

$$(p_o+4p_1)a_4\frac{x_k{}^5}{5}+.........+(p_o+Np_1)a_N\frac{x_k{}^{N+1}}{N+1}+G(a,x_k)=F(x_k),$$

Where,

$$x_i = a + \frac{(b-a)k}{N+1}, \quad k=1.\,2, \quad ,N+1$$

Thus (22) is put in matrix form as

$$Ax=b \qquad (23)$$

Where,

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & . \,........ & A_{1N} \\ A_{21} & A_{22} & A_{23} & ............ & A_{2N} \\ A_{31} & A_{32} & A_{33} & .\,............ & A_{3N} \\ . & . & . & . & . \\ . & . & . & . & . \\ A_{N1} & A_{N2} & A_{N3} & ......... & A_{NN} \end{bmatrix}, x=\begin{bmatrix} a_o \\ a_1 \\ a_2 \\ a_N \end{bmatrix} \text{ and } b=\begin{bmatrix} F(x_1) \\ F(x_2) \\ F(x_3) \\ . \\ . \\ F(x_{N+1}) \end{bmatrix}$$

Where,

$$A_{11}=p_0x_1+G(x_1), \ \backslash$$

$$A_{12}=(p_0+p_1)\frac{x_1{}^2}{2}+G(x_1)$$

$$A_{13}=(P_O+2P_1)\frac{x_1{}^3}{3}+G(x_1)$$

:

$$A_{1N}=(p_o+Np_1)\frac{x_1{}^{N+1}}{N+1}+G(x_1)$$

$$A_{21}=p_ox_2+G(x_2)$$

$$A_{22}=(p_o+p_1)\frac{x_2{}^2}{2}+G(x_2)$$

$A_{23} = (p_o + 2p_1)\dfrac{x_2^{\ 3}}{3} + G(x_2)$

:

$A_{2N} + (p_o + Np_1)\dfrac{x_2^{\ N+1}}{N+1} + G(x_2)$

$A_{31} = p_o x_3 + G(x_3) \quad A_{32} = (p_o + p_1)\dfrac{x_3^{\ 2}}{2} + G(x_3)$

$A_{33} = (p_o + 2p_1)\dfrac{x_3^{\ 3}}{3} G(x_3)$

:

$A_{3\mathrm{N}} = (p_o + NP_1)\dfrac{x_3^{\ N+1}}{\mathrm{N}+1} + G(x_3)$

:

:

:

:

$A_{N1} = p_o x_N + G(x_N)$

:

:

:

$A_{NN} = (p_o + p_1)\dfrac{x_N^{\ 2}}{2} \;\; G(x_N)$

Remarks, The matrix is solved by Gaussian Elimination method to obtain the unknown constants $a_i (i \geq o)$ which are then substituted into the approximate solution (6)

**Demonstration of method for case n=2**

$$p_o y(x) + p_1 x y'(x) + p_2 x^2 y''(x) + \int_a^b k(x,t) y(t) dt = f(x) \tag{24}$$

We integrated the terms in (24) to have

$$p_o \int_o^x \int_o^u y(t) dtdu + p_1 \int_o^x \int_o^u t y'(t) dtdu + p_2 \int_o^x \int_o^u t^2 y''(t) dtdu \tag{25}$$

$$+ \int_o^x \int_o^u \int_a^b k(z,t) y(t) dtdudz = \int_o^x \int_o^u f(z) dtdu$$

We simplified each term of (25) to have

$$p_o \int_o^x \int_o^u y(t) dtdu = p_o \int_o^x \int_o^u \sum_{r=o}^{N} a_r t^r dt = p_o \int_o^x \sum_{r=o}^{N} a_r \frac{u^{r+1}}{r+1} = p_o \sum_{r=o}^{N} a_r \frac{u^{r+1}}{(r+1)(r+2)} \tag{26}$$

$$p_1 \int_0^z \int_0^u t y'(t) dtdu \quad = \quad p_1 \sum_{r=o}^{N} a_r \frac{x^{r+2}}{(r+1)(r+2)} \tag{27}$$

$$p_2 \int \int t^2 y''(t) dtdu = p_2 \sum_{r=o}^{N} \frac{(r^2 - r)}{(r+1)(r+2)} a_r x^{r+2} \tag{28}$$

