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The Place of Group Theory in Decision-Making in Organizational Management A case of 16- Puzzle

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Abstract: *Ourmainconcern, in this paper is how group theory quietly involve in decision making , Problem solving and critical thinking in games which could also be helpful to organizational management, policy making, Politics, Business, Science, Health care , Career choice and many other fields, yet unknowingly to many. We show in this work how Group theory can be applied to16- puzzles very effectively. The game we present here is organizational puzzleon management decision making.*
Keyword: *Decision- making, Group Theory, Games, Organizational Management, 16- Puzzle.*

I. Introduction

1.0 Introduction

Group theory is the branch of pure mathematics which emanated from abstract algebra.Due to its abstract nature, it was seen, as if were to be an arts subject rather than a science subject. In fact it was considered pure abstract and not practical.Even students of group theory, after being introduced to the course seem not to believe as to whether the subject has any practical application in real life situations because of its abstract nature.This problem prompt the researchers to study the different ways in which groups can be expressed concretely (its groups representation) both from theoretical and practical point of view, with intention of bringing its real life application in problem solving and critical thinking and decision making in an organizational management. we are considering here the game of 16 – Puzzle, one of the recreational games for better understanding of the subject.[1] State that the modern concept of abstract group developed out of several fields of mathematics.In fact the idea of group theory although developed from the concept of abstract algebra yet can be applied in many other mathematical areas and other fields in sciencesas well as puzzles. Group theory can be applied to puzzles very effectively. A permutation representation can be used in many puzzles to define a group, which often reveals interesting aspect of the puzzle,[2].

In this case, we will start our reader off with organizational management issues, then further present the 16-puzzle in terms of group theory.

Everyone experiences problems from time to time. Some of our problems are big and complicated, while others may be more easily solved. There is no shortage, of challenges and issues that can arise on the job.

Whether in an office or on a construction site , experiencing difficulties with the tasks at hand or with co-workers, the work place presents on going challenges on a daily basis. Whether these problems are large or small, they need to be dealt with constructively and fairly. Problem solving and critical thinking refers to the ability to use knowledge, facts, and data to effectively take decision that will solve problems. This doesn't mean you need to have an immediate answer, it means you have to be able to think on your feet, asses problems and find solutions. The ability to develop a well thought out solution within a reasonable time frame, however, is a skill that employers value greatly. According to [3] saidthat Employers say they need a workforce fully equipped with skills beyond the basics of reading , writing and arithmetic to grow their businesses. These skills including critical thinking and problem solving, according to a 2010 critical skill survey by the American Management Association and others.

II. Methodology

2.0 Method

Group are useful in modelling games that involve a series of discrete moves with each move leading to a change in the board state. The loose idea is to identify each possible move with an element of a certain huge group, where the effect of performing a sequence of moves corresponds to the product of those elements. This sequence of moves gives some rearrangement (permutation) of the group. Thus the group modelling this game is the group of permutations,[4].

2.1 Groups of Transformations

According to [5] said the notion of a group is one of the most important and ubiquitous notions in the entire field of mathematics. One of its primary functions is to describe symmetry. For this reason, one of the most common ways in which groups arise in nature is as groups of transformations, or symmetries, of various mathematical objects.

Definition 2.1 Let X be a set, and Let G be a subset of the set of all invertible transformation (i.e, bijections) $f:X \rightarrow X$. One says that G is a group .

1, If G is closed under composition, i.e $fog \in G$ for f, g, $\in G$;

2, If $id \in G$; and

3, If $g \in G$ then $g^{-1} \in G$

Definition 2.2 if G is a finite group, then the order /G/ of G is the number of elements in G.

2.2 The Symmetric and Alternating Groups.

The Symmetric or alternating group is the most obvious example of a group of transformations (or permutations) of X. This group is of special interest if X is a finite set: $X = \{1,\ldots,n\}$. In this case the group perm (X) is called the symmetric group on letters S_n and is denoted by S_n. The order of this group is n!.

The simplest example of a permutation which is not identity is a transposition (*ij*), $1 \leq i < j \leq n$. This permutation switches i and j and keeps all other elements fixed. In particular, if $j = i + 1$, then (ij) is called a transposition of neighbours. It is clear that any permutation is a composition of transpositions of neighbours.

