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Exact Analytical Expression for Outgoing Intensity from the Top of the Atmosphere in a Flat Homogeneous Atmosphere–Ocean System using Analytical Discrete Ordinate method for the Solution of Polarized Radiation Transfer Equation.

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Abstract: *This research is a part of the work devoted on the application of analytical Discrete Ordinate (ADO) method to the polarized monochromatic radiative transfer equation undergoing anisotropic scattering with source function matrix in a finite coupled Atmosphere –Ocean media having flat interface boundary conditions involving specular reflection and transmission matrix. Discontinuities in the derivatives of the Stokes vector with respect to the cosine of the polar angle at smooth interface between the two media with different refractive indices (air and water) is tackled by using a suitable quadrature scheme devised earlier. Atmosphere and ocean are assumed to be homogeneous. No stratification is adopted in the two media. Exact expression for the emergent radiation intensity vector from the top of the atmosphere is derived. Exact expressions for the emergent polarized radiation intensity vector from the air-water interface as well as from any point of the two medium in any direction can also be derived in terms of eigenvectors and eigenvalues.*

Keywords: *Polarized Radiative transfer, Eigenvectors, Eigenvalues, Green function, specular.*

I. Introduction

Most of the fundamental problems that include scattering of polarized light in radiative transfer theory were formulated by Chandrasekhar [1], Kuscer and Riberic [2] for plane parallel media. Reformulation in terms of real parameters instead of complex parameters [3] made the problems easier for computation and numerical works. Basic theory of radiative transfer in natural water bodies (such as lakes, ponds or calm and rough oceans) was formulated by several authors [4, 5, 6, 7, 8, 9, 10 and 38]. There are several substantive contributions in this field including polarized radiation field and vector radiative transfer equation with rough surface modeling[17,18,19,20,21,22,23,24,25,26]. The majority of the methods adopted for solution of the problems stated therein include matrix-operator method, method of successive order of scattering, discrete ordinate method and Monte Carlo Method. Chandrasekhar's celebrated discrete ordinate method (DOM) was successfully applied to the coupled atmosphere-Ocean system by Knut and Stamens [11].In 2000 Siewert [33, 34] discovered a variation of DOM and applied it to a basic polarized radiation transfer problem [12]. The method is now popularly known as analytical discrete ordinate method (A.D.O). We have applied this method in simple coupled homogeneous Atmosphere-Ocean system. No stratification in either system is adopted to keep the mathematics simple although there exits well defined schemes for treating problems where changes in refractive index occurs between layers of the two media [37]. The radiation field considered is polarized.

Boundary conditions are specified at the top of the atmosphere and at the bottom of the ocean. However at the two interfaces i.e. at the junction of air and water two interface conditions have been introduced depending on the direction of light (one from air to water and another from water to air) with specular reflection and transmission. Appropriate specular reflection and transmission matrices are introduced in the interface conditions at flat interfaces of atmosphere and ocean. Exact analytical expression for exit intensity in the upward direction from the top of the atmosphere is derived in terms of known integral terms. Some integrals have been calculated in detail.

II. The basic equation of transfer and reduction

2.1: Basic equations:

Following [2] we introduce the specific intensity vector (Polarized radiation intensity vector) $\mathbf{I}_{At/Oc}(z,\mu,\varphi)$ as a column vector comprising stokes components in the following form

$$: \mathbf{I}_{At/Oc}(z,\mu,\varphi) = [\mathbf{L}_{At/Oc}(z,\mu,\varphi), \mathbf{Q}_{At/Oc}(z,\mu,\varphi), \mathbf{U}_{At/Oc}(z,\mu,\varphi), \mathbf{V}_{At/Oc}(z,\mu,\varphi)]^{T} \quad (1)$$