We let

$$\int_o^x \int_o^u \int_a^b k(z,t) \sum_{r=o}^{N} a_r t^r dtdudz = G(a,x) \tag{29}$$

And

$$\int_o^x \int_o^u f(z)dzdu = f(x) \qquad (30)$$

We substituted (26) - (30) into (25), we have

$$p_o \sum_{r=o}^{N} a_r \frac{U^{r+2}}{(r+1)(r+2)} + p_1 \sum_{r=o}^{N} a_r \frac{x^{r+2}}{(r+1)(r+2)} + p_2 \sum_{r=o}^{N} \frac{(r^2-r)}{(r+1)(r+2)} a_r x^{r+2} + G(a,x) = F(x)$$

Thus, from (30) we have

$$p_o \left[ \frac{x^2}{2} a_o + \frac{x^3}{6} a_1 + \frac{x^4}{12} a_2 + \frac{x^5}{20} a_3 + ....... + \frac{x^{N+2}}{(N+1)(N+2)} a_N \right] \qquad (31)$$

$$+ p_2 \left[ \frac{x^4}{6} a_2 + \frac{3x^5}{10} a_3 + \frac{2}{5} a_4 + ..... + \frac{(N^2-N)x^{N+2}}{(N+1)(N+2)} a_N \right]$$

$$+ G(a,x) = F(x)$$

Further simplification of (31) gives

$$p_o \frac{x^2{}_k}{2} a_o + (p_o + p_1) \frac{x^3{}_k}{6} a_1 + \frac{5}{12}(p_o + p_1 + p_2) x^4{}_k a_2 + (p_o + p_1 + p_2) \frac{x^5{}_k}{2} a_3$$

$$+ \quad .......... + (p_o + p_1 + p_2) \frac{(N^2-N)x^{N+2}}{(N+1)(N+2)} a_N + G(a,x) = F(x)$$

(32)

Hence, we collocated (32) at the point $x = x_k$ to have

$$p_o \frac{x_k^{\ 2}}{2} a_o + (p_o + p_1) \frac{x_k^{\ 3}}{6} a_1 + \frac{5}{12}(p_o + p_1 + p_2) x_k^{\ 4} a_2 + (p_o + p_1 + p_2) \frac{x_k^{\ 5}}{2} a_3 +$$

$$........ + (p_o + p_1 + p_2) \frac{(N^2-N)x_k^{N+2}}{(N+1)(N+2)} a_N + G(a,x_k) = F(x_k) \qquad (33)$$

Where $x_k = a + \frac{(b-a)k}{N+1}, \ \ k = 1,2,3, .......\ \mathrm{N}+1.$

Thus, (33) is put in matrix form as described in (23), we have

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \ ... & A_{1N} \\ A_{21} & A_{22} & A_{23} \ ... & A_{2N} \\ A_{31} & A_{32} & A_{33} \ ... & A_{3\mathrm{N}} \\ : & : & : & : \\ : & : & : & : \\ A_{N1} & A_{N2} & A_{N3} \ ... & A_{NN} \end{bmatrix}, \ X = \begin{bmatrix} a_o \\ a_1 \\ a_2 \\ : \\ : \\ a_N \end{bmatrix} \quad and \quad b = \begin{bmatrix} F(x_1) \\ F(x_2) \\ F(x_3) \\ : \\ : \\ F(x_{N+1}) \end{bmatrix}$$

Where,

$$A_{11} = \frac{p_0}{2} x_1^{\ 2} + G(x_1),$$

$$A_{12} = (p_0 + p_1) x_1^{\ 3} + G(x_1)$$

$$A_{13} = \frac{5}{12}(p_0 + p_1 + p_2) x_1^{\ 4} + G(x_1)$$

:

:

$$A_{1N} = \frac{(p_0 + p_1 + p_2)(N^2 + 1)}{(N+1)(N+2)} \quad x_1^{\;N+1} + G(x_1)$$

$$A_{21} = \frac{p_0}{2} x_2^{\;2} + G(x_2)$$

$$A_{22} = (P_0 + p_1)x_2^{\;3} + G(x_2)$$

$$A_{23} = \frac{5}{12}(p_0 + p_1 + p_2)x^4{}_2 + G(x_2)$$

:

