IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728,p-ISSN: 2319-765X, Volume 7, Issue 6 (Sep. - Oct. 2013), PP 01-10
www.iosrjournals.org

Application of Plantri Graph:All Combinatorial Structure of Orderable And Deformable Compact Coxeter Hyperbolic Polyhedra

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Abstract: *By Andreev's theorem and Choi's theorem, we proved that the degree of each vertex is three and the number of vertices of orderable compact Coxeter polyhedral is at most 10. Therefore a combinatorial polyhedron is a 3-connected planner graph. From the Plantri program, we found that the number of 3-connected planner graphs with at most 10 vertices of degree 3 is 9. We find that only five planner graphs among these 9 graphs satisfy the properties of orderable compact Coxeter polyhedra. Then we verify the polyhedra which are associated with these 5 planner graphs are orderable. Therefore the number of combinatorial polyhedra of orderable and deformable compact hyperbolic Coxeter polyhedra is five up to symmetry.*

I. Introduction

A *n-dimensional orbifold* is a topological space with a structure based on the quotient space of R^n by a finite group action. An orbifold is called *good* if its universal cover is a manifold. We will concentrate only on good orbifolds.

To give a *hyperbolic structure* on an orbifold, we model it locally by the orbit spaces of finite subgroups of $PO(1,n)$ acting on open subsets of H^n. Similarly, to put a real projective structure on an orbifold, we model it locally by the orbit spaces of finite subgroups of $PGL(n+1,R)$ acting on open subsets of RP^n.

A real projective structure on an orbifold *M* implies that *M* has a universal cover $\tilde{M}$ and the deck transformation group $\pi_1(M)$ acting on $\tilde{M}$ so that $\frac{\tilde{M}}{\pi_1(M)}$ is homeomorphic to *M*.

A *convex* set in RP^n is a convex set in an affine patch. If we use Klein's model of a *n*-dimensional hyperbolic space, then is an open ball in RP^n and $PO(1,n)$ is a subgroup of $PGL(n+1,\square)$ preserving H^n. Therefore H^n can be imbedded in an $(n+1)$-dimensional real vector space *V* as an upper part of hyperboloid

$$-x_1^2+x_2^2+...+x_{n+1}^2=-1$$

Hence hyperbolic orbifolds naturally have real projective structures. But a real projective structure of an orbifold may not have hyperbolic structure.

We will concentrate on 3-dimensional compact hyperbolic orbifolds whose base spaces are homeomorphic to a convex polyhedron and whose sides are silvered and each edge is given an order. If the dihedral angle of an edge of a compact hyperbolic polyhedron is $\frac{\pi}{n}$ then we say that the order of the edge is n where n is a positive number.

Definition 1.0.1. Let X be S^3, E^3, or H^3. Let *Isom(X)* denotes the group of isometries of *X.* A *Coxeter polyhedron* is a convex polytope in *X* whose dihedral angles are all integer sub-multiples of π. Let *P* be a 3-dimensional Coxeter polyhedron and Γ be the group generated by the reflections in the faces of *P*. Then Γ is a discrete group of Isom(*X*) and *P* is its fundamental polyhedron. Conversely, every discrete group Γ of Isom(*X*) can be obtained from a Coxeter polyhedron *P* such that *P* is its fundamental polyhedron. The number of faces intersect at vertex is called the *degree* of that vertex. Also the edge order of edges of a Coxeter polyhedron are positive integers.

Definition 1.0.2. Let *P* be a fixed convex polyhedron. Let us assign orders at each edge. Let *e* be the number of edges and e_2 be the numbers of order-two. Let f be number of sides.
We remove any vertex of *P* which has more than three edges ending or with orders of the edges ending there is not of the form

$$(2,2,n), n \geq 2, (2,3,3), (2,3,4), (2,3,5),$$

i.e., orders of spherical triangular groups. This make *P* into an open 3-dimensional orbifold.
Let $\hat{P}$ denote the differential orbifold with sides silvered and the edge orders realized as assigned from *P* with vertices removed. We say that $\hat{P}$ has a *Coxeter orbifold structure*.

