IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728,p-ISSN: 2319-765X, Volume 7, Issue 5 (Jul. - Aug. 2013), PP 30-37
www.iosrjournals.org

On the Derivation of High Order Formulae with Interpolants for Solution of Eight-Point Second Order Ordinary Differential Equations

Awari, Y. Sani
Department of Mathematics/Statistics Bingham University, Km 26 Keffi-Abuja expressway, P.M.B 005 Karu, Nigeria

Abstract: *In this Paper, we extend the idea of collocation of linear multistep methods to develop an eight-point Continuous Block method of order (7, 7, 7, 7, 7, 7, 7, 7)* ^{T} *for direct solution of the second order ordinary differential equations. The methods are derived by interpolating the continuous formulation at* $x = x_{n+j}$, $j = k$ *and collocating the first and second derivative of the continuous interpolant at* x_{n+j} , $j = 0,1,2,(k)$ *and* $j = 2,3,(k)$ *respectively. This approach yielded the multi discrete schemes that form a self-starting uniform order 7 block methods. The convergence analysis of the methods were discussed and the absolute stability regions shown. Two numerical experiments were used to demonstrate the efficiency of the new methods.*
Keywords: *Linear multistep methods (LMM), zero-stable, block method, interpolation and collocation.*

I. Introduction

We are concerned with the analysis of a general family of eight–step collocation methods for the numerical integration of second order Ordinary Differential Equations (ODEs) of the form:

$y'' = f(x, y, y')$, $y(0) = \propto$, $y'(0) = \beta$

(1)

These initial value problems are important mathematical models in many applications of molecular dynamics, orbital mechanics, seismology, and so on. In many situations they may have large dimension; when the response time is extremely important, for example in simulation processes, there is the need of obtaining an accurate solution in a reasonable time frame. Therefore there is a great demand of efficient methods for problem (1). Moreover the system of ODEs (1) may be stiff, and therefore the method has to be implicit and highly stable. Many numerical methods for the direct integration of (1) appeared in the literature: see for example [2], [3], [7], [10], [11] and the bibliography therein contained.

We approximate the exact solution y(x) by seeking the continuous method $\overline{y}(x)$ of the form

$\overline{y}(x) = \sum_{j=o}^{s-1} \propto_j (x) y_{n+j} + h^2 \sum_{j=o}^{r-1} \beta_j (x) f_{n+j}$

(2)

Where $x \epsilon [a, b]$ and the following notations are introduced. The positive integer $k \geq 2$ denotes the step number of the method (2), which is applied directly to provide the solution to (1).

II. Construction Of The Methods

We propose an approximate solution to (1) in the form:

$y_k(x) = \sum_{j=o}^{s+r-1} a_j x^j$

(3)

$y'_k(x) = \sum_{j=o}^{s+r-1} j a_j x^{(j-1)}$

(4)

$y''_k(x) = \sum_{j=o}^{s+r-1} j(j-1) a_j x^{(j-2)} = f(x, y, y')$

(5)

We consider, $k = 8$, interpolating (3) at x_{n+j} , $j = k$ and collocating (5) at x_{n+j} , $j = 2, 3, \ldots, k$ leads to a system of equations written in the form.

$a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5 + a_6 x_n^6 + a_7 x_n^7 + a_8 x_n^8 = y_n$
$a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + a_4 x_{n+1}^4 + a_5 x_{n+1}^5 + a_6 x_{n+1}^6 + a_7 x_{n+1}^7 + a_8 x_{n+1}^8 = y_{n+1}$
$a_0 + a_1 x_{n+2} + a_2 x_{n+2}^2 + a_3 x_{n+2}^3 + a_4 x_{n+2}^4 + a_5 x_{n+2}^5 + a_6 x_{n+2}^6 + a_7 x_{n+2}^7 + a_8 x_{n+2}^8 = y_{n+2}$
$a_0 + a_1 x_{n+3} + a_2 x_{n+3}^2 + a_3 x_{n+3}^3 + a_4 x_{n+3}^4 + a_5 x_{n+3}^5 + a_6 x_{n+3}^6 + a_7 x_{n+3}^7 + a_8 x_{n+3}^8 = y_{n+3}$
$a_0 + a_1 x_{n+4} + a_2 x_{n+4}^2 + a_3 x_{n+4}^3 + a_4 x_{n+4}^4 + a_5 x_{n+4}^5 + a_6 x_{n+4}^6 + a_7 x_{n+4}^7 + a_8 x_{n+4}^8 = y_{n+4}$
$a_0 + a_1 x_{n+5} + a_2 x_{n+5}^2 + a_3 x_{n+5}^3 + a_4 x_{n+5}^4 + a_5 x_{n+5}^5 + a_6 x_{n+5}^6 + a_7 x_{n+5}^7 + a_8 x_{n+5}^8 = y_{n+5}$
$a_0 + a_1 x_{n+6} + a_2 x_{n+6}^2 + a_3 x_{n+6}^3 + a_4 x_{n+6}^4 + a_5 x_{n+6}^5 + a_6 x_{n+6}^6 + a_7 x_{n+6}^7 + a_8 x_{n+6}^8 = y_{n+6}$

$a_0 + a_1 x_{n+7} + a_2 x_{n+7}^2 + a_3 x_{n+7}^3 + a_4 x_{n+7}^4 + a_5 x_{n+7}^5 + a_6 x_{n+7}^6 + a_7 x_{n+7}^7 + a_8 x_{n+7}^8 = y_{n+7}$

$2a_2 + 6a_3 x_{n+8} + 12 a_4 x_{n+8}^2 + 20 a_5 x_{n+8}^3 + 30 a_6 x_{n+8}^4 + 42 a_7 x_{n+8}^5 + 56 a_8 x_{n+8}^6 = f_{n+6}$

(6)

Where $a_j{}'_s$ are the parameters to be determined?