:

$$A_{2N} = \frac{(P_0 + P_1 + p_2)(N^2 + 1)}{(N+1)(N+2)} x_2^{\mathrm{N+1}} + G(x_2)$$

$$A_{31} = \frac{p_o}{2} x\frac{2}{3} + G(x_3)$$

$$A_{32} = (p_o + p_1)x\frac{3}{3} + G(x_3)$$

$$A_{33} = \frac{5}{12}(p_o + p_1 + p_2)x\frac{4}{3} + G(x_3)$$

:

:

$$A_{3\mathrm{N}} = \frac{(p_o + p_1 + p_2)(N^{2+1}}{(N+1)(N+2)} x_3^{\mathrm{N+1}} + G(x_3)$$

$$A_{N1} = \frac{p_o}{2} x_{N+1}^2 + G(x_N + 1)$$

$$A_{N2} = (p_o + p_1)x_R^3 + G(x_N + 1)$$

$$A_{N3} = \frac{5}{12}(p_o + p_1 + p_2)x_{N+1}^4 + G(x_{N+1})$$

:

:

$$A_{NN} = \frac{(p_o + p_1 + p_2)(N^2 + 1)}{(N+1)(N+2)} x_{N+1}^{N+1} + G(x_{N+1})$$

Remark:

The matrix is solved by Gaussian Elimination method to obtain the unknown constants $a_i (i \geq 0)$ which are then substituted into the approximate solution (6)

### 2.1.2 Perturbed Integral Collocation Method

An attempt to improve the accuracy of the standard integral collocation approximation $y_N(x)$ in section (2.1.1) is the focus of this section.

In this method, the approximate solution (6) is substituted into a slightly perturbed equation (5) to get

$$\int\int\int \ldots\ldots n \ldots\ldots \int \sum_{i=o}^{n} \left( p_i x^i \frac{d^i}{dx^i} \right) y_N(x)dx = \qquad (34)$$

$$\int\int\int \ldots\ldots n \ldots\ldots \int \left[ f(x) - \int_a^b k(x,t)dt \right] dx + H_N(x)$$

Together with the condition

$$L^* y_N(x_{rk}) = \alpha_k \quad , k = 1(1)n \tag{35}$$

And where,

$$H_N(x) = \sum_{i=o}^{N} \tau_r T_r(x) \tag{36}$$

Substituting (36) into (38), we obtain

$$\int\int\int ......n........\int \sum_{i=o}^{n} \left( p_i x^i \frac{d^i}{dx^i} \right) y_N(x)dx$$
$$= \int\int\int ......n.........\int \left[ f(x) - \int_a^b k(x,t)y(t)dt \right] dx + \tau_1 T_1(x) + ..... + \tau_2 T_N(x) \tag{37}$$

Thus, (37) is collocated at the point $x = x_{k,}$ we have

$$\int\int\int ......n........\int \sum_{i=o}^{n} \left( p_i {x_k}^i \frac{d^i}{dx^i} \right) y_N(x_k)dx$$
$$= \int\int\int ......n.........\int \left[ f(x_k) - \int_a^b k(x_k,t)y(t)dt \right] dx + \tau_1 T_1(x_k) + ..... + \tau_2 T_N(x_k)$$

Where,

$$x_k = a + \frac{(b-a)k}{N+1} \quad , k = 1,2,3,........\text{N} + \text{n} + 1 \tag{39}$$

Remarks

The conditions are taken care after the evaluation of the integral in (38). Altogether, we obtained (N+n+1) algebraic systems of linear equations in (N+n+1) unknown constants $(a_i (i \geq o) \;\; \tau_1 \tau_{2......,} \tau_N)$. These linear equation are then solved by Gaussian Elimination method to obtain (N+n+1) unknown constants which are substituted into (6) to obtain the approximation solution for the value of N.