We have used for optical depth $z \in (0, z_w)$ in atmosphere and $z \in (z_w, z_b)$ in Ocean respectively. In equation (1) $\omega_{At/Oc}$ are the albedos for atmosphere and Ocean respectively for single scattering whereas $\mu \in (-1,1)$,is the cosine of polar angle measured from the normal drawn on the ocean surface with direction increasing in the downward direction from the top of the atmosphere at $z = 0$. The azimuthal angle is represented by $\varphi \in (0, 2\pi)$. The two radiative transfer equations for homogeneous atmosphere and ocean can be written compactly

$$: \mu \frac{dI_{At/Oc}(z,\mu,\varphi)}{dz} + I_{At/Oc}(z,\mu,\varphi) = \frac{\omega_{At/Oc}}{4\pi} \int_0^{2\pi} \int_{-1}^{1} P_{At/Oc}(\mu,\varphi,\mu',\varphi') I_{At/Oc}(z,\mu',\varphi') d\mu'\varphi' + S_{At/Oc}(z,\mu,\varphi) \quad (2)$$

The source vector $S_{At/Oc}(z,\mu,\varphi)$ contains actual internal source for atmosphere and ocean. The atmosphere is illuminated by the solar beam with stokes parameter described in section (1.3) below in detail. Here we did not consider thermal, infrared microwave or fluorescent emission in the source function. However this can be introduce without much effort.

2.2: Fourier Decomposition:

One of the usual procedures of solving the transfer equations is to consider Fourier series expansion of the intensity vector in the following form

$$I_{At/Oc}(z,\mu,\varphi) = \begin{bmatrix} L_{At/Oc}(z,\mu,\varphi) \\ Q_{At/Oc}(z,\mu,\varphi) \\ U_{At/Oc}(z,\mu,\varphi) \\ V_{At/Oc}(z,\mu,\varphi) \end{bmatrix} = \sum_{S=0}^{L} (2 - \delta_{0,S}) \begin{bmatrix} L^S_{At/Oc}(z,\mu) \cos S(\varphi - \varphi_s) \\ Q^S_{At/Oc}(z,\mu) \cos S(\varphi - \varphi_s) \\ U^S_{At/Oc}(z,\mu) \sin S(\varphi - \varphi_s) \\ V^S_{At/Oc}(z,\mu) \sin S(\varphi - \varphi_s) \end{bmatrix} \quad (3)$$

However we shall use following analytic Fourier series representation of phase functions [28].

$$P_{At/Oc}(\mu,\varphi,\mu',\varphi') = \sum_{S=0}^{L} \frac{1}{2}(2 - \delta_{0,S})[\cos(S(\varphi - \varphi'))P^S_{Atc/Occ}(\mu,\mu') + \sin(S(\varphi - \varphi'))P^S_{Ats/Ocs}(\mu,\mu')] \quad (4)$$

The above mentioned two forms of Fourier representations for two different types of vector functions induce some simplification in the following developments required for the reduction of the equation of transfer to an azimuth independent form.

The cosine and sine components of the phase functions for atmosphere are derived from Deuze et all [28]

$$P^S_{Atc}(\mu,\mu') = \begin{bmatrix} \sum_{J=S}^{M} \beta_J P_J^S(\mu) P_J^S(\mu') & \sum_{J=S}^{M} \gamma_J P_J^S(\mu) R_J^S(\mu') & 0 & 0 \\ \sum_{J=S}^{M} \gamma_J R_J^S(\mu) P_J^S(\mu') & \sum_{J=S}^{M} (\alpha_J R_J^S(\mu) R_J^S(\mu') + \xi_J T_J^S(\mu) T_J^S(\mu')) & 0 & 0 \\ 0 & 0 & \sum_{J=S}^{M} (\alpha_J T_J^S(\mu) T_J^S(\mu') + \xi_J R_J^S(\mu) R_J^S(\mu')) & -\sum_{J=S}^{M} \varepsilon_J R_J^S(\mu) P_J^S(\mu') \\ 0 & 0 & \sum_{J=S}^{M} \varepsilon_J P_J^S(\mu) R_J^S(\mu') & \sum_{J=S}^{M} \delta_J P_J^S(\mu) P_J^S(\mu') \end{bmatrix}. \quad (5)$$