When re-arranging (6) in a matrix form $AX = B$, we obtained

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 \\ 1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 & x_{n+4}^8 \\ 1 & x_{n+5} & x_{n+5}^2 & x_{n+5}^3 & x_{n+5}^4 & x_{n+5}^5 & x_{n+5}^6 & x_{n+5}^7 & x_{n+5}^8 \\ 1 & x_{n+6} & x_{n+6}^2 & x_{n+6}^3 & x_{n+6}^4 & x_{n+6}^5 & x_{n+6}^6 & x_{n+6}^7 & x_{n+6}^8 \\ 1 & x_{n+7} & x_{n+7}^2 & x_{n+7}^3 & x_{n+7}^4 & x_{n+7}^5 & x_{n+7}^6 & x_{n+7}^7 & x_{n+7}^8 \\ 0 & 0 & 2 & 6x_{n+8} & 12\,x_{n+8}^2 & 20\,x_{n+8}^3 & 30\,x_{n+8}^4 & 42\,x_{n+8}^5 & 56\,x_{n+8}^6 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \\ y_{n+7} \\ f_{n+8} \end{pmatrix} \qquad (7)$$

Where $a_j{}'_s$ are obtained as continuous coefficients of $\propto_j(x)$ and $\beta_j(x)$

Specifically, from (2) the proposed solution takes the form

$$y(x) = \propto_0(x)\, y_n + \propto_1(x)\, y_{n+1} + \propto_2(x)\, y_{n+2} + \propto_3(x)\, y_{n+3} + \propto_4(x)\, y_{n+4} + \propto_5(x)\, y_{n+5} + \propto_6(x)\, y_{n+6} + \propto_7(x)\, y_{n+7} + h^2[\beta_8(x)\, f_{n+8}] \qquad (8)$$

A mathematical package (Maple 13) is used to obtained the inverse of the matrix of equation (7), where values for $a_j{}'_s$ were established. After some manipulation to the inverse, we obtain the desired continuous formulation.

Evaluating the continuous formulation at $x_{n+j},\ j = k$ and its second derivative evaluated at $x_{n+j},\ j = 2, \dots, k$ while its 1st derivative is evaluated at $x_{n+j},\ j = 0$ yields the following set of discrete equations.

$$\frac{29531}{5040} y_{n+8} - \frac{962}{35} y_{n+7} + \frac{621}{10} y_{n+6} - \frac{4006}{45} y_{n+5} + \frac{691}{8} y_{n+4} - \frac{282}{5} y_{n+3} + \frac{2143}{90} y_{n+2} - \frac{206}{35} y_{n+1} + \frac{363}{560} y_n = h^2[f_{n+8}]$$

$$\frac{6197}{5} y_{n+7} - \frac{148343}{20} y_{n+6} + 20097 y_{n+5} - \frac{116651}{4} y_{n+4} - 8557 y_{n+3} + \frac{1108467}{20} y_{n+2} - \frac{165913}{5} y_{n+1} + \frac{6239}{4} y_n = h^2[47 f_{n+8} - 29531 f_{n+2}]$$

$$\frac{349}{2} y_{n+7} - \frac{21223}{36} y_{n+6} - \frac{9111}{4} y_{n+5} + \frac{82765}{2} y_{n+4} - \frac{701500}{9} y_{n+3} + \frac{172137}{4} y_{n+2} - \frac{16243}{4} y_{n+1} + \frac{2531}{9} y_n = h^2[9 f_{n+8} + 29531 f_{n+3}]$$

$$\frac{22618}{45} y_{n+7} - \frac{53473}{10} y_{n+6} + \frac{232242}{5} y_{n+5} - \frac{1499471}{18} y_{n+4} + 46742 y_{n+3} - \frac{56919}{10} y_{n+2} + \frac{31366}{45} y_{n+1} - \frac{469}{10} y_n = h^2[9 f_{n+8} + 29531 f_{n+4}]$$

$$\frac{54319}{20} y_{n+7} - \frac{159605}{4} y_{n+6} + \frac{366264}{5} y_{n+5} - \frac{73091}{2} y_{n+4} - \frac{4697}{4} y_{n+3} + \frac{45351}{20} y_{n+2} - \frac{1397}{2} y_{n+1} + \frac{416}{5} y_n = h^2[47 f_{n+8} - 29531 f_{n+5}]$$

$$\frac{128661}{5} y_{n+7} - \frac{6857303}{180} y_{n+6} - \frac{86643}{5} y_{n+5} + \frac{223065}{4} y_{n+4} - \frac{351013}{9} y_{n+3} + \frac{331011}{20} y_{n+2} - \frac{20337}{5} y_{n+1} + \frac{80023}{180} y_n = h^2[261 f_{n+8} + 29531 f_{n+6}]$$