To demonstrate this method further, we consider case n=1 in (1). After all simplification as in solution (2.1.1), we slightly perturbed (20).to obtain

$$\sum_{r=0}^{N} (p_0 + p_1) a_r \frac{x^{r+1}}{r+1} + G(a,x) = F(x) + \tau_1 T_1(x) + ....... + \tau_N T_N(x) \tag{40}$$

Where $G(a,x)$ and $F(x)$ are as defined in (6).

Thus, we have

$$P_0 a_0 x + (P_0 + P_1) a_1 \frac{x^2}{2} + (P_0 + 2P_1) a_2 \frac{x^3}{3} + (P_0 + 3\text{P}_1) a_3 \frac{x^4}{4} + (P_0 + 4P_1) a_4 \frac{x^5}{5} + ...... + \tag{41}$$
$$(p_o + Np_1) a_N \frac{x_k^{N+1}}{N+1} + G(a, x_k) = F(x_k) + \tau_1 T_1(x_k) + ........... + \tau_N T_N(x_k)$$

Where

$$x_k = a + \frac{(b-a)k}{N+n+1\ ,} \quad k = 1,2,3,...........\text{N} + n \tag{42}$$

Thus (41) is put in matrix form described in (23) where

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \ldots & A_{1N} \\ A_{21} & A_{22} & A_{23} \ldots & A_{2N} \\ A_{31} & A_{32} & A_{33} \ldots & A_{3\text{N}} \\ : & : & : & : \\ : & : & : & : \\ A_{N1} & A_{N2} & A_{N3} \ldots & A_{NN} \end{bmatrix}, \; X = \begin{bmatrix} a_o \\ a_1 \\ a_2 \\ : \\ : \\ a_N \end{bmatrix} \; and \;\; b = \begin{bmatrix} F(x_1) \\ F(x_2) \\ F(x_3) \\ : \\ : \\ F(x_{N+n}) \end{bmatrix}$$

Where,,

$$A_{11} = p_o x_1 + G(x_1),$$

$$A_{12} = (p_o + p_1)\frac{x_1^2}{2} + G(x_1)$$

$$A_{13} = (p_o + 2p_1)\frac{x_1^3}{3} + G(x_1)$$

$$A_{1N} = (p_o + Np_1\frac{x_1^{N+1}}{N+1} + G(x_1) - (C_o^{(1)} + C_1^{(1)} x_1)$$

$$A_{21} = p_o x_2 + G(x_2)$$

$$A_{22} = (p_o + p_1)\frac{x_2^2}{2} + G(x_2)$$

$$A_{23} = \left(p_o + 2p_1\right)\frac{x_2^3}{3} + G\left(x_2\right)$$

$$A_{31} = p_o x_3 + G\left(x_3\right)$$

$$A_{32} = (p_o + p_1)\frac{x_3^2}{2} + G\left(x_3\right)$$

$$A_{33} = \left(p_o + 2p_1\right)\frac{x_3^3}{3} + G\left(x_3\right)$$

$$A_{3\mathrm{N}} = \left(p_o + Np_1\right)\frac{x_3^{\mathrm{N}+1}}{N+1} + G\left(x_3\right) - \left(C_o^{(1)}\right) - \left(C_o^{(1)} x_3\right)$$

$$A_{N1} = p_o x_N + G\left(x_N\right)$$

$$A_{N2} = \left(p_o + p_1\right)\frac{x_N^2}{2} + G\left(x_N\right)$$

$$A_{N3} = \left(p_o + 2p_1\right)\frac{x_N^3}{3} + G\left(x_N\right)$$

$$A_{NN} = \left(p_o + Np_1\right)\frac{x_N^{N+1}}{N+1} + G\left(x_N\right) - \left(C_o^{(1)}\right) + \left(C_1^{(1)} x_N\right)$$

$$A_{N+n,1} = P_0 x_R + G(x_{N+n})$$

$$A_{N+n,2} = (P_0 + P_1)\frac{x_R^2}{2} + G(x_{N+n})$$

$$A_{N+n,3} = \left(P_0 + 2P_1\right)\frac{x_\mathrm{R}^3}{3} + G(x_{N+n})$$

$$A_{N+n,N} = (P_0 + NP_1)\frac{x_{N+n}^{N+1}}{N+1} + G(x_{N+n}) - (C_0^{(1)}) + (C_1^{(1)} x_{N+n})$$