Definition 1.0.3. The deformation space $\hat{P}$ of projective structures on an orbifold $\hat{P}$ is the space of all projective structures on $\hat{P}$ quotient by isotopy group actions of $\hat{P}$.

Definition 1.0.4. We say *P* is *orderable* if we can order the sides of *P* so that each sides meets sides meets sides of higher index in less than or equal to 3 edges.
Example 1.0.5. Cube and dodecahedron are not satisfying orderability condition.

Definition 1.0.6. Let $\hat{P}$ be the orbifold structure of a 3-dimensional polyhedron *P*. We say that the orbifold structure $\hat{P}$ is *orderable* if the sides of *P* can be ordered so that each side has no more than three edges which are either of order 2 or included in a side of higher index. $\hat{P}$ is trivalent if each side *F* has three or less number of edges of order two or edges belonging to sides of higher class than *F*.

Definition 1.0.7. A combinatorial polyhedron is a 3-ball whose boundary sphere S^2 is equipped with a cell complex whose 0-cells, 1-cells and 2-cells will also be called vertices, edges and faces respectively, and which can be realized as a convex polyhedron. Topologically, a compact polyhedron *P* is a combinatorial polyhedron. A polyhedron is called *trivalent* if degree of each vertex is 3.
Remark 1.0.8. For our convenient, we will use notation in short as:
(1) A compact hyperbolic Coxeter polyhedron is as CH-Coxeter polyhedron.
(2) An orderable and projectively deformable compact hyperbolic Coxeter orbifold is as ODCH-Coxeter orbifold.

Theorem 1.0.9. *Let P be a 3-dimensional CH-Coxeter polyhedron and* $\hat{P}$ *be its Coxeter orbifold structure. Suppose that* $\hat{P}$ *is orderable and projectively deformable. Then the total number of combinatorial polyhedral of such P is 5 and P is one of the combinatorial polyhedral in figure 1.*

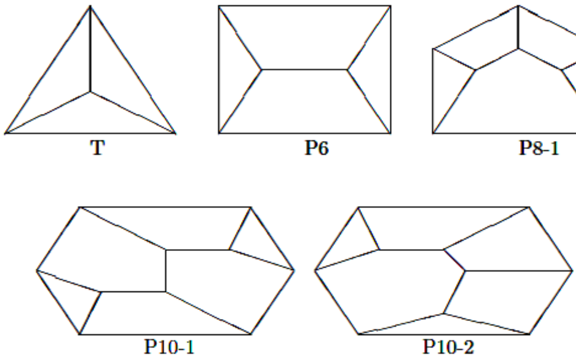

FIGURE 1

Proof. Using Andreev's theorem and Choi's theorem, we proved that such combinatorial polyhedron has vertices not more than 10. It is proved that the degree of each vertex of ODCH Coxeter polyhedron is 3. Each polyhedron is associated with a planner graph. Using Plantri graph, we find the 3-connected graphs having not more than 10 vertices and the number of such graph is 9. Among these 9 graphs, only five graphs satisfy the properties of ODCH Coxeter orbifold structure. Finally we find that each polyhedron associated with these five graphs has ODCH Coxeter orbifold structure. Therefore the number of combinatorial polyhedron is five.

Acknowledgments. I like to thank Dr. Ashish Kumar Upadhyay, IIT, Panta for his helpful comment to run Plantri program.

II. Preliminary

2.1. **Andreev's Theorem.** In 1970, E.M. Andreev provides a complete characterization of 3-dimensional compact hyperbolic polyhedral having non-obtuse dihedral angles on his article [2]. Therefore Andreev's theorem is a fundamental tool for classification of $3-$dimensional compact hyperbolic Coxeter polyhedron. Some elementary faces about polyhedral are essential before we state Andreev's theorem.