And

$$P^S_{Ats}(\mu,\mu')$$

$$= \begin{bmatrix} 0 & 0 & \sum_{J=S}^{M} \gamma_J P_J^S(\mu) T_J^S(\mu') & 0 \\ 0 & 0 & \sum_{J=S}^{M} (\alpha_J R_J^S(\mu) T_J^S(\mu') + \xi_J T_J^S(\mu) R_J^S(\mu')) & -\sum_{J=S}^{M} \varepsilon_J T_J^S(\mu) P_J^S(\mu') \\ -\sum_{J=S}^{M} \gamma_J T_J^S(\mu) P_J^S(\mu') & -\sum_{J=S}^{M} (\alpha_J T_J^S(\mu) R_J^S(\mu') + \xi_J R_J^S(\mu) T_J^S(\mu')) & 0 & 0 \\ 0 & -\sum_{J=S}^{M} \varepsilon_J P_J^S(\mu) T_J^S(\mu') & 0 & 0 \end{bmatrix}. \tag{6}$$

In the above formulation the functions $\mathbf{P_J^S}$ with arguments prime and unprimed symbols are normalized associated Legendre functions which include the Legendre functions $\mathbf{P_J(\mu)}$

$$\mathbf{P_J^S(\mu) = \left[\frac{(S-J)!}{(s+J)!}\right]^{1/2} (1-\mu^2)^{S/2} \frac{d^S}{d\mu^S} P_J(\mu)} \tag{7}$$

The functions $\mathbf{R_J^S}, \mathbf{T_J^S}$ with prime and unprimed arguments can be expressed as linear combinations of generalized Legendre functions [29], [30] and Mee (1990) [31].

$$\mathbf{P_{m,n}^r(\mu) = A_{m,n}^r (i)^{n-m} (1-\mu)^{-\frac{(m-n)}{2}} (1+\mu)^{-\frac{(n+m)}{2}} \frac{d^{r-n}}{d\mu^{r-n}} [(1-\mu)^{r-m} (1+\mu)^{r+m}]} \tag{8}$$

$$\mathbf{A_{m,n}^r = \frac{(-1)^{r-m}}{2^r (r-m)!} \sqrt{\frac{(r-m)!(r+n)!}{(r+m)!(r-n)!}}} \tag{9}$$

Unprimed $\boldsymbol{\alpha, \beta, \gamma, \delta}$ are coefficients to be computed for atmosphere. The corresponding phase functions for ocean can be written down by replacing the unprimed coefficients by primed one. Since we are considering homogeneous atmosphere and ocean we shall not consider layered atmosphere. Substitution of (3), (4) and use of (5) & (6) in (2) separates the equation of transfer into a set of azimuth independent equations for each $\mathbf{S}$

$$\mathbf{\mu \frac{dI_{At/Oc}^S(z,\mu)}{dz} = -I_{At/Oc}(z,\mu) + \frac{\omega_{At/Oc}}{4} \int_{-1}^{+1} P_{At/Oc}^S(\mu,\mu') \, I_{At/Oc}^S(z,\mu') d\mu' + S_{At/Oc}^S(z,\mu)} \tag{10}$$

Where we have defined

$$\mathbf{P_{At}^S(\mu,\mu') =}$$

$$= \sum_{J=S}^{M} \begin{bmatrix} P_J^S(\mu) & 0 & 0 & 0 \\ 0 & R_J^S(\mu) & -T_J^S(\mu) & 0 \\ 0 & -T_J^S(\mu) & R_J^S(\mu) & 0 \\ 0 & 0 & 0 & P_J^S(\mu) \end{bmatrix} \begin{bmatrix} \beta_J & \gamma_J & 0 & 0 \\ \gamma_J & \alpha_J & 0 & 0 \\ 0 & 0 & \xi_J & -\varepsilon_J \\ 0 & 0 & \varepsilon_J & \delta_J \end{bmatrix}$$

$$\begin{bmatrix} P_J^S(\mu') & 0 & 0 & 0 \\ 0 & R_J^S(\mu') & -T_J^S(\mu') & 0 \\ 0 & -T_J^S(\mu') & R_J^S(\mu') & 0 \\ 0 & 0 & 0 & P_J^S(\mu') \end{bmatrix}$$

