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A Redefined Riemann Zeta Function Represented via Functional Equations Using the Osborne's Rule

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Abstract: *Intrigues most researchers about the Riemann zeta hypothesis is the ability to employ cum different approaches with instinctive mindset to obtain some very interesting results. Motivated by their style of reasoning, the result obtained in this work of redefining or re-representation of Riemann zeta function in different forms by employing different techniques on two functional equations made the results better, simpler and concise new representations of Riemann zeta function.*
Keywords: *Analytic Continuation, Osborne's rule, Riemann Zeta Function*

THE OSBORNE'S RULE APPROACH

Osborne's rule gave a more comprehensive method for hyperbolic functions interms of the Euler method.

$recall\ from\ coshx = \frac{e^{x}+e^{-x}}{2}$ and the fact that $e^{\theta i} = cos\theta + isin\theta$

$Implying\ that\ \Re(e^{x}) = coshx$ (1.0)

$\Re\left(e^{\frac{s}{2}lnx}\right) = \cosh(\frac{s}{2}\ lnx)$ (1.1)

From the Riemann zeta function

$$\zeta(s) = \frac{4}{\frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}\left\{\int_1^\infty \sum_{n=1}^\infty e^{-nn\pi x}\ \left((nn)^2\pi^2 x^{\frac{5}{4}} - \frac{3}{2}nn\pi x^{\frac{1}{4}}\right)\cosh\left(\frac{s}{2}\ lnx\right)dx\right\} \quad (1.2)$$

Substituting (1.0), into equation (1.1), then the equation (1.2) becomes

$$\zeta(s) = \frac{4}{\frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}\left\{\int_1^\infty \sum_{n=1}^\infty e^{-nn\pi x}\ \left((nn)^2\pi^2 x^{\frac{5}{4}} - \frac{3}{2}nn\pi x^{\frac{1}{4}}\right)e^{\frac{s}{2}lnx}\,dx\right\} \quad (1.3)$$

considering the right arm of the r.h.s side of the equation(1.3)

$$\int_1^\infty \sum_{n=1}^\infty e^{-nn\pi x}\ \left((nn)^2\pi^2 x^{\frac{5}{4}} - \frac{3}{2}nn\pi x^{\frac{1}{4}}\right)e^{\frac{s}{2}lnx}\,dx \quad (1.4)$$

$$= \int_1^\infty \sum_{n=1}^\infty \left((nn)^2\pi^2 x^{\frac{5}{4}} - \frac{3}{2}nn\pi x^{\frac{1}{4}}\right)e^{-nn\pi x + \frac{s}{2}lnx}\,dx \quad (1.5)$$

$$= \sum_{n=1}^\infty \left(\int_1^\infty((nn)^2\pi^2 x^{\frac{5}{4}}e^{-nn\pi x + \frac{s}{2}lnx}\,dx - \int_1^\infty \frac{3}{2}nn\pi x^{\frac{1}{4}}e^{-nn\pi x + \frac{s}{2}lnx}\,dx\right) \quad (1.6)$$

considering the left hand side of the equation (1.6) i.e

$$\sum_{n=1}^\infty (nn)^2\pi^2 \int_1^\infty x^{\frac{5}{4}}e^{-nn\pi x + \frac{s}{2}lnx}\,dx. \quad (1.7)$$

$Let\ \mathrm{I} = \int_1^\infty x^{\frac{5}{4}}e^{-nn\pi x + \frac{s}{2}lnx}\,dx$ and introducing the integration by part:

$$then\ , I = (\frac{14}{18+4s})\left[\frac{4}{9}x^{\frac{9}{4}}e^{-nn\pi x + \frac{s}{2}lnx} - \frac{4}{9}(nn)^3\pi^3 \int_1^\infty x^{\frac{9}{4}}e^{-nn\pi x + \frac{s}{2}lnx}\,dx\right] \quad (1.8)$$

From equation (1.8)

$$Suppose\ M_1 = \int_1^\infty x^{\frac{9}{4}}e^{-nn\pi x + lnx}\,dx, by\ furthering\ the\ integration\ by\ part\ from \quad (1.9),$$

$$M_1 = \frac{4}{9}(nn)^3\pi^3\left[\frac{4}{13}x^{\frac{13}{4}}e^{-nn\pi x + \frac{s}{2}lnx} + \frac{4}{13}nn\pi\int_1^\infty x^{\frac{13}{4}}e^{-nn\pi x + \frac{s}{2}lnx}\,dx - \frac{4s}{26}M_1\right] \quad (2.0)$$

