IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728,p-ISSN: 2319-765X, Volume 7, Issue 3 (Jul. - Aug. 2013), PP 72-76
www.iosrjournals.org

On The Homogeneous Quintic Equation with Five Unknowns

$$x^5 - y^5 + xy(x^3 - y^3) = 34((x+y)(z^2 - w^2)P^2$$

M.A. Gopalan[1], S.Vidhyalakshmi[2], S.Mallika,[3]
.Department of Mathematics, Shrimati Indira Gandhi college,Trichirappalli.620002

Abstract: *The quintic Diophantine equation with five unknowns given by* $x^5 - y^5 + xy(x^3 - y^3) = 34((x+y)(z^2 - w^2)P^2$ *is analyzed for its infinitely many non-zero distinct integral solutions. A few interesting relations between the solutions and special numbers namely, centered polygonal numbers, centered pyramidal numbers, jacobsthal numbers, lucas numbers and kynea numbers are presented.*

Keywords: *Quintic equation with five unknowns, Integral solutions, centered polygonal numbers, centered pyramidal numbers.*

I. Introduction

The theory of Diophantine equations offer a rich variety of fascinating problems.In particular quintic equations homogeneous or non-homogeneous have aroused the interest of numerous mathematicians since antiquity[1,2,3].For illustration ,one may refer [4-14],for quintic equations with three ,four and five unknowns. This paper concerns with the problem of determining integral solutions of the non-homogeneous quintic equation with five unknowns given by $x^5 - y^5 + xy(x^3 - y^3) = 34((x+y)(z^2 - w^2)P^2$. A few relations between the solutions and the special numbers are presented..

Notations

$t_{m,n} = n\left(1 + \frac{(n-1)(m-2)}{2}\right)$.-Polygonal number of rank n with size m.

$P_n^m = \left(\frac{n(n+1)}{6}\right)[(m-2)n + (5-m)]$ -Pyramidal number of rank n with size m.

$Pt_n = \frac{n(n+1)(n+2)(n+3)}{24}$ - pentatope number of rank n.

$SO_n = n(2n^2 - 1)$ - Stella octangular number of rank n.

$S_n = 6n(n-1) + 1$ - Star number of rank n.

$\Pr_n = n(n+1)$ - Pronic number of rank n.

$J_n = \frac{1}{3}\left(2^n - (-1)^n\right)$-Jacobsthal number of rank n.

$j_n = (2^n + (-1)^n)$ - Jacobsthal lucas number of rank n.

$Ky_n = (2^n + 1)^n - 2$ - Kynea number.

$F_{4,m,3} = \frac{n(n+1)(n+2)(n+3)}{4!}$ -Four dimensional figurative number of rank n whose generating polygon is a triangle.

$F_{5,m,3} = \frac{n(n+1)(n+2)(n+3)(n+4)}{5!}$ -Five dimensional figurative number of rank n whose generating polygon is a triangle.

$CP_n^m = \frac{mn(n-1)}{2} + 1$ - Centered polygonal number of rank n with size m.

II. Method of Analysis

The Diophantine equation representing the quintic with five unknowns under consideration is

$$x^5 - y^5 + xy(x^3 - y^3) = 34((x+y)(z^2 - w^2)P^2 \quad (1)$$

Introducing the transformations

$$x = u+v, y = u-v, z = uv+1, w = uv-1 \quad (2)$$

in (1),it leads to

$$u^2 + v^2 + = 17p^2 \quad (3)$$

which is solved in different ways leading to different solution patterns to (1).

2.1 Pattern : I

Assume $p = a^2 + b^2$ (4)

write 17 as

$$17 = (1+4i)(1-4i) \quad (5)$$

Substituting (4) and (5) in (3) and employing the method of factorization ,define

$$u + iv = (1+4i)(a+ib)^2 \quad (6)$$

Equating real and imaginary parts, we get

$$u = a^2 - b^2 - 8ab$$

$$v = 4a^2 - 4b^2 + 2ab$$

Thus, in view of (2), the non-zero distinct integral solutions of (1) are given by

$$x(a,b) = 5a^2 - 5b^2 - 6ab$$

$$y(a,b) = -3a^2 + 3b^2 - 10ab$$

$$z(a,b) = (a^2 - b^2 - 8ab)(4a^2 - 4b^2 + 2ab) + 1$$

$$w(a,b) = (a^2 - b^2 - 8ab)(4a^2 - 4b^2 + 2ab) - 1$$

$$p(a,b) = a^2 + b^2$$

2.1.2 Properties

1) $x(a,a+1) + y(a,a+1) - P(a,a+1) + 36t_{3,a} \equiv 1(\mathrm{mod}[a+1])$

2) $z(a,1) + w(a,1) - 96Pt_a - 48f_{4,a,2} + 50So_a + 2t_{114,a} \equiv 0(\mathrm{mod}\, a)$

3) $x(2^n,1) + y(2^n,1) + P(2^n,1) - 3Ky_n \equiv 0(\mathrm{mod}\, 2)$

4($x(1,2^n) - y(1,2^n) - P(1,2^n) + j_{2n} + 8Ky_n = 0$

5) $x(a,1)P(a,1) + 3So_a - 24f_{4,a,7} + 42P_a^4 + 28t_{4,a} + 5 = 0$

6) $x(a(a+1),a) - y(a(a+1),a) - 32(t_{3,a})^2 + 8t_{4,a} - 8P_a^5 = 0$

7) $z(1,b) + w(1,b) - 96f_{4,b,2} + 42OH_b \equiv 0(\mathrm{mod}\, 2)$

2.2 Pattern: II

Instead of (5) write 17 as

$$17 = (4+i)(4-i)$$

For this choice, after performing calculations similar to pattern.I ,the corresponding non-zero integral solutions to (1) are found to be