Definition 2.1.1. A cell complex on S^2 is called *trivalent* if each vertex is the intersection of three faces. A 3-dimensional *combinatorial polyhedron* is a cell complex *C* on S^2 that satisfied the following condition:
(1) Every edges of *C* is the intersection of exactly two faces.
(2) Anon-empty intersection of two faces is either an edge or a vertex.
(3) Every faces contains not fewer than 3 edges. If a face contains *n* edges then *n* is called the *length* of the face.
Suppose *C** be the dual complex of *C* in S^2. Then *C** is a simplicial complex which embed in the same S^2 so that the vertex correspond to face of *C*, etc. A simple closed curve Γ in *C** is called *k-circuit* if it is formed by *k* edges of *C**. A *k*-circuit Γ is called *prismatic k-circuit* if the intersection of any two edges of *C* intersected by Γ is empty. If a prismatic *k*-circuit meets the edges $e_1, e_2, ..., e_k$ of *C* successively then we say that the edges $F_1, F_2, ..., F_k$ are an *k-prismatic element* of *C*.

Theorem 2.1.2 (Andreev, 1970), *Let C be an combinatorial polyhedron such that C is not a simplex and suppose that non-obtuse angles* $0 < \alpha_{ij} \leq \frac{\pi}{2}$ *are given corresponding to each edge* $F_{ij} = F_i \cap F_j$ *of C where* F_i *and* F_j *are the faces of C. Then there exist a compact hyperbolic* *polyhedron P in 3-dimensional hyperbolic space which realize C with dihedral angles* α_{ij} *at the edge* F_{ij} *if and only if the following five conditions hold:*

(1) *C is trivalent.*

(2) *If* $F_{ijk} = F_i \cap F_j \cap F_k$ *is a vertex of C then*

$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} > \pi$.

(3) *If* Γ *is a prismatic 3-circuit which intersects edges* F_{ij}, F_{jk}, F_{ki} *of* C *then*

$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} < \pi$

(4) *If* Γ *is a prismatic 4-circuit which intersects edges* $F_{ij}, F_{jk}, F_{ki}, F_{li}$ *of* C *then*

$\alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} < 2\pi$.

(5) *If* F_s *is a four sides faces of C with edges* $F_{is}, F_{js}, F_{ks}, F_{ls}$ *enumerated successively, then*

$\alpha_{is} + \alpha_{ks} + \alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} < 3\pi$

$\alpha_{js} + \alpha_{ls} + \alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} < 3\pi$

Furthermore, this polyhedron is unique up to hyperbolic isometries. Also Roeder, Hubbard and Dunbar proved that if *C* is not a triangular prism, then condition (5) is a consequence of (3) and (4) (Sec [14]). Andreev's restriction to non-obtuse dihedral angles is necessary to ensure that *P* be convex. Without this restriction of dihedral angles, compact hyperbolic polyhedral realizing a given abstract polyhedron may not be convex. Since dihedral angles of Coxeter polyhedron is non-obtuse, Andreev's theorem provide a complete characterization of 3-dimensional hyperbolic Coxeter polyhedron having more than four faces.

2.2. Choi's Theorem. Prof. Choi concentrated a class of Coxeter orbifolds which is called orderable Coxeter orbifolds and a certain type of orbifolds known as normal type orbifolds. In this class of orbifolds, we understand the restricted deformation space of orbifolds in real projective space from his article [6].

Definitation 2.2.1. We denote *k(P)* the dimension of the projective group acting on a convex polyhedron *P*.

$$k(P) = \begin{cases} 3 \text{ if } P \text{ is a tetrahedron,} \\ 1 \text{ if } P \text{ is a cone with base} \\ \quad \text{a convex polygon which is no} \\ 0 \text{ otherwise} \end{cases}$$

Definition 2.2.2. A Coxeter group Γ is an abstract group define by a group presentation of form

$$\left(R_{i,};\left(R_iR_j\right)^{n_{ij}}\right), i, j \in I.$$

Where *I* is a countable index set, $n_{ij} \in N$ is symmetric for i, j and $n_{ij} = 1$.