$$-\frac{4074562}{45} y_{n+7} + \frac{6508687}{20} y_{n+6} - \frac{2066991}{4} y_{n+5} + \frac{8984843}{18} y_{n+4} - \frac{646109}{2} y_{n+3} + \frac{2707527}{20} y_{n+2} - \frac{5988163}{180} y_{n+1} + 3647 y_n = h^2[3267 f_{n+8} - 29531 f_{n+7}]$$

$$\frac{57518}{2205} y_{n+7} - \frac{59213}{378} y_{n+6} + \frac{228886}{525} y_{n+5} - \frac{2589307}{3528} y_{n+4} + \frac{154714}{189} y_{n+3} - \frac{398433}{630} y_{n+2} + \frac{23254}{63} y_{n+1} - \frac{11179373}{88200} y_n - \frac{29531}{630} h z_n = h^2[f_{n+8}] \qquad (9)$$

Where $Z_n = y_n'$

Equation (9) is the proposed six-step block method. Furthermore, the first derivative of the continuous formulation was evaluated at x_{n+j}, $j = 1, 2, (k)$ which yields the following discrete schemes.

$$z_{n+1}' = \frac{1}{12403020h}[-1208210 y_{n+7} + 6334734 y_{n+6} - 18340245 y_{n+5} + 32910500 y_{n+4} - 41278650 y_{n+3} + 42510510 y_{n+2} - 19533759 y_{n+1} - 1574880 y_n + 37800 h^2[f_{n+8}]$$

$$z_{n+2}' = \frac{-1}{12403020h}[-362424 y_{n+7} + 2318295 y_{n+6} - 7147000 y_{n+5} + 14415450 y_{n+4} - 24095400 y_{n+3} + 11482849 y_{n+2} + 3617880 y_{n+1} - 229650 y_n + 12600 h^2[f_{n+8}]$$

$$z_{n+3}' = \frac{1}{12403020h}[-235173 y_{n+7} + 1597694 y_{n+6} - 5528502 y_{n+5} + 14850780 y_{n+4} - 5154975 y_{n+3} - 6381522 y_{n+2} + 930426 y_{n+1} - 78728 y_n + 7560 h^2[f_{n+8}]$$

$$z'_{n+4} = \frac{-1}{12403020h}[-264704y_{n+7} + 2011128y_{n+6} - 9249408y_{n+5} - 652995y_{n+4} + 10348800y_{n+3} - 2660616y_{n+2} + 516992y_{n+1} - 49197y_n + 7560h^2[f_{n+8}]$$

$$z'_{n+5} = \frac{1}{12403020h}[-539610y_{n+7} + 5419050y_{n+6} + 6703039y_{n+5} - 16592100y_{n+4} + 6912150y_{n+3} - 2367190y_{n+2} + 517125y_{n+1} - 52464y_n + 12600h^2[f_{n+8}]$$

$$z'_{n+6} = \frac{-1}{12403020h}[-2504760y_{n+7} - 14130249y_{n+6} + 28171080y_{n+5} - 18768750y_{n+4} + 10400600y_{n+3} - 4000815y_{n+2} + 931224y_{n+1} - 98330y_n + 37800h^2[f_{n+8}]$$

$$z'_{n+7} = \frac{1}{12403020h}[27028959y_{n+7} - 59842230y_{n+6} + 66965850y_{n+5} - 59030300y_{n+4} + 36628725y_{n+3} - 14982534y_{n+2} + 3624530y_{n+1} - 393000y_n + 264600h^2[f_{n+8}]$$

$$z'_{n+8} = \frac{1}{12403020\,h}[58905552y_{n+7} - 183628956y_{n+6} + 280636048y_{n+5} - 279876345y_{n+4} + 185564400y_{n+3} - 79108372y_{n+2} + 19686576y_{n+1} - 2178903y_n + 5753160h^2[f_{n+8}] \qquad (10)$$

The application of the block integrators (9) using equation (10) for n = 0 simultaneously, give the values of $y_1, y_2, \ldots, y_k$ directly without the use of starting values.

III. Convergence Analysis Of The Methods

Definition 1: determination of order of a Block Method

The LMM (2) can be expanded into the system

$$\begin{pmatrix} \propto_{01} & \propto_{11} & \propto_{21} & \cdots & \propto_{k1} \\ \propto_{01} & \propto_{12} & \propto_{22} & \cdots & \propto_{k2} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \propto_{0k} & \propto_{1k} & \propto_{2k} & \cdots & \propto_{kk} \end{pmatrix} \begin{pmatrix} y_n \\ y_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ y_{n+k} \end{pmatrix} = h^2 \begin{pmatrix} \beta_{01} & \beta_{11} & \beta_{21} & \cdots & \beta_{k1} \\ \beta_{02} & \beta_{12} & \beta_{22} & \cdots & \beta_{k2} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \beta_{0k} & \beta_{1k} & \beta_{2k} & \cdots & \beta_{kk} \end{pmatrix} \begin{pmatrix} f_n \\ f_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ f_{n+k} \end{pmatrix} \qquad (11)$$

This is equivalent to

$$\sum_{i=0}^{k} \overrightarrow{\alpha_k}\, y_{n+k} = h^2 \sum_{i=0}^{k} \overrightarrow{\beta_k}\, f_{n+k}$$
(11)

Where

$\overrightarrow{\alpha_0} = (\propto_{01} \;\; \propto_{02} \ldots \;\; \propto_{0k})^T$

$\overrightarrow{\alpha_1} = (\propto_{11} \;\; \propto_{12} \ldots \;\; \propto_{1k})^T$

.