$$x(a,b) = 5a^2 - 5b^2 + 6ab$$

$$y(a,b) = 3a^2 - 3b^2 + 10ab$$

$$z(a,b) = (4a^2 - 4b^2 - 2ab)(a^2 - b^2 + 8ab) + 1$$

$$w(a,b) = (4a^2 - 4b^2 - 2ab)(a^2 - b^2 + 8ab) - 1$$

$$P(a,b) = (a^2 + b^2)$$

2.3 Pattern: III

Rewrite (3) as

$$17P^2 - v^2 = u^2 * 1 \quad \textbf{(7)}$$

Assume

$$u = 17a^2 - b^2 \quad (8)$$

Write 1 as

$$1 = [\sqrt{17} + 4][\sqrt{17} - 4] \quad (9)$$

Using (8) and (9) in (7) and employing the method of factorization, define

$$(\sqrt{17}P + v) = (\sqrt{17} + 4)(\sqrt{17}a + b)^2$$

Equating rational and irrational parts, we get

$$P = 17a^2 + b^2 + 8ab$$
$$v = 68a^2 + 4b^2 + 34ab)$$

Substituting the values of u and v in (2),the non-zero distinct integral solutions of (1) are as follows.

$$x(a,b) = 85a^2 + 3b^2 + 34ab\,,$$
$$y(a,b) = -51a^2 - 5b^2 - 34ab$$
$$z(a,b) = [17a^2 - b^2][68a^2 + 4b^2 + 34ab] + 1$$
$$w(a,b) = [17a^2 - b^2][68a^2 + 4b^2 + 34ab] - 1$$
$$P(a,b) = 17a^2 + b^2 + 8ab$$

2.3.1 Properties

1. $3[x(2^n,1) + y(2^n,1) - 102J_{2n}]$ is a nasty number.
2. $x(1,b+1) - y(1,b+1) + P(1,b+1) - S_b - 6t_{3,b} \equiv 43 (\mathrm{mod}\, 97)$
3. $z(a,1) - 6936 f_{4,a,5} + 4624P_a^5 + 289So_a \equiv 4(\mathrm{mod}\, a).$
4. $x(a,1) - P(a,1) - 136t_{3,a}$ is divisible by 2.
5. $x(1,2^n) + y(1,2^n) - P(1,2^n) + 3Ky_n + 3J_n + j_n = 17$

2.4 Pattern :IV

Instead of (9) , one may write 1 as

$$1 = \frac{[\sqrt{17} + 1][\sqrt{17} - 1]}{16} \quad (10)$$

Substituting (8) and (10) in (7) and employing the method of factorization, define

$$\sqrt{17}P + v = [\sqrt{17}a + b]^2 \frac{[\sqrt{17} + 1]}{4}$$

Equating rational and irrational parts ,we get

$$P = \frac{17a^2 + b^2 + 2ab}{4}$$
$$v = \frac{17a^2 + b^2 + 34ab}{4}$$

Since our interest is on finding integral solutions , it is possible to choose a and b so that P and v are integers.

2.4.1 Choice: 1

Let $a = 2A, b = 2B$

Then

$$P = 17A^2 + B^2 + 2AB$$
$$v = 17A^2 + B^2 + 34AB$$

$$u = 68A^2 - 4B^2$$

Substituting these values in (2) the corresponding integral solutions to (1) are given by,

$$x(A,B) = 85A^2 - 3B^2 + 34AB$$

$$y(A,B) = 51A^2 - 5B^2 - 34AB$$

$$z(A,B) = [68A^2 - 4B^2][17A^2 + B^2 + 34AB] + 1$$

$$w(A,B) = [68A^2 - 4B^2][17A^2 + B^2 + 34AB] - 1$$

$$P(A,B) = 17A^2 + B^2 + 2AB$$

NOTE; Suppose we choose A,B such that $A \succ B \succ 0$ then $u \succ v$.Considering u, v to be the generators of a Pythagorean triangle, .then its area is represented by $xy[z+w]$

2.4.2 Choice :II

Let $a = (2k-1)b$

Then

$$P = [17k^2 - 16k + 4]b^2$$

$$v = [17k^2 + 4]b^2$$

$$u = [68k^2 - 68k + 16]b^2$$

Substituting these values in (2),the non-zero distinct integer solutions of (1) is found to be,

$$x = [85k^2 - 68k + 20]b^2$$

$$y = [51k^2 - 68k + 12]b^2$$

$$z = [68k^2 - 68k + 16][17k^2 + 4]b^4 + 1$$

$$w = [68k^2 - 68k + 16k][17k^2 + 4]b^4 - 1$$

$$P = [17k^2 - 16k + 4]b^2$$

2.4.3 Choice: III

Let $b = (2k+1)a$

Then

$$P = [k^2 + 2k + 5]a^2$$

$$v = [k^2 + 18k + 13]a^2$$

$$u = [16 - 4k^2 - 4k]a^2$$

Then the corresponding non-zero distinct integral solutions of (1) are given by,

$$x = [-3k^2 + 14k + 29]a^2$$

$$y = [-5k^2 - 22k + 3]a^2$$

$$z = [16 - 4k^2 - 4k][k^2 + 18k + 13]a^4 + 1$$

$$w = [16 - 4k^2 - 4k][k^2 + 18k + 13]a^4 - 1$$

$$P = [k^2 + 2k + 5]a^2$$

.**Remark:** It is worth mentioning here that ,the triple (x, y, z) and (x, y, w) obtained from any of the above patterns satisfy respectively the following hyperbolic paraboloids. $x^2 - y^2 = 4(z-1)$ and $x^2 - y^2 = 4(w+1)$.

III. Conclusion

One may search for other choices of solutions to (1) along with the corresponding properties.

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