$substituting$ (2.0) $into$ (1.8), $we\ obtain$:

$$I = (\frac{18}{18+4s})[\frac{4}{9}x^{\frac{9}{4}}e^{-nn\pi x + \frac{s}{2}lnx} - \frac{4}{9}(nn)^3\pi^3\left[M + \frac{4s}{26}M = \frac{4}{13}x^{\frac{13}{4}}e^{-nn\pi x + \frac{s}{2}lnx} + \frac{4}{13}nn\pi\int_1^\infty x^{\frac{13}{4}}e^{-nn\pi x + \frac{s}{2}lnx}\,dx\right]$$
(2.1)

$$From\ (2.0), we\ have\ \ M_1 = \left(\frac{26}{26+4s}\right)[\frac{4}{13}x^{\frac{13}{4}}e^{-nn\pi x + \frac{s}{2}lnx} + \frac{4}{13}nn\pi\int_1^\infty x^{\frac{13}{4}}e^{-nn\pi x + \frac{s}{2}lnx}\,dx\ ,$$

$substituting\ 2.0\ into\ 1.8$

$$I=\left(\frac{18(nn)^2\pi^2}{18+4s}\right)\frac{4}{9}x^{\frac{9}{4}}e^{-nn\pi x+\frac{s}{2}lnx}-\frac{4(nn)^2\pi^2}{9}\left(\frac{18}{18+4s}\right)\left[\left(\frac{26}{26+4s}\right)\left[\frac{4}{13}x^{\frac{13}{4}}e^{-nn\pi x+\frac{s}{2}lnx}+\frac{4}{13}nn\pi\int_1^\infty x^{\frac{13}{4}}e^{-nn\pi x+\frac{s}{2}lnx}dx\right]\right.\qquad(2.1)$$

$$=\left(\frac{18(nn)^2\pi^2}{18+4s}\right)\frac{4}{9}x^{\frac{9}{4}}e^{-nn\pi x+\frac{s}{2}lnx}-\frac{468(nn)^2\pi^2}{1053+396s+36s^2}\left(\frac{4}{13}x^{\frac{13}{4}}e^{-nn\pi x+\frac{s}{2}lnx}\right)-\frac{144}{1053+234s}(nn)^4\pi^4\int_1^\infty x^{\frac{13}{4}}e^{-nn\pi x+\frac{s}{2}lnx}dx$$
(2.2)

$considering\ the\ right\ hand\ side\ of\ the\ equation\ (1.6)i.e$

$$=\sum_{n=1}^{\infty}\int_1^\infty\frac{3}{2}nn\pi x^{\frac{1}{4}}e^{-nn\pi x+\frac{s}{2}lnx}dx=\frac{3}{2}nn\pi\int_1^\infty x^{\frac{1}{4}}e^{-nn\pi x+\frac{s}{2}lnx}dx\qquad(2.3)$$

$$Suppose\ M_2=\int_1^\infty x^{\frac{1}{4}}e^{-nn\pi x+\frac{s}{2}lnx}dx\qquad(2.4)$$

$applying\ integration\ by\ part\ on\ equation\ (2.4)$

$$M_2=\left(\frac{5}{5+2s}\right)\left[\frac{4}{5}x^{\frac{5}{4}}e^{-nn\pi x+\frac{s}{2}lnx}+\frac{16nn\pi}{45}x^{\frac{9}{4}}e^{-nn\pi x+\frac{s}{2}lnx}+\frac{16}{45}(nn)^2\pi^2\int_1^\infty x^{\frac{9}{4}}e^{-nn\pi x+\frac{s}{2}lnx}dx-\right.$$

16nnπs901∞x54e−nnπx+s2lnxdx]

(2.5)

$$recall\ M_2=\int_1^\infty x^{\frac{1}{4}}e^{-nn\pi x+\frac{s}{2}lnx}dx\quad then\ from\ (1.6)\ M_2=\left(\frac{2}{3nn\pi}\right)\int_1^\infty x^{\frac{1}{4}}e^{-nn\pi x+\frac{s}{2}lnx}dx, we\ obtain$$

$$M_2=\frac{10}{nn\pi(15+6s)}\left[\frac{4}{5}x^{\frac{5}{4}}e^{-nn\pi x+\frac{s}{2}lnx}+\frac{16nn\pi}{45}x^{\frac{9}{4}}e^{-nn\pi x+\frac{s}{2}lnx}+\frac{16}{45}(nn)^2\pi^2\int_1^\infty x^{\frac{9}{4}}e^{-nn\pi x+\frac{s}{2}lnx}dx-\right.$$