Remark:

The matrix is solved by Gaussian Elimination method to obtain the unknown constants $(a_i (i \geq 0) \;\; \tau_1, \tau_2, ...., \tau_N)$ which are then substituted into the approximate solution (6)

**Demonstration of method for case n =2**

Thus after all simplifications as in section (2.1.1), (32) is slightly perturbed to obtain

$$p_o \frac{x^2}{2} a_o + (p_o + p_1)\frac{x^3}{2} a_1 + \frac{5}{12}(p_o + p_1 + p_2)x^4 a_2 + (p_o + p_1 + p_2)\frac{x^5}{2} a_3 \quad + .......$$

$$+ (p_o + p_1 + p_2)\frac{(N^2+1)x^{N+1}}{(N+1)(N+2)} a_N + G(a,x) = F(x) + \tau_2 T_N(x) + .......\tau_N T_N(x) \qquad (43)$$

Hence, we collocated (43) at the point $x = x_k$ to have

$$P_0 \frac{x^2}{2} a_0 + (P_0 + P_1)\frac{x_k^3}{6} a_1 + \frac{5}{12}(P_0 + P_1 + P_2)x_k^4 a_2 + (P_0 + P_1 + P_2)\frac{x_k^5}{2} a_3 + ......$$

$$+ (P_0 + P_1 + P_2)\frac{(N^2+1)x_k^{N+2}}{(N+1)(N+2)} a_N + G(a, x_k) = F(x_k) + \tau_1 T_1(x_k) + ...... + \tau_N T_N(x_k) \qquad (44)$$

Where $x_k = a + \frac{(b-a)k}{N+3}, \qquad k = 1,2,3,..., \mathrm{N}+2.$ Thus, (44) is put in matrix form as described in (23), where

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \ldots & A_{1N} \\ A_{21} & A_{22} & A_{23} \ldots & A_{2N} \\ A_{31} & A_{32} & A_{33} \ldots & A_{3\mathrm{N}} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ A_{N1} & A_{N2} & A_{N3} \ldots & A_{NN} \end{bmatrix}, \quad X = \begin{bmatrix} a_o \\ a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_N \end{bmatrix} \; and \, b = \begin{bmatrix} F(x_1) \\ F(x_2) \\ F(x_3) \\ \vdots \\ \vdots \\ F(x_{N+2}) \end{bmatrix}$$

and

$$A_{11} = \frac{p_o}{2} x_1^2 + G(x_1),$$

$$A_{12} = (p_o + p_1)\frac{x_1^2}{6} G(x_1)$$

$$A_{13} = \frac{5}{12}(p_o + p_1 + p_2)x_1^4 + G(x_1)$$

$$A_{14} = (p_o + p_1 + p_2)\frac{x_1^5}{2} + G(x_1)$$

$$A_{1N} = \frac{(p_o + p_1 + p_2)(N^2+1)}{(N+1)(N+2)} x_1^{N+1} + G(x_1) - \left(C_o^{(1)} + C_1^{(1)} x_1\right) - \left(C_o^{(2)} + C_1^{(2)} x_1 + C_2^{(2)} x_1^2\right)$$

$$A_{21} = \frac{p_o}{2} x_2^2 + G(x_1)$$

$$A_{22} = (p_o + p_1)\frac{x_2^3}{6} + G(x_1)$$

$$A_{23} = \frac{5}{12}(p_o + p_1 + p_2)x_2^4 + G(x_1)$$

$$A_{24} = (p_o + p_1 + p_2)\frac{x_2^5}{2} + G(x_1)$$

$$A_{2N} = \frac{(p_o + p_1 + p_2)(N^2+1)}{(N+1)(N+2)} x_2^{N+2} + G(x_2) - \left(C_o^{(1)} + C_1^{(1)} x_2\right) - \left(C_o^{(2)} + C_1^{(2)} x_2 + C_2^{(2)} x_2^2\right)$$