For every permutation $s \in S_n$, we have a notion of the sign of s. Namely, let, inv(s) be the number of inversion of s, i.e the number of pairs $i < j$ such that $s(i) > s(j)$. Then s is said to be even (respectively, odd) if invs(s) is even (respectively, odd), and one defines sign(s) to b $(-1)^{inv(s)}$

Proposition 2.1

For any representation of s as a composition of n transpositions of neighbours, $sign(s) = (-1)^n$

Proof

Right multiplication of a permutation by a transposition of neighbours either creates a new inversion or kills an existing one. So n equals the number of inversions module 2.

Corollary 2.1

One has sign $(s \circ t)$ = sign(s) sign(t). In particular, even permutations form a group, which for $n \geq 2$ has order $n!/2$.

The group of even permutations is called the alternating group and denoted by A_n, [5].

Examples

Let $(a_i,..,a_m) \in \{1,\ldots,n\}$ be distinct elements. Denote by $(a_i\ldots,a_m)$ the cyclic permutation of $a_i\ldots,a_m$ i.e $a_i \mapsto a_2 \mapsto - - - \mapsto a_m \mapsto a_i$. it can be shown that any permutation can be uniquely represented as a composition of cycles on disjoint collections of elements (up to order of composition). This representation is called the cycle decomposition, and is a convenient way of recording permutations.

Also, any cycle of even length (in particular, any transposition) is an odd permutation, while any cycle of odd length is even permutation. Thus, the sign of any permutation s is $(-1)^r$, where r is the number of cycles of even length in the cycle decomposition of s.

2.3 Permutations and Parity

According to [6] State Every finite group is (isomorphic to) a subset of S_n for some integer n by Cayley's theorem. Now we will now look in more detail at what a permutation does when it acts on the elements of the set $A = \{1, 2, 3\ldots n\}$. Even though infinite groups are also permutations (on an infinite set), we will assume G is finite from now on.

Suppose g in G acts on some distinct elements $a_1, a_2,.., a_n$ of A as a permutation in the following way:
$a_{ig} = a_{i+1}$ for all i<n,
$a_n g = a_1$
$ag = a$
for all other elements a in A. Then we call g an n – cycle, and denote it by $(a_1, a_2, \ldots, a_n)$. A cycle rotates a number of elements around in a loop, and does nothing else. In particular, any permutation that just swaps two elements a and b is a 2 – cycles denoted (ab), and any permutation that moves exactly three elements will be a 3 – cycles. Note that a permutation that moves exactly 4 elements need not be a 4 cycle (abcd) as it might instead be a product of two 2 – cycles such as (ab) (cd) which swaps elements a with b, and g with d.

Lemma 2.1
Every permutation is a product of disjoint cycles.
Proof
Let g be any permutation of set A. Take any a in A that is permutated by g. The elements, a, ag, ag^2etc, form a cycle because if $ag^n = ag^m$ then $ag^{n-m} = a$. If g does not permute any other elements in A, then g is simply this one cycle. Otherwise we can take an element b of A that does not lie in this cycle and form a new cycle (b bg bg^2...). These cycles are disjoint, because if $bg^n = ag^m$ then $b = ag^{m-n}$ contrary to our choice of b. This process can be repeated to enumerate all the disjoint cycles, and it is obvious that g acts the same on A as the product of these cycles. Therefore g is the product of these disjoint cycles.

The representation of a permutation as a product of disjoint cycles is essentially unique, as it can only differ in the order of the product, cyclically permuting each cycle.

Lemma 2.2
Every permutation is a product of 2-cycles, which can all be of the form (1 a). In other words, S_n is generated by the permutations (1 a)
Proof
A cycle $(a_1, a_2, \ldots, a_k)$ can be split as a product $(1\ a_1), (1\ a_2), (1\ a_3), \ldots, (1\ a_k)\ (1\ a_1)$. A cycle involving the element 1 can be split into 2 – cycles in a similar way.

Therefore any product of cycles is also a product of such 2 – cycles, and hence any permutation.
Note: If on a puzzle you could swap any two pieces at will, then it is easy to solve any position with each swap you can put (at least) one piece in its correct position. If each swap you make has to involve the piece at one particular place, then it is still easy.

2.4 Symmetric Group
Given a set A, the Symmetric Group S_n is the set of all bijectives from A to A, which forms a group under composition. If A is the finite set {1, 2, 3, - - -, n}, then its symmetric group is denoted by S_n and the elements of this group are called permutations.