.

.

$\overrightarrow{\alpha_k} = (\propto_{k1} \;\; \propto_{k2} \ldots \;\; \propto_{kk})^T$

and

$\overrightarrow{\beta_0} = (\beta_{01} \;\; \beta_{02} \ldots \;\; \beta_{0k})^T$

$\overrightarrow{\beta_1} = (\beta_{11} \;\; \beta_{12} \ldots \;\; \beta_{1k})^T$

.

.

.

$\overrightarrow{\beta_k} = (\beta_{k1} \;\; \beta_{k2} \ldots \;\; \beta_{kk})^T$

Following Fatunla [4, 5] and Lambert [8, 9] we define the local truncation error associated with the conventional form of (2) to be the linear difference operator.

$$L[y(x); h] = \sum_{j=o}^{k} \propto_j \; y(x+jh) - h^2 \beta_j y''(x+jh) \qquad (12)$$

Where the constant coefficients C_q, $q = 2,3, \ldots$ are given as follows:

$\overrightarrow{c_0} = \sum_{j=0}^{k} \overrightarrow{\alpha_j}$

$\overrightarrow{c_1} = \sum_{j=0}^{k} j\overrightarrow{\alpha_j}$

.

.

.

$\overrightarrow{c_q} = \sum_{j=0}^{k} \left[\frac{1}{q!} j^q \, \overrightarrow{\alpha_j} - \frac{1}{(q-2)!} \, j^{q-2} \overrightarrow{\beta_j}\right], q = 2, 3, \ \ldots$

We then have that the block method (11) is of order P if

$\overrightarrow{c_0} = \overrightarrow{c_1} = \overrightarrow{c_2} = \cdots = \overrightarrow{c_{p+1}} = 0, \ \overrightarrow{c_{p+2}} \neq 0 \qquad (13)$

Applying (11) – (13), we obtain the order and error constant of the block methods (9) in the following way:

$$c_2 = \frac{1}{2!}\left\{\begin{pmatrix} \frac{-206}{35} \\ \frac{-165913}{5} \\ \frac{-16243}{4} \\ \frac{31366}{45} \\ \frac{-1397}{2} \\ \frac{-20337}{5} \\ \frac{-5988163}{180} \\ \frac{23254}{63} \end{pmatrix} + 2^2 \begin{pmatrix} \frac{2143}{90} \\ \frac{1108467}{20} \\ \frac{172137}{4} \\ \frac{-56919}{10} \\ \frac{45351}{20} \\ \frac{331011}{20} \\ \frac{2707527}{20} \\ \frac{-398433}{630} \end{pmatrix} + 3^2 \begin{pmatrix} \frac{-282}{5} \\ -8557 \\ \frac{-701500}{9} \\ 46742 \\ \frac{-4697}{4} \\ \frac{-351013}{9} \\ \frac{-646109}{2} \\ \frac{154714}{189} \end{pmatrix} + 4^2 \begin{pmatrix} \frac{691}{8} \\ \frac{-116651}{4} \\ \frac{82765}{2} \\ \frac{-1499471}{18} \\ \frac{-73091}{2} \\ \frac{223065}{4} \\ \frac{8984843}{18} \\ \frac{-2589307}{3528} \end{pmatrix} + 5^2 \begin{pmatrix} \frac{-4006}{45} \\ 20097 \\ \frac{-9111}{4} \\ \frac{232242}{5} \\ \frac{366264}{5} \\ \frac{-86643}{5} \\ \frac{-2066991}{4} \\ \frac{228886}{525} \end{pmatrix} + 6^2 \begin{pmatrix} \frac{621}{10} \\ \frac{-148343}{20} \\ \frac{-21223}{36} \\ \frac{-53473}{10} \\ \frac{-159605}{4} \\ \frac{-6857303}{180} \\ \frac{6508687}{20} \\ \frac{-59213}{378} \end{pmatrix} + 7^2 \begin{pmatrix} \frac{-962}{35} \\ \frac{6197}{5} \\ \frac{349}{2} \\ \frac{22618}{45} \\ \frac{54319}{20} \\ \frac{128661}{5} \\ \frac{-4074562}{45} \\ \frac{57518}{2205} \end{pmatrix} + 8^2 \begin{pmatrix} \frac{29531}{5040} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}$$

$$-\frac{1}{(2-1)!}\left\{\begin{pmatrix} \frac{-206}{35} \\ \frac{-165913}{5} \\ \frac{-16243}{4} \\ \frac{31366}{45} \\ \frac{-1397}{2} \\ \frac{-20337}{5} \\ \frac{-5988163}{180} \\ \frac{23254}{63} \end{pmatrix} + 2^{(2-1)} \begin{pmatrix} \frac{2143}{90} \\ \frac{1108467}{20} \\ \frac{172137}{4} \\ \frac{-56919}{10} \\ \frac{45351}{20} \\ \frac{331011}{20} \\ \frac{2707527}{20} \\ \frac{-398433}{630} \end{pmatrix} + 3^{(2-1)} \begin{pmatrix} \frac{-282}{5} \\ -8557 \\ \frac{-701500}{9} \\ 46742 \\ \frac{-4697}{4} \\ \frac{-351013}{9} \\ \frac{-646109}{2} \\ \frac{154714}{189} \end{pmatrix} + 4^{(2-1)} \begin{pmatrix} \frac{691}{8} \\ \frac{-116651}{4} \\ \frac{82765}{2} \\ \frac{-1499471}{18} \\ \frac{-73091}{2} \\ \frac{223065}{4} \\ \frac{8984843}{18} \\ \frac{-2589307}{3528} \end{pmatrix}\right.$$