The fundamental group of the orbifold will be a Coxeter group with a presentation

$$R_i, i = 1, 2, \ldots f, \left(R_iR_j\right)^{n_{ij}} = 1$$

where R_i is associated with silvered sides and $R_{i,j}$ has order $n_{i,j}$ associated with the edge formed by the *i-th* and *j-th* side meeting.

A Coxeter orbifold whose polytope has a side *F* and a vertex *v* where all other sides are adjacent triangles to *F* and contains *v* and all edge orders of *F* are 2 is called a *cone-type* Coxeter orbifold. A

Coxeter orbifold whose polyhedron is topologically a polygon times an interval and edges orders of top and bottom sides are 2 is called a *product-type* Coxeter orbifold. If $\hat{P}$ is not above type and has not finite fundamental group, then $\hat{P}$ is said to be a *normal-type* Coxeter orbifold.

Theorem 2.2.3 (Choi, 2006). *Let P be a convex polyhedron and $\hat{P}$ be given a normal type Coxeter orbifold structure. Let k(P) be the dimension of the group of projective automorphisms acting on P. Suppose that $\hat{P}$ is orderable. Then the restricted deformation space of projective structures on the orbifold $\hat{P}$ is a smooth manifold of dimension $3f-e-e_2-k(P)$ if it is not empty.*

Corollary 2.2.4. *Let P be a convex polyhedron and $\hat{P}$ be given a normal type Coxeter orbifold structure. If $3f-e-e_2<0$ and $\hat{P}$ is orderable, then the restricted deformation space is empty.*

Remark 2.2.5. Let *P* be a convex polyhedron and $\hat{P}$ be given a normal type Coxeter orbifold structure. Let *k(P)* be the dimension of the group of projective automorphisms acting on *P*. Suppose that $\hat{P}$ is orderable. Then $\hat{P}$ is projectively deformable if and only if $3f-e-e_2-k(P)>0$.

2.3. Planar Graphs. The study of graphs is very important to understand the combinatorial structure of a polyhedron. We will discuss about the basis relation between graph theory and the 3-dimensional convex polyhedron.

Definition 2.3.1. A *planar graph* is a graph that can be drawn on the sphere(or the plane) without edge crossings. Two edges of a graph are *parallel* if they have the same endpoints. A *loop* is an edge whose endpoints are the same vertex. If there are neither parallel edges nor loops, a graph is called simple. A simple graph is called *k-connected* if the removal of any *k*-1 or fewer vertices (all the edges they are incident with) leaves a connected graph. The *dual graph* of a plane graph is a plane graph obtained from the original graph by exchanging the vertices and faces. The dual graph of a graph is *k*-connected if and only if the graph is *k*-connected. If all the faces of planar graph is triangles then the graph is called *triangulation*. The dual of a triangulation is a trivalent planar graph. A triangulation with *n* vertices has exactly $3n-6$ edges and $2n-4$ faces.

Definition 2.3.2. Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be two graphs imbedded on the sphere such that V_1,V_2 be the set of vertices of G_1,G_2 and E_1,E_2 be the sets of edges of G_1,G_2 . An *isomorphism* from G_1 to G_2 is a pair of bijections $\varphi:V_1\to V_2$ and $\varphi:E_1\to E_2$ which preserve the vertex-edge incidence relationship.

Definition 2.3.3. Let *P* be a convex polyhedron. The vertices and the edges of *P* from an abstract, finite, simple graph, called the graph of *P* and denoted by *G(P)*. Thus, *G(P)* is an abstract graph defined on the set of vertices vert *(P)* of *P*. Two vertices *u* and *v* in vert *(P)* are adjacent in *G(P)* if and only if *[u,v]* is an edge of *P*.

Definition 2.3.4. A 3-dimensional polyhedron is called *simplicial* polyhedron if every face contain exactly 3 vertices. A 3-dimensional polyhedron is called a simple polyhedron if each vertex is the intersection of exactly 3 faces.