Lemma 2.3

$$/S_n/ = n! = n(n-1)\ldots 2.1 \qquad 2.1$$

Proof
An informal proof runs as follows. Each permutation is a bijection on A. Suppose we were to specify where each element of A is mapped to by a permutation. The element 1 can be mapped to any of the n elements of A. Element 2 cannot be mapped to the same element as 1, so can be mapped to any of the remaining n-1 elements. Similarly 3 is mapped to anyone of the remaining n-2 elements, and so on. Clearly there are n.(n-1).(n-2). . .2.1 possibilities, each of which specify a unique permutation. Therefore $/S_n/ = n!$.
Note: Any puzzle which has n distinct pieces, and on which any two pieces can be swapped (though this may not be easy!) will have n! permutations and hence n! positions.

III. Result

3. 0 Introduction

We took this puzzle from organizational management Puzzles we name it 16- puzzle because it has similar feature to 15-puzzle. They are both on 4 X 4 squares. The solution of this puzzle gives solution to managerial problems that involve organizational management and decision making.

We suggest to our reader, first of all, attempt solving the puzzle before going to our results. This way one can appreciate the puzzle better, because it is not as easy as you can think.

3.1 The Physical application of real life situation of groups to the game of 16- puzzle.

The Problem

Sixteen prisoner's each in his cell, one died in his cell and as a prison warder you were ask to take the corpse and show it at least once to each prisoner as a mark of last respect before taking it out finally for funeral.

How can you do this task without repeating it twice to at least any prisoner in the cell?

Assumptions

1. The entire prison cell is a 4 x 4 closed prison with only one exit.
2. Each cell is connected with the next close to it
3. The exit cell and the cell to which the prisoner died must always be at the extreme ends and diagonal to each other.

The question at hand is : Can one get all possible rearrangement of the corpse from 16^{th} cell to 1^{st} cell?

A permutation is a function that rearranges the elements of a given set say S_n

Now we can consider the idea of cycle.

For the set $S_n=\{1,2,3,---,n\}$, a k-cycle is X_1, X_2, - - - X_k. This means X_1 goes to where X_2 was, X_2 goes to where X_3 was, etc. Finally, X_k goes to where X_1 was. A permutation is just a product of cycles. While A transposition is a 2-cycle.

Theorem 3.1

Every permutation can be written as the product of transpositions.

Proof

We only need to look at cycles. Let $\pi = (X_1\ X_2\ \text{- - -}\ X_n)$. We can rewrite this as $\pi = (X_1X_n)\ (X_1X_{n-1})$ - - - (X_1X_2)

Theorem 3.2

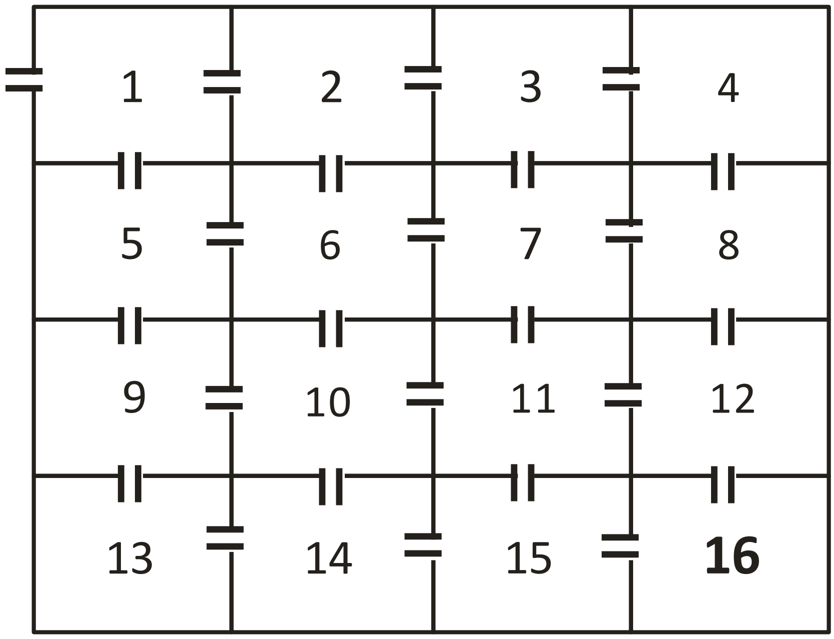
Every permutation is even or odd

What does this mean for the puzzle ?

- Every "State" the puzzle is in, is just a permutation of the set $S_{16}= \{1, 2, \text{- - -} 15, 16\}$ where 16 represents the corpse to be carried through.
- Every move is just a transformation 16 and one of its neighbours
- Every possible state must be permutation that can decompose into a sequence of moves.
- This sequence of moves forms composition of mapping.
- The composition of general mapping obeys all the four axioms of a group.