$$\left. + 5^{(2-1)} \begin{pmatrix} \frac{-4006}{45} \\ 20097 \\ \frac{-9111}{4} \\ \frac{232242}{5} \\ \frac{366264}{5} \\ \frac{-86643}{5} \\ \frac{-2066991}{4} \\ \frac{228886}{525} \end{pmatrix} + 6^{(2-1)} \begin{pmatrix} \frac{621}{10} \\ \frac{-148343}{20} \\ \frac{-21223}{36} \\ \frac{-53473}{10} \\ \frac{-159605}{4} \\ \frac{-6857303}{180} \\ \frac{6508687}{20} \\ \frac{-59213}{378} \end{pmatrix} + 7^{(2-1)} \begin{pmatrix} \frac{-962}{35} \\ \frac{6197}{5} \\ \frac{349}{2} \\ \frac{22618}{45} \\ \frac{54319}{20} \\ \frac{128661}{5} \\ \frac{-4074562}{45} \\ \frac{57518}{2205} \end{pmatrix} + 8^{(2-1)} \begin{pmatrix} \frac{29531}{5040} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$c_3 = \frac{1}{3!}\left\{\begin{pmatrix} \frac{-206}{35} \\ \frac{-165913}{5} \\ \frac{-16243}{4} \\ \frac{31366}{45} \\ \frac{-1397}{2} \\ \frac{-20337}{5} \\ \frac{-5988163}{180} \\ \frac{23254}{63} \end{pmatrix} + 2^3 \begin{pmatrix} \frac{2143}{90} \\ \frac{1108467}{20} \\ \frac{172137}{4} \\ \frac{-56919}{10} \\ \frac{45351}{20} \\ \frac{331011}{20} \\ \frac{2707527}{20} \\ \frac{-398433}{630} \end{pmatrix} + 3^3 \begin{pmatrix} \frac{-282}{5} \\ -8557 \\ \frac{-701500}{9} \\ 46742 \\ \frac{-4697}{4} \\ \frac{-351013}{9} \\ \frac{-646109}{2} \\ \frac{154714}{189} \end{pmatrix} + 4^3 \begin{pmatrix} \frac{691}{8} \\ \frac{-116651}{4} \\ \frac{82765}{2} \\ \frac{-1499471}{18} \\ \frac{-73091}{2} \\ \frac{223065}{4} \\ \frac{8984843}{18} \\ \frac{-2589307}{3528} \end{pmatrix} + 5^3 \begin{pmatrix} \frac{-4006}{45} \\ 20097 \\ \frac{-9111}{4} \\ \frac{232242}{5} \\ \frac{366264}{5} \\ \frac{-86643}{5} \\ \frac{-2066991}{4} \\ \frac{228886}{525} \end{pmatrix} + 6^3 \begin{pmatrix} \frac{621}{10} \\ \frac{-148343}{20} \\ \frac{-21223}{36} \\ \frac{-53473}{10} \\ \frac{-159605}{4} \\ \frac{-6857303}{180} \\ \frac{6508687}{20} \\ \frac{-59213}{378} \end{pmatrix} + 7^3 \begin{pmatrix} \frac{-962}{35} \\ \frac{6197}{5} \\ \frac{349}{2} \\ \frac{22618}{45} \\ \frac{54319}{20} \\ \frac{128661}{5} \\ \frac{-4074562}{45} \\ \frac{57518}{2205} \end{pmatrix} + 8^3 \begin{pmatrix} \frac{29531}{5040} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}$$

$$-\frac{1}{(3-1)!}\left\{\begin{pmatrix} \frac{-206}{35} \\ \frac{-165913}{5} \\ \frac{-16243}{4} \\ \frac{31366}{45} \\ \frac{-1397}{2} \\ \frac{-20337}{5} \\ \frac{-5988163}{180} \\ \frac{23254}{63} \end{pmatrix} + 2^{(3-1)} \begin{pmatrix} \frac{2143}{90} \\ \frac{1108467}{20} \\ \frac{172137}{4} \\ \frac{-56919}{10} \\ \frac{45351}{20} \\ \frac{331011}{20} \\ \frac{2707527}{20} \\ \frac{-398433}{630} \end{pmatrix} + 3^{(3-1)} \begin{pmatrix} \frac{-282}{5} \\ -8557 \\ \frac{-701500}{9} \\ 46742 \\ \frac{-4697}{4} \\ \frac{-351013}{9} \\ \frac{-646109}{2} \\ \frac{154714}{189} \end{pmatrix} + 4^{(3-1)} \begin{pmatrix} \frac{691}{8} \\ \frac{-116651}{4} \\ \frac{82765}{2} \\ \frac{-1499471}{18} \\ \frac{-73091}{2} \\ \frac{223065}{4} \\ \frac{8984843}{18} \\ \frac{-2589307}{3528} \end{pmatrix}\right.$$