Theorem 2.3.5 (Blind and Mani). *If P is convex polyhedron, then the graph G(P) determines the entire combinatorial structure of P.*
In other words, if two simple polyhedral have isomorphic graphs, then their combinatorial polyhedral are isomorphic as well.

Steinitz established the following basic theory for 3-dimensional polyhedron.

Theorem 2.3.6 (Steinitz). *G(P) is the graph of a 3-dimensional polyhedron P if and only if it is simple, planar and 3-connected.*

Corollary 2.3.7. *Every 3-connected planar graph has a representation in the plane such that all edges are straight, and all the bounded regions determined by it, as well as the union of all the bounded regions, are convex polygons.*

Since the compact hyperbolic polyhedron is simple, the combinatorial polyhedron of a compact hyperbolic polyhedron can be known from 3-connected planar graph of the polyhedron.

2.4. Plantri Program. The program *plantri* is one of the fastest C program which generates certain type of graphs that are imbedded on the sphere or the plane. Exactly one graph of each isomorphism class is output. A 3-dimensional convex polyhedron can be represented by a 3-connected simple planar graph. Plantri program generates the list of 3-connected simple planar graphs of finite number of vertices and hence generates the list of 3-dimensional polyhedrons of finite number of vertices up to isomorphism (See[11]) .
Ascii code is a human readable version of planar code. The vertices of the graph are named as Ascii characters starting with 'a'. Each line of output represents a graph.

Example 2.4.1. The output format of a graph is
10 bcd, aef, agd, ach, bif, beg, cfj, dji, ehj, gih

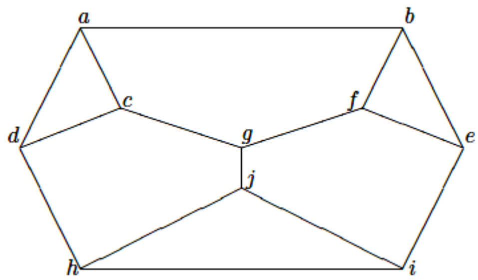

FIGURE 2

This is a graph with 10 vertices a, b, c, d, e, f, g, h, i, j. The neighboring vertices of 'a' in clockwise order are b, c, d; and so on. Starting neighboring vertex of a vertex in output format is the lowest ascii characher. For example: 'b' is the lowest ascii character among the neighbors of 'a' and hence neighbors of 'a' is started from 'b' in clockwide direction, etc.

III. Results

3.1. **Known Results from previous article.** In my previous article [The graphical investigation of orderable and deformable compact Coxeter polyhedral in hyperbolic space], we found the following theorems and propositions:

Let *P* be a CH-Coxeter polyhedron and $\hat{P}$ be its Coxeter orbifold structure of *P*.

Proposition 3.1.1. *If* $\hat{P}$ *is orderable, then P is also orderable.*

Remark 3.1.2. If *P* is not orderable, then $\hat{P}$ is also not orderable.
Proposition 3.1.3. *Let P be an orderable CH-Coxeter polyhedron. Assume that* $F_1, F_2, F_3, \ldots$ *be the face order of the faces of P. Then the number of edges in the face* F_i , i.e., *the length of* F_i *can't more than* $i+2$.

Corollary 3.1.4. *If P is a 3-dimensional orderable polyhedron then P has at least one triangular face.*

Corollary 3.1.5. *Let P be orderable 3-dimensional polyhedron. Suppose P has exactly one face of length* $\leq i$. *Then P has at least one face of length* $i+1$.

Let e be the number of edges of *P* and e_2 be the number of edges of edge order 2. Let v be the number of vertices of *P* and f be the number of faces of *P*.