$$= \sum_{J=S}^{M} \mathbf{P}_J^S(\mu) B_J^{AT} \mathbf{P}_J^S(\mu') \tag{11}$$

$$\mathbf{P_{Oc}^S(\mu,\mu') =}$$

$$= \sum_{J=S}^{M} \begin{bmatrix} P_J^S(\mu) & 0 & 0 & 0 \\ 0 & R_J^S(\mu) & -T_J^S(\mu) & 0 \\ 0 & -T_J^S(\mu) & R_J^S(\mu) & 0 \\ 0 & 0 & 0 & P_J^S(\mu) \end{bmatrix} \begin{bmatrix} \beta_J' & \gamma_J' & 0 & 0 \\ \gamma_J' & \alpha_J' & 0 & 0 \\ 0 & 0 & \xi_J' & -\varepsilon_J' \\ 0 & 0 & \varepsilon_J' & \delta_J' \end{bmatrix}$$

$$\begin{bmatrix} P_J^S(\mu') & 0 & 0 & 0 \\ 0 & R_J^S(\mu') & -T_J^S(\mu') & 0 \\ 0 & -T_J^S(\mu') & R_J^S(\mu') & 0 \\ 0 & 0 & 0 & P_J^S(\mu') \end{bmatrix}$$

$$= \sum_{J=S}^{M} \mathbf{P}_J^S(\mu) B_J^{OC} \mathbf{P}_J^S(\mu') \quad (12)$$

It can be shown that [29]

$$P_{Atc/Occ}^S = \mathbf{P}_J^S(\mu) B_J^{At/Oc} \mathbf{P}_J^S(\mu') + D\mathbf{P}_J^S(\mu) B_J^{At/Oc} \mathbf{P}_J^S(\mu') D \quad (13)$$

$$P_{Ats/Ocs}^S = \mathbf{P}_J^S(\mu) B_J^{At/Oc} \mathbf{P}_J^S(\mu') D - D\mathbf{P}_J^S(\mu) B_J^{At/Oc} \mathbf{P}_J^S(\mu') \quad (14)$$

The required order S for the above development depends mainly on the constituent particles of the medium. The functions $R_J^S(\mu)$, $T_J^S(\mu)$ are to be determined from the expressions given in [29].

2.3: The exclusive expressions for source matrix vector, reflection and transmission matrix function:

We shall now specify the solar-beam source matrix vector $S_{At/Oc}(z,\mu,\varphi)$ in equation (2). For atmosphere this source matrix may be obtained using Snell-Descartes law to model the reflection and transmission matrix at the air-seawater interface. The atmospheric source matrix contains two distinct terms. The first term (downwelling) describes the contribution from the downward beam source vector $F_0 = [F_{01}, F_{02}, 0, 0]^T$ attenuated exponentially while the second term (reflected) expresses the contribution from beam source reflected upward **specularly** at the air-water interface following Fresnel reflection law and gets attenuated.

$$S_{At}(z,\mu,\varphi) = \frac{\omega_{At}}{4\pi} F_0 P_{AT}(\mu,\varphi,-\mu_0,\varphi_0) \exp\left(\frac{-z}{\mu_0}\right) +$$

$$\frac{\omega_{At}}{4\pi} F_0 R_{At}(n,-\mu_0^{at},\mu^{at}) P_{AT}(\mu,\varphi,-\mu_0,\varphi_0) \exp\left(\frac{-(2z_\omega - z)}{\mu_0}\right) \quad (15)$$

The source matrix for ocean, suitably modified adopted from Zhonghai and Stamnes [32], essentially desribes the fact that downward direct solar beam source vector, incident in the direction (μ_0, φ_0), reach at depth z after getting transmitted, scattered and attenuated.

$$S_{OC}(z,\mu,\varphi) = \frac{\omega_{Oc}}{4\pi} \frac{\mu_0^{at}}{\mu_{0n}^{oc}} F_0 T_{Oc}(n,-\mu_0^{at},-\mu_{0n}^{oc})$$

$$\times P_{OC}(\mu,\varphi,-\mu_{0n}^{oc},\varphi_0) \exp\left(-\frac{z_\omega}{\mu_0^{at}}\right) \times \exp\left(-\frac{(z - z_\omega)}{\mu_{0n}^{oc}}\right) \quad (16)$$

The exact expressions for stokes component of the source matrix can be computed from expressions (15) and (16) using equations (4), (11) and (12) and expression for specular reflection and transmission matrix given below.