$$\left. + 5^{(3-1)} \begin{pmatrix} \frac{-4006}{45} \\ 20097 \\ \frac{-9111}{4} \\ \frac{232242}{5} \\ \frac{366264}{5} \\ \frac{-86643}{5} \\ \frac{-2066991}{4} \\ \frac{228886}{525} \end{pmatrix} + 6^{(3-1)} \begin{pmatrix} \frac{621}{10} \\ \frac{-148343}{20} \\ \frac{-21223}{36} \\ \frac{-53473}{10} \\ \frac{-159605}{4} \\ \frac{-6857303}{180} \\ \frac{6508687}{20} \\ \frac{-59213}{378} \end{pmatrix} + 7^{(3-1)} \begin{pmatrix} \frac{-962}{35} \\ \frac{6197}{5} \\ \frac{349}{2} \\ \frac{22618}{45} \\ \frac{54319}{20} \\ \frac{128661}{5} \\ \frac{-4074562}{45} \\ \frac{57518}{2205} \end{pmatrix} + 8^{(3-1)} \begin{pmatrix} \frac{29531}{5040} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

.
.
.

$$c_9 = \frac{1}{9!}\left\{\begin{pmatrix}\frac{-206}{35}\\ \frac{-165913}{5}\\ \frac{-16243}{4}\\ \frac{31366}{45}\\ \frac{-1397}{2}\\ \frac{-20337}{5}\\ \frac{-5988163}{180}\\ \frac{23254}{63}\end{pmatrix} + 2^9\begin{pmatrix}\frac{2143}{90}\\ \frac{1108467}{20}\\ \frac{172137}{4}\\ \frac{-56919}{10}\\ \frac{45351}{20}\\ \frac{331011}{20}\\ \frac{2707527}{20}\\ \frac{-398433}{630}\end{pmatrix} + 3^9\begin{pmatrix}\frac{-282}{5}\\ -8557\\ \frac{-701500}{9}\\ 46742\\ \frac{-4697}{4}\\ \frac{-351013}{9}\\ \frac{-646109}{2}\\ \frac{154714}{189}\end{pmatrix} + 4^9\begin{pmatrix}\frac{691}{8}\\ \frac{-116651}{4}\\ \frac{82765}{2}\\ \frac{-1499471}{18}\\ \frac{-73091}{2}\\ \frac{223065}{4}\\ \frac{8984843}{18}\\ \frac{-2589307}{3528}\end{pmatrix} + 5^9\begin{pmatrix}\frac{-4006}{45}\\ 20097\\ \frac{-9111}{4}\\ \frac{232242}{5}\\ \frac{366264}{5}\\ \frac{-86643}{5}\\ \frac{-2066991}{4}\\ \frac{228886}{525}\end{pmatrix} + 6^9\begin{pmatrix}\frac{621}{10}\\ \frac{-148343}{20}\\ \frac{-21223}{36}\\ \frac{-53473}{10}\\ \frac{-159605}{4}\\ \frac{-6857303}{180}\\ \frac{6508687}{20}\\ \frac{-59213}{378}\end{pmatrix} + 7^9\begin{pmatrix}\frac{-962}{35}\\ \frac{6197}{5}\\ \frac{349}{2}\\ \frac{22618}{45}\\ \frac{54319}{20}\\ \frac{128661}{5}\\ \frac{-4074562}{45}\\ \frac{57518}{2205}\end{pmatrix} + 8^9\begin{pmatrix}\frac{29531}{5040}\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix}\right\}$$

$$\frac{-5988163}{180}\ \frac{23254}{63}\quad \frac{2707527}{20}\ \frac{-398433}{630}\quad \frac{-646109}{2}\ \frac{154714}{189}\quad \frac{8984843}{18}\ \frac{-2589307}{3528}\quad \frac{-2066991}{4}\ \frac{228886}{525}\quad \frac{6508687}{20}\ \frac{-59213}{378}\quad \frac{-4074562}{45}\ \frac{57518}{2205}\quad 0\ 0$$

$$-\frac{1}{(9-1)!}\left\{\begin{pmatrix}\frac{-206}{35}\\ \frac{-165913}{5}\\ \frac{-16243}{4}\\ \frac{31366}{45}\\ \frac{-1397}{2}\\ \frac{-20337}{5}\\ \frac{-5988163}{180}\\ \frac{23254}{63}\end{pmatrix} + 2^{(9-1)}\begin{pmatrix}\frac{2143}{90}\\ \frac{1108467}{20}\\ \frac{172137}{4}\\ \frac{-56919}{10}\\ \frac{45351}{20}\\ \frac{331011}{20}\\ \frac{2707527}{20}\\ \frac{-398433}{630}\end{pmatrix} + 3^{(9-1)}\begin{pmatrix}\frac{-282}{5}\\ -8557\\ \frac{-701500}{9}\\ 46742\\ \frac{-4697}{4}\\ \frac{-351013}{9}\\ \frac{-646109}{2}\\ \frac{154714}{189}\end{pmatrix} + 4^{(9-1)}\begin{pmatrix}\frac{691}{8}\\ \frac{-116651}{4}\\ \frac{82765}{2}\\ \frac{-1499471}{18}\\ \frac{-73091}{2}\\ \frac{223065}{4}\\ \frac{8984843}{18}\\ \frac{-2589307}{3528}\end{pmatrix}\right.$$