$$A_{31} = \frac{p_o}{2} x_3^2 + G(x_3)$$

$$A_{32} = (p_o + p_1)\frac{x_3^3}{6} + G(x_3)$$

$$A_{33} = \frac{5}{12}(p_o + p_1 + p_2)x_3^4 + G(x_3)$$

$$A_{34} = (p_o + p_1 + p_2)\frac{x_3^5}{2} + G(x_3)$$

$$A_{3N} = \frac{(p_o + p_1 + p_2)(N^2+1)}{(N+1)(N+2)}x_3^{N+2} + G(x_3) - \left(C_o^{(1)} + C_1^{(1)}x_3\right) - \left(C_o^{(2)} + C_1^{(2)}x_3 + C_2^{(2)}x_3^2\right)$$

$$A_{N1} = \frac{p_o}{2}x_N^2 + G(x_N)$$

$$A_{N2} = (p_o + p_1 +)\frac{x_2^3}{6} + G(x_N)$$

$$A_{N3} = \frac{5}{12}(p_o + p_1 + p_2)x_N^4 + G(x_N)$$

$$A_{N4} = A_{N4}$$

$$A_{NN} = \frac{(p_o + p_1 + p_2)(N^2+1)}{(N+1)(N+2)}x_N^{N+2} + G(x_N) - \left(C_o^{(1)} + C_1^{(1)}x_N\right) - \left(C_o^{(2)} + C_1^{(2)}x_N + C_2^{(2)}x_N^2\right)$$

$$A_{N+2,1} = \frac{p_o}{2}x_{N+2}^2 + G(x_{N+2})$$

$$A_{N+2,2} = \frac{5}{12}(p_o + p_1 + p_2)x_{N+2}^4 + G(x_{N+2})$$

:

:

$$A_{N+2,N} = (p_o + p_1 + p_2)\frac{x_{N+2}^5}{2} + G(x_{N+2})$$

Remarks

The above matrix is solved by Gaussian Elimination method to obtain the unknown constants $a_i(i \geq o)$ which are then substituted into the approximate solution (6)

## III. Numerical Demonstration.

In this section, we have demonstrated the method discussed here on four examples, two first and two second orders integro differential equations. We have defined error as

Error = $Error = |y(x) - y_N(x)|$ , $a \leq x \leq b$ (45)

**Example 1:**

We consider the following first order integro differential equation of the form

$$y'(x) - \int_0^1 3xt\, y(t)dt = 3e^{3x} - \frac{1}{3}(2x^3+1)$$

With condition given as $y(0)=1$ and the exact solution as $y(x) = e^{3x}$

**Example 2:**

$$y'(x) - y(x) - \int_0^1 Sin(4\pi x - 2\pi t)y(t)dt = -Cos(2\pi x) - 2\pi Sin 2\pi x - \frac{1}{2}Sin(4\pi x)$$

With the condition $y(0)=1$ and the exact solution is given as $y(x) = Cos(2\pi x)$

**Example 3:**

$$y''(x) + \int_0^{\frac{\pi}{2}} xt\, y(t)dt = x - Sin\, x, \qquad 0 \le x \le \frac{\pi}{2}$$

With the conditions given as

$y(0) = 0 \quad and \quad y'(0) = 1$

The exact solution is given as $y(x) = Sinx$

**Example 4:**

$$y''(x) + 4xy(x) + 2\int_0^1 \frac{t^2+1}{x^2+1} y(t)dt = -\frac{8x^4}{(x^3+1)^3}, \qquad 0 \le x \le 1$$

With the conditions $y(0) = 1 \quad and \quad y'(0) = \frac{1}{2}$ The exact solution is $y(x) = (x^2+1)^{-1}$