Obviously, with this type of puzzle, we are concerned with permutations of 16 different cells,thusapplying the group of S_{16}. We can number the slots 1 through 16 along a Hamiltonian path. We can create a relatively short list of all valid moves on 16 – puzzle and immediately realize that all such representative elements in $S_{16}$are k-cycleswhere k is odd ,[7]. More important is the realization that each of these elements are members of the subgroup S_{16} which represents all of the even permutations in S_{16} . Since S_{16} is a group and is closed under group operation of function composition, then using any of the valid moves on the 16 – puzzle and composing them together will only generate elements that exist in the group S_{16}.

Table XII The 4x4 Prison Cells, each cell connected next to it, 16^{th} being the corpse and A permutation Matrix of the transformation



$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 15 & 6 & 2 & 3 & 1 & 10 & 8 & 4 & 5 & 14 & 7 & 16 & 9 & 13 & 11 & 12 \end{pmatrix}$$

TheHamiltonian path of the transformation of this problem is as below

P_1 : 16→12→16→15→11→7→8→4→3→2→6→10→14→13→9→5→1→exit

Which gives the mapping.

$$\left.\begin{array}{ll} f_1(16) \to 12, & f_6(7) \to 8 \\ f_2(12) \to 16, & . \\ f_3(16) \to 15, & . \\ f_4(15) \to 11, & . \\ f_5(11) \to 7, & f_{16}(5) \to 1 \end{array}\right\} . \qquad 2.2$$

If S be a set then a one – one onto mapping f:S →S is said to be a transformation or in case S is finite, f is said to be a permutation.

The composition of 2.2 gives the complete four axioms of the group S_{16} as follows

$f_1 \, O \, f_2 \in S_{16}$ Composition of mapping is closed

$(f_3 0 f_4) 0 f_5 = f_3 0 (f_4 0 f_5)$ Composition of mapping is associative

Identity exist by doing nothing

Inverse exist, since $f_1 0 f_2 =$ I which implies that $f_2 = f^{'1}{}_1$

By the definition of abstract group this has form a group

Other possible Hamiltonian path are as follows

P_2 : 16→12→16→15→11→10→14→13→9→5→6→7→8→4→3→2→1→exit

P_3: 16→12→16→15→14→13→9→10→11→7→8→4→3→2→6→5→1→exit

P_4: 16→12→16→15→14→13→9→5→6→10→11→7→8→4→3→2→1→exit

P_5 : 16→15→16→12→11→10→14→13→9→5→6→7→8→4→3→2→1→exit

P_6: 16→15→16→12→8→4→3→7→11→10→14→13→9→5→6→2→1→exit

P_7: 16→15→16→12→11→7→8→4→3→2→6→10→14→13→9→5→1→exit

P_8: 16→15→16→12→8→4→3→7→11→10→14→13→9→5→6→2→1→exit

- Each mapping has an inverse in another path
- Composition of mapping is associative
- Doing nothing is identity

The sixteen transformation exhausted in two cycles from P_1 which is written as a product of cycles.(12 16) (15 11 7 8 4 3 2 6 10 14 13 9 5 1) 2.3

This is a 14-cycle times a 2-cycle , which refers to odd permutation times odd permutation, overall this rearrangement is an even permutation as in lemma 2.1.

The transposition of the corpse is as below from the Hamiltonian path P_1

(12 16)(16 16)(15 16)(11 16)(7 16)(8 16)(4 16)(3 16)(2 16)(6 16)(10 16)(14 16)(13 16)(9 16)(5 16)(1 16)2.42.4 is according to theorem 3.1

Therefore it is possible to take the corpse successfully through, only once to each of the prisoner in their cells.

According to [8]In a permutation group G, every element of G can be express as a product of disjoint cycles as in 2.3 above.

The set of all even permutations of degree n is known as an alternating set or group and is denoted by say $S_n$as in corollary 2.1

IV. Conclusion

We have shown that the corpse can be successfully taken in only once without repeating to any prisoner twice using group theory.The beauty of this puzzle to any organizational management, is when solving a difficult organizational problem you need to resort back to the root cause of the problem, as in repeating 16^{th} cell twice is the only way you can solve this 16 – puzzle successfully. Group theory is interesting, we are yet to discover many other puzzles that can be applied to it successfully.

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