$$\left. + 5^{(9-1)}\begin{pmatrix}\frac{-4006}{45}\\ 20097\\ \frac{-9111}{4}\\ \frac{232242}{5}\\ \frac{366264}{5}\\ \frac{-86643}{5}\\ \frac{-2066991}{4}\\ \frac{228886}{525}\end{pmatrix} + 6^{(9-1)}\begin{pmatrix}\frac{621}{10}\\ \frac{-148343}{20}\\ \frac{-21223}{36}\\ \frac{-53473}{10}\\ \frac{-159605}{4}\\ \frac{-6857303}{180}\\ \frac{6508687}{20}\\ \frac{-59213}{378}\end{pmatrix} + 7^{(9-1)}\begin{pmatrix}\frac{-962}{35}\\ \frac{6197}{5}\\ \frac{349}{2}\\ \frac{22618}{45}\\ \frac{54319}{20}\\ \frac{128661}{5}\\ \frac{-4074562}{45}\\ \frac{57518}{2205}\end{pmatrix} + 8^{(9-1)}\begin{pmatrix}\frac{29531}{5040}\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix}\right\} = \begin{pmatrix}\frac{-761}{1260}\\ \frac{-632623}{2520}\\ \frac{-6515}{112}\\ \frac{-761}{140}\\ \frac{-408847}{5040}\\ \frac{-958331}{2520}\\ \frac{-16530161}{5040}\\ \frac{-65911}{11340}\end{pmatrix}$$

Hence our proposed block methods (9) have uniform order $p = 7$ and error constant given by the vector $c_{p+2} = \left(-\frac{761}{1260}, -\frac{632623}{2520}, -\frac{6515}{112}, -\frac{761}{140}, -\frac{408847}{5040}, -\frac{958331}{2520}, -\frac{16530161}{5040}, -\frac{65911}{11340}\right)^T$.

IV. Zero-Stability of the Block Method

The block methods shown in (9) can be represented by a matrix finite difference equation in the form:

$$IY_{w+1} = A^0 Y_{w-1} + h^2\left[B^1 F_{w+1} + B^0 F_{w-1}\right]$$
(14)

Where

$Y_{w+1} = (y_{n+1}, \ldots, y_{n+8})^T, \quad Y_{w-1} = (y_{n-7}, \ldots, y_n)^T,$

$F_{w+1} = (f_{n+1}, \dots, f_{n+8})^T, \; F_{w-1} = (f_{n-7}, \dots, f_n)^T,$
And w = 0, 1, 2 , … and n is the grid index
And
I = Identity matrix

$$A^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B^1 = \begin{pmatrix} 0 & \frac{1152169}{120960} & -\frac{242653}{6720} & \frac{296053}{4480} & \frac{2102243}{30240} & \frac{115747}{2688} & -\frac{32863}{2240} & \frac{36799}{17280} \\ 0 & \frac{736}{27} & -\frac{6226}{63} & \frac{112339}{630} & -\frac{175996}{945} & \frac{36179}{315} & -\frac{2458}{63} & \frac{10681}{1890} \\ 0 & \frac{207807}{4480} & -\frac{10437}{64} & \frac{1313433}{4480} & -\frac{342201}{1120} & \frac{843009}{4480} & -\frac{28611}{448} & \frac{8283}{896} \\ 0 & \frac{61976}{945} & -\frac{23776}{105} & 408 & -\frac{401344}{945} & \frac{27448}{105} & -\frac{3104}{35} & \frac{12128}{945} \\ 0 & \frac{2050925}{24192} & -\frac{1168075}{4032} & \frac{4222625}{8064} & -\frac{469825}{864} & \frac{2698625}{8064} & -\frac{457675}{4032} & \frac{397325}{24192} \\ 0 & \frac{3639}{35} & -\frac{12354}{35} & \frac{44757}{70} & -\frac{23172}{35} & 408 & -\frac{4842}{35} & \frac{1401}{70} \\ 0 & \frac{425663}{3456} & -\frac{79919}{192} & \frac{483287}{640} & -\frac{3370661}{4320} & \frac{925757}{1920} & -\frac{10437}{64} & \frac{407827}{17280} \\ 0 & \frac{134528}{945} & -\frac{30208}{63} & \frac{274304}{315} & -\frac{848896}{945} & \frac{175232}{315} & \frac{11776}{63} & \frac{736}{27} \end{pmatrix}$$

And $B^0 = 0$

It is worth noting that zero-stability is concerned with the stability of the difference system in the limit as h tends to zero. Thus, as $h \to 0$, the method (14) tends to the difference system.

$IY_{w+1} - A^0 Y_{w-1} = 0$

Whose first characteristic polynomial $\rho(\lambda)$ is given by

$\rho(\lambda) = \det(\lambda I - A^0)$
$= \lambda^7(\lambda - 1)$
(15)

Following Fatunla [5], the block method (9) is zero-stable, since from (15), $\rho(\lambda) = 0$ satisfy $|\lambda_j| \le 1, \; j = 1, \dots, k$ and for those roots with $|\lambda_j| = 1$, the multiplicity does not exceed 2. The block method (9) is consistent as it has order P > 1. Accordingly, following Henrici [6], we assert the convergence of the block method (12).