Proposition 3.1.6. *Let P be a 3*-dimensional compact hyperbolic Coxeter polyhedron and $\hat{P}$ be Coxeter orbifold structure of *P*. Suppose $\hat{P}$ is orderable and projectively deformable. Then

a. *Every vertex is incident with exactly three edges.*

b. *Every vertex is incident with at least one edge of edge order 2.*

c. $\frac{v}{2} \leq e_2 \leq 5 \Rightarrow v \leq 10$.

d. *v is even.*

Proposition 3.1.7. *Let* $\hat{P}$ *be Coxeter orbifold structure of P. Assume that* $\hat{P}$ *is ODCH-Coxeter polyhedron. If* $v=10$ *, then edge orders of the three edges at each vertex is one of the following form*

$$(2,3,3),(2,3,4),(2,3,5),$$

i.e, no of the form $(2,2,n), n \geq 2$.

Proposition 3.1.8. *Let P be a 3-dimensional CH-Coxeter polyhedron and* $\hat{P}$ *be Coxeter orbifold structure of P. Suppose that P is not a tetrahedron. If* $\hat{P}$ *is orderable and projectively deformable, then P has not more one triangular face.*

Proposition 3.1.9. *Let P be a CH-Coxeter polyhedron and* $\hat{P}$ *be its Coxeter orbifold structure of P. Suppose that P is not a tetrahedron. If* $\hat{P}$ *is orderable and projectively deformable, then P has more one triangular face.*

Corollary 3.1.10. *Let P be a CH-Coxeter polyhedron and* $\hat{P}$ *be its Coxeter orbifold structure of P. Suppose that P is not a tetrahedron. If* $\hat{P}$ *is orderable and projectively deformable, then P has exactly two triangular face and both are disjoint.*

3.2 **Main Results.** Now we are ready to establish the main results.

Theorem 3.2.1. *Let P be a 3-dimensional CH-Coxeter polyhedron and* $\hat{P}$ *be its Coxeter orbifold structure of P. Suppose that P is not a tetrahedron. If* $\hat{P}$ *is orderable and projectively deformable. Then the total number of combinatorial polyhedron of such P is at most 5 and P is one of the combinatorial polyhedral in figure 1.*

Proof. By proposition 3.1.6, $v \leq 10$ and the polyhedral are trivalent. ODCH-Coxeter polyhedral are By Steinitz's theorem 2.3.6, the combinatorial polyhedral of ODCH-Coxeter polyhedral are 3-connected simple planar graphs. Using Plantri program,

Output of Plantri Program		
Name	Vertex	3-connected trivalent graphs
T	4	bcd, adc, abd, acb,
P6	6	bcd, aef, afd, ace, bdf, bec,
P8-1	8	bcd, aef, afg, agh, bhf, bec, chd, dge,
P8-2	8	bcd, aef, afg, age, bdh, bhc, chd,egf,
P10-1	10	bcd, aef, afg, ahi, bjf, bec, cjh, dgi,dhj, eig,
P10-2	10	bcd, aef, afd, acg, bhi, bic, djh, egj, ejf, gih,
P10-3	10	bcd, aef, agd, ach, bif, beg, cf, dji, ehj, gih,
P10-4	10	bcd, aef, afg, agh, bhi, bic, cjd, dje, ejf, gih,
P10-5	10	bcd, aef, agh, ahe, bdi, bjg, cfj, cid, eh, fig,

Table 1. Total 9 graph with $v \leq 10$

we obtain all the trivalent 3-connected simple planar graph with $v \leq 10$ as follows:

We obtain total 9 graphs as in table 1.

- **P8-2** and **P10-4** don't have triangular faces, therefore by proposition 3.1.9, **P8-2** and **P10-4** are not orderable.
- **P10-5** has only one triangular face. Therefore, by corollary 3.1.10, **P10-5** is not orderable.
- **P10-3** has 3 triangular faces. Therefore, by corollary 3.1.10, **P10-3** is not orderable.

Therefore the combinatorial polyhedron is one of **T, P6, P8-1, P10-1, P10-2**.

Theorem 3.2.2. *The numb*er of combinatorial polyhedral of ODCH Coxeter polyhedral is five.