## IV. Tables of Results

**Table 1: Errors obtained from example 1 for different values of N**

| X | HAM[6] N=10 | HAM[6] N=20 | Standard integral collocation method N=10 | Standard integral collocation method N=20 | Perturbed integral collocation method N=10 | Perturbed integral collocation method N=20 |
|---|---|---|---|---|---|---|
| **0** | **0** | **0** | **0** | **0** | **0** | **0** |
| **0.2** | **9.50054E-9** | **5.25769E-9** | **6.12604E-6** | 3.14032E-9 | **8.057**35E-7 | **4.** 32105E-10 |
| **0.4** | 3.82421E-4 | **2.10**307E-8 | **8.0**3261E-5 | 1.86232E-8 | **7.** 32561E-5 | **2.005**32E-9 |
| **0.6** | **8.60448E-4** | **4.7**3192E-8 | **1.6129**3E-4 | 3.17031E-8 | **6.005**32E-5 | **1.278**32E-8 |
| **0.8** | **1.52968E-5** | **8.41**230E-8 | **9.04215E-5** | **2.14652E-8** | **7.** 32406E-4 | **2.57**342E-9 |
| **1** | **2.**30013E-3 | **1.**33442E-7 | **1.76112E**-3 | **2.01142E-8** | **1.25672E-4** | **1.11456E-8** |

**Table 2: Errors obtained from example 2 for different values of N**

| X | HAM[6] N=10 | HAM[6] N=20 | Standard integral collocation method N=10 | Standard integral collocation method N=20 | Perturbed integral collocation method N=10 | Perturbed integral collocation method N=20 |
|---|---|---|---|---|---|---|
| **0.** | **0** | **0** | **0** | **0** | **0** | **0** |
| **0.2** | **2.5701E-4** | **4.2117E-6** | **2.1**326E-4 | **2.** 3204E-6 | **2.0756E-4** | **7.5461E-7** |
| **0.4** | **4.8001**E-4 | **5.8797E-9** | **4.100**3E-4 | 7.1325E-9 | 3.9321E-4 | **8.5**323E-10 |
| **0.6** | 9.7723E-3 | **4.2112E-8** | **2.9412E-5** | **9.7214E-9** | **5.** 3451E-5 | **5.0176E-9** |
| **0.8** | **9.772**3E-3 | **6.412**3E-8 | **1.7**325E-3 | **5.0**321E-8 | **8.146**3E-4 | 3. 3742E-8 |
| **1** | **1.**3021E-3 | **1.2**314E-8 | **1.0027E**-3 | **1.1**327E-8 | **9.** 3241E-4 | **1.1**343E-9 |

Table 3: Maximum errors obtained for example 3

| Method N | Chebyshev Polynomial Approximation [10] | Chebyshev Polynomial Approximation [10] | Standard Integral Collocation Method | Perturbed Integral Collocation Method |
|---|---|---|---|---|
| 6 | 1.32E-5 | 1.75E-4 | 1.16E-4 | 2.24E-6 |
| 8 | 1.17E-9 | 3.08E-8 | 1.03E-9 | 2.56E-10 |
| 10 | 1.25E-11 | 1.35E-9 | 2.04E-11 | 3.16E-12 |

Table 4: Errors of Example 4

| X | Wavelet Galerkin{8} N=6 | Standard Integral Collocation Method N=6 | Perturbed Integral Collocation Method N=6 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0.125 | 2.7E-4 | 1. 3E-4 | 8. 6E-5 |
| 0.250 | 3.1E-5 | 2. 8E-5 | 2. 6E-5 |
| 0. 375 | 2.6E-4 | 7. 4E-5 | 1. 7E-5 |
| 0.500 | 4. 3E-4 | 3. 2E-4 | 3. 9E-6 |
| 0.625 | 5. 6E-4 | 5. 1E-4 | 6. 3E-5 |
| 0.750 | 6. 6E-4 | 1. 9E-5 | 4. 6E-6 |
| 0.875 | 7. 2E-4 | 6. 2E-5 | 3.5E-5 |
| 1 | 0 | 0 | 0 |

## V. Conclusion

This paper has employed successfully standard and perturbed integral collocation method to solve special first and second orders linear integro-differential equation.

Power series form of approximation is used as basis function, the application of standard and perturbed integral collocation methods on some problems including linear first and second orders are considered. The most important ones are the simplicity of the methods.

Furthermore, these methods yield the desired accuracy when compared the results obtain with the exact solutions.

All these advantage of the Standard and Perturbed integral collocation methods to solve first and second orders linear integro-differential equation assert that the methods are convenient, reliable and powerful tools for the classes of the problem considered.

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