V. IMPLEMENTATION STRATEGIES

Equation (10) was substituted into Equation (9) and when solved simultaneously provides for $y_1, y_2, \dots, y_k$ at once without recourse to any Predictors [1]. In this section, we test the performance of our eight-point block methods on some numerical problems. We present results in tabular form where $YEXT$ is the exact solution, YAN the numerical solution and $ERR = |YEXT - YAN|$ is the absolute errors respectively.

Example 1.1: We consider the IVP for the step-size $h = 0.01$

$y'' - 100y = 0, y(0) = 1, y'(0) = -10$

Table of results and absolute errors for problem 1.1

X	$YEXT$	YAN	ERR
0	1.0000000000	1.0000000000	0.00e+0
0.01	0.9048374180	0.9048374165	1.500e-9
0.02	0.8187307531	0.8187307491	4.000e-9
0.03	0.7408182207	0.7408182141	6.600e-9
0.04	0.6703200460	0.6703200369	9.100e-9
0.05	0.6065306597	0.6065306478	1.190e-8
0.06	0.5488116364	0.5488116214	1.500e-8

0.07	0.4965853038	0.4965852861	1.770e-8
0.08	0.4493289641	0.4493289433	2.080e-8
0.09	0.4065696597	0.4065696402	1.950e-8
0.10	0.3678794412	0.3678794223	1.890e-8
0.11	0.3328710837	0.3328710653	1.840e-8
0.12	0.3011942119	0.3011941938	1.810e-8

$ERR = |YEXT - YAN|$, for example 1.1, where $YEXT = e^{-10x}$ and ERR = Absolute errors
Example 1.2: We consider the IVP for the step-size $h = 0.1$
$y'' + y = 0, y(0) = 1, y'(0) = 1$

Table of results and absolute errors for problem 1.2

X	$YEXT$	YAN	ERR
0	1.0000000000	1.0000000000	0.000e-0
0.1	1.0948375819	1. 0948375831	-1.200e-9
0.2	1.1787359086	1. 1787359116	-3.000e-9
0.3	1.2508566958	1. 2508567005	-4.700e-9
0.4	1.3104793363	1. 3104793428	-6.500e-9
0.5	1.3570081005	1. 3570081087	-8.200e-9
0.6	1.3899780883	1. 3899780981	-9.800e-9
0.7	1.4090598745	1. 4090598858	-1.130e-8
0.8	1.4140628003	1. 4140628129	-1.260e-8
0.9	1.4049368779	1. 4049368906	-1.270e-8
1.0	1.3417732907	1. 3417733023	-1.160e-8
1.1	1.3448034815	1. 3448034918	-1.030e-8
1.2	1.2943968404	1. 2943968495	-9.100e-9

$ERR = |YEXT - YAN|$, for example 1.2, where $YEXT = Cosx + Sinx$

VI. Conclusions

We have proposed an eight-step block LMM with continuous coefficients from which multiple finite difference methods were obtained and applied as simultaneous numerical integrators ,without first adapting the ODE to an equivalent first order system. The method is derived through interpolation and collocation procedures by the matrix inverse approach. We conclude that our new eight-step block method of uniform order 7 is suitable for direct solution of general second order differential equations. The new block methods are self-starting and all the discrete schemes used were obtained from the single continuous Formulation and its derivative which are of uniform order of accuracy. The results were obtained in block form which speeds up the computational process and the result obtained from the two numerical examples converges with the theoretical solutions.

References

[1] Awoyemi,D.O , A Class of Continuous Methods for General Second order Initial Value Problems in Ordinary Differential Equations: International Journal of Computer Mathematics, 72:29-37,(1999) .

[2] Carpentieri M., Paternoster B., Stability regions of one step mixed collocation methods for y = f(x, y), vol.53 (2-4), p.201-212, Appl. Num. Math., 2005.

[3] Coleman J.P., Duxbury S.C., Mixed collocation methods fory = f(x, y), vol.126, p.47-75, J. Comput. Appl. Math., 2000.

[4] Fatunla,S.O , Block methods for second order IVP's. Inter.J.Comp.Maths.72,No.1 ,(1991).

[5] Fatunla,S.O , Higher order parallel methods for second order ODE's.Scientific Computing.Proceeding of fifth International Conference on Scientific Computing, (1994).

[6] Henrici,P., Discrete variable methods for ODE's.John Wiley, New York ,(1962).

[7] Ixaru L.Gr., Paternoster B., Function fitting two–step BDF algorithms for ODEs, Part IV, p.443-450 in Computational Science - ICCS 2004, Lecture Notes in Computer Science 3039,M.Bubak, G.D. van Albada, P.M.A.Sloot, J.J.Dongarra Eds., Springer Verlag, 2004.

[8] Lambert,J.D , Computational methods for ordinary differential equations.(John Wiley,New York, 1973).

[9] Lambert,J.D , Numerical methods for ordinary differential systems.John Wiley,New York ,(1991).

[10] Paternoster B., Two step Runge-Kutta-Nystrom methods for y = f(x, y) and P-stability, Part III, p. 459-466 in Computational Science - ICCS 2002, Lecture Notes in Computer Science 2331, P.M.A.Sloot, C.J.K.Tan, J.J.Dongarra, A.G.Hoekstra Eds., Springer Verlag, Amsterdam, 2002.

[11] Petzold L.R., Jay L.O., Yen J., Numerical solution of highly oscillatory ordinary differential equations, p.437-483, Acta Nu-merica, Cambridge University Press, 1997.