Proof. By the theorem 3.2.1, the combinatorial polyhedron of ODCH Coxeter orbifold is one of the polyhedral **T, P6, P8-1, P10-1** and **P10-2**. If we can assign some order in each edge of the polyhedron, then each polyhedron possesses ODCH Coxeter orbifold structure.
We assign the order to the edges of the polyhedron **T, P6, P8-1, P10-1, P10-2** as in the following figures and then we assign the order of the faces. We verify that each of the following figure with edge orders satisfy orderability condition and hence they have ODCH Coxeter with orbifold structure.
In the tetrahedron *T*, each face of tetrahedron *T* adjacent with exactly three other faces, we can assign face order arbitrarily to make *T* orderable.

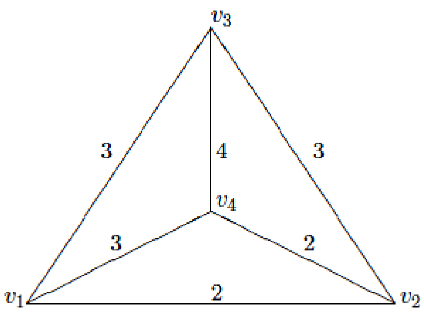

FIGURE 3. The orders of the edges of tetrahedron T

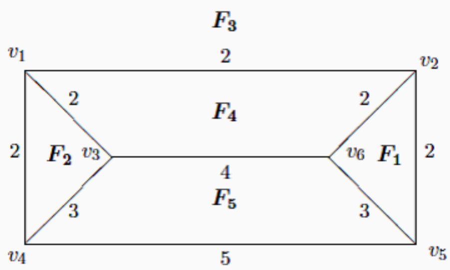

FIGURE 4. The order of the edges and the faces of Prism P6

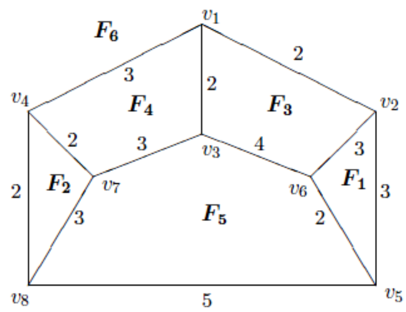

FIGURE 5. The order of the edges and the faces of P8-1

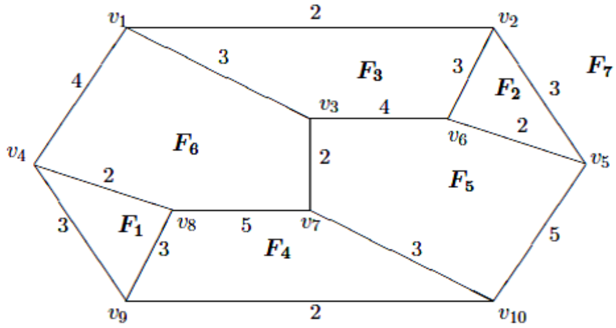

FIGURE 6. The orders of the edges and the faces of P10-1

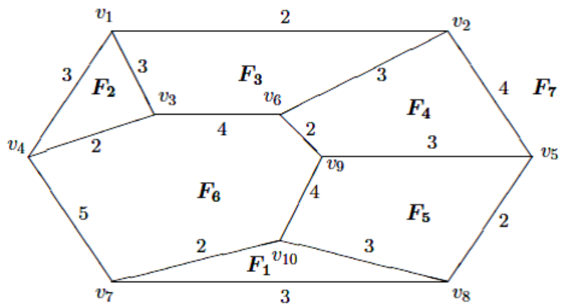

FIGURE 7. The orders of the edges and the faces of P10-2

IV. Conclusion

In this article, we proved that the number of combinatorial polyhedral of oderable and deformable compact hyperbolic Coxeter polyhedral in real projective space is exactly five. Form these combinatorial polyhedral, it can be extended to find all the 3-dimensional compact hyperbolic Coxeter polyhedral which are orderable and deformable in real projective space.

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