16nnπs901∞x54e−nnπx+s2lnxdx,

(2.6)

$from\ the\ equation\ (1.6), substituting\ I\ and\ M_2, to\ obtain$

$Equation\ (1.6), becomes$

$$=\left(\frac{18(nn)^2\pi^2}{18+4s}\right)\frac{4}{9}x^{\frac{9}{4}}e^{-nn\pi x+\frac{s}{2}lnx}-\frac{468(nn)^2\pi^2}{1053+396s+36s^2}\left(\frac{4}{13}x^{\frac{13}{4}}e^{-nn\pi x+\frac{s}{2}lnx}\right)-\frac{144}{1053+234s}(nn)^4\pi^4\int_1^\infty x^{\frac{13}{4}}e^{-nn\pi x+\frac{s}{2}lnx}dx-$$

10nnπ15+6s45x54e−nnπx+s2lnx+16nnπ45x94e−nnπx+s2lnx+1645nn2π21∞x94e−nnπx+s2lnxdx −16nnπs901∞x54e−nnπx+s2lnxdx (2.7)

$$recall\ \ M_2=\int_1^\infty x^{\frac{1}{4}}e^{-nn\pi x+\frac{s}{2}lnx}dx\ , I=\int_1^\infty x^{\frac{5}{4}}e^{-nn\pi x+\frac{s}{2}lnx}dx\ , M_1=\int_1^\infty x^{\frac{9}{4}}e^{-nn\pi x+\frac{s}{2}lnx}dx$$

$$=\left(\frac{18(nn)^2\pi^2}{18+4s}\right)\frac{4}{9}x^{\frac{9}{4}}e^{-nn\pi x+\frac{s}{2}lnx}-\frac{468(nn)^2\pi^2}{(1053+396s+36s^2)\left(\frac{4}{13}x^{\frac{13}{4}}e^{-nn\pi x+\frac{s}{2}lnx}\right)}-\frac{144}{1053+234s}(nn)^4\pi^4\int_1^\infty x^{\frac{13}{4}}e^{-nn\pi x+\frac{s}{2}lnx}dx-$$

40x54e−nnπx+s2lnxnnπ75+30s+160nnπx94e−nnπx+s2lnxnnπ675+3270s−16nn2π2nnπ675+270s M1+16nnπsnnπ1375+540sI (2.8)

$collecting\ like\ terms, we\ have:$

$$=\left(\frac{18(nn)^2\pi^2}{18+4s}\right)\frac{4}{9}x^{\frac{9}{4}}e^{-nn\pi x+\frac{s}{2}lnx}-\frac{468(nn)^2\pi^2}{1053+396s+36s^2}\left(\frac{4}{13}x^{\frac{13}{4}}e^{-nn\pi x+\frac{s}{2}lnx}\right)-\frac{40x^{\frac{5}{4}}e^{-nn\pi x+\frac{s}{2}lnx}}{nn\pi(75+30s)}+\frac{160nn\pi x^{\frac{9}{4}}e^{-nn\pi x+\frac{s}{2}lnx}}{nn\pi(675+3270s)}-\frac{144}{1053+234s}(nn)^4\pi^4\int_1^\infty x^{\frac{13}{4}}e^{-nn\pi x+\frac{s}{2}lnx}dx-\frac{16(nn)^2\pi^2}{nn\pi(675+270s)}M_1+\frac{16nn\pi s}{nn\pi(1375+540s)}I\qquad(2.9)$$

$recall\ from\ (1.6), by\ simplifying\ we\ obtain$

$$=\sum_{n=1}^{\infty}\left[\left(\frac{18(nn)^2\pi^2}{18+4s}\right)\frac{4}{9}x^{\frac{9}{4}}e^{-nn\pi x+\frac{s}{2}lnx}-\frac{468(nn)^2\pi^2}{1053+396s+36s^2}\left(\frac{4}{13}x^{\frac{13}{4}}e^{-nn\pi x+\frac{s}{2}lnx}\right)-\frac{40x^{\frac{5}{4}}e^{-nn\pi x+\frac{s}{2}lnx}}{nn\pi(75+30s)}+\right.$$

160nnπx94e−nnπx+s2lnxnnπ(675+3270s)−1441053+234s(nn)4π41∞x134e−nnπx+s2lnxdx−16(nn)2π2nnπ(675+270s)M1+16nnπsnnπ(1375+540s)I

(3.0)

$$=\sum_{n=1}^{\infty}e^{-nn\pi x}\left[\left(\frac{18(nn)^2\pi^2}{18+4s}\right)\frac{4}{9}x^{\frac{9}{4}}e^{\frac{s}{2}lnx}-\frac{468(nn)^2\pi^2}{1053+396s+36s^2}\left(\frac{4}{13}x^{\frac{13}{4}}e^{\frac{s}{2}lnx}\right)-\frac{40x^{\frac{5}{4}}e^{\frac{s}{2}lnx}}{nn\pi(75+30s)}+\frac{160nn\pi x^{\frac{9}{4}}e^{\frac{s}{2}lnx}}{nn\pi(675+3270s)}\right]-$$

1441053+234s(nn)4π41∞x134e−nnπx+s2lnxdx−16nn2π2nnπ(675+270s)M1+16nnπsnnπ(1375+540s)I

(3.1)

$$Let \sum_{n=1}^{\infty} e^{-nn\pi x} = \psi(x) and\ e^{\ln x^{\frac{s}{2}}} = x^{\frac{s}{2}}$$

Then equations 3.1 becomes

$$= \psi(x)x^{\frac{s}{2}}\left[\left(\frac{18(nn)^2\pi^2}{18+4s}\right)\frac{4}{9}x^{\frac{9}{4}} - \frac{468(nn)^2\pi^2}{1053+396s+36s^2}\left(\frac{4}{13}x^{\frac{13}{4}}\right) - \frac{40x^{\frac{5}{4}}}{nn\pi\,(75+30s)} + \frac{160nn\pi\, x^{\frac{9}{4}}}{nn\pi\,(675+3270\,s)}\right] - \frac{144}{1053+234s}(nn)^4\pi^4\int_1^{\infty} x^{\frac{13}{4}}e^{-nn\pi x+\frac{s}{2}\ln x}\,dx - \frac{16(nn)^2\pi^2}{nn\pi\,(675+270s)}M_1 + \frac{16nn\pi s}{nn\pi\,(1375+540s)}I \qquad (3.2)$$

$Substituting\ (3.0)\ into\ (1.2)$

$$\zeta(s) = \frac{4}{\frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}\left\{\psi(x)x^{\frac{s}{2}}\left[\left(\frac{18(nn)^2\pi^2}{18+4s}\right)\frac{4}{9}x^{\frac{9}{4}} - \frac{468(nn)^2\pi^2}{1053+396s+36s^2}\left(\frac{4}{13}x^{\frac{13}{4}}\right) - \frac{40x^{\frac{5}{4}}}{nn\pi\,(75+30s)} + \frac{160nn\pi\, x^{\frac{9}{4}}}{nn\pi\,(675+3270s)}\right] - \right.$$

1441053+234s(nn)4π41∞x134e−nnπx+s2lnxdx−16(nn)2π2nnπ(675+270s)M1+16nnπsnnπ(1375+540s) I . (3.3)

from the above represenation the ζ(s)is valid for $s > 1$

Conclusion

This representation really shows how the complex part of the equation could be eliminated by introducing a hyperbolic function from Enoch and Adeyeye(2012) result which was the reason why employ the technique of the Osborne's Rule approach. Also from the result obtained, by observation one see how much primes numbers are been played around the synthetic process. This actually obeys the rules guiding the Number theory (every integer is a product of prime). This method helps in removing the "i"in the result using the Euler's Method though we could not see how the integers are been expressed in the products of primes much but this validate some of the claims of researchers like Euler (1748), Odgers (2004). Thus, the Osborne's approach is more preferable.

References

[1] E. Bombieri, (2000), Problems *of the millennium: The Riemann hypothesis, Clay* mathematical Institute.
[2] O. Enoch, (2012), A New Representation of the Riemann Zeta Function Via Its Functional Equation
[3] O. Enoch, (2012), A General Representation of the Zeros of the Riemann Zeta Function via Fourier series Expansion
[4] O.O.A. Enoch and F.J. Adeyeye, (2012), A Validation of the Real Zeros of the Riemann Zeta Function via the Continuation Formula of the Zeta Function, *Journal of Basic & Applied Sciences, 2012, 8, 1*
[5] P. Sarnak, (2004), Problems of the Millennium: The Riemann Hypothesis by Princeton University and courant Institute of Mathematics.