The form of reflection and transmission matrix can be written using Snell-Descartes laws with cosine of the solar zenith angle μ_{0n} in the ocean related to the incident polar direction μ_0^{at} given by

$$\mu_{0n}^{oc} = \sqrt{1 - \frac{(1 - {\mu_0^{at}}^2)}{n^2}} \quad (17).$$

We have defined $\mathbf{R_{At/Oc}(n,\pm\mu^{at/oc})}$ and $\mathbf{T_{At/Oc}(n,\pm\mu^{at/oc})}$ as the azimuth independent **specular reflection and transmission matrix** of the invariant intensity respectively with appropriate suffix 'At' and 'Oc' for atmosphere and Ocean respectively. The plus-minus sign applies for the downward and the upward directions respectively. If the sea surface is absolutely calm a single image of the sun can be seen on a specular surface. In this case the general specular reflection matrix can be expressed in the Chandrasekhar's representation of stokes vector [1], [15], [14].

$$\mathbf{R_{At}(n,-\mu^{at},\mu^{oc})} = \begin{bmatrix} r_P^2 & 0 & 0 & 0 \\ 0 & r_L^2 & 0 & 0 \\ 0 & 0 & -r_P r_L & 0 \\ 0 & 0 & 0 & -r_P r_L \end{bmatrix} = \mathbf{R_{Oc}(n,+\mu^{oc},\mu^{at})} \quad (18)$$

$$r_P = -\frac{n\mu^{at}-\mu^{oc}}{n\mu^{at}+\mu^{oc}} \quad (19); \qquad r_L = \frac{\mu^{at}-n\mu^{oc}}{\mu^{at}+n\mu^{oc}} \quad (20)$$

In the same manner the general specular transmission matrix can be expressed as

$$\mathbf{T_{Oc}(n,\mu^{oc},\mu^{at})} = |n| \left[\frac{\mu^{oc}}{\mu^{at}}\right] \begin{bmatrix} t_P^2 & 0 & 0 & 0 \\ 0 & t_L^2 & 0 & 0 \\ 0 & 0 & -t_P t_L & 0 \\ 0 & 0 & 0 & -t_P t_L \end{bmatrix} = \mathbf{T_{At}(n,-\mu^{at},\mu^{oc})} \quad (21)$$

$$t_P = \left[\frac{2\mu^{oc}}{n\mu^{at}+\mu^{oc}}\right] \quad (22); \qquad t_L = \left[\frac{2\mu^{at}}{n\mu^{oc}+\mu^{at}}\right] \quad (23)$$

The polar angles μ^{oc} and μ^{at} are related by the Snells law given in equation (17).

The ratio of the refractive index of the ocean to that of atmosphere is expressed as $\mathbf{n}$. In case of air-water interface there arises a fundamental problem when one considers numerical evaluation. To understand this we shall refer to [16].Let us assume that we have a discrete set of N cosines of the polar angle for the ocean such that $\mu_i^{oc} = \cos\theta_i > 0$. Next we can a set of points for which $\mu_i^{at} = -\cos\theta_i < 0$ which will label the upper hemisphere. Out of this set of N points for the ocean, only M of them can be mapped into the atmosphere for which $\mu_i^{oc} > \cos\theta_c$ where θ_c is the critical angle defined by $\sin(\theta_c) = \frac{1}{n}$ where $\mathbf{n}$ is the refractive index of water. The remaining N–M points are restricted to the ocean and cannot be mapped into the atmosphere. These points specify the region of total internal reflection. The matter has been explained in the following figure [2] adopted from [16] without any change. Let us consider a radiance vector $\mathbf{I(z,\xi^{at})}$ striking the interface at some angle α and entering the ocean at the refracted angle θ .The relationship connecting these two set of points is simply Snell's law. With reference to the figure [2] this well known relation in quadrature form is

$$\sin(\alpha_i) = n\sin(\theta_i)$$

Or more precisely on decretizing $\xi_i^{at} = \cos(\alpha_i) = (1-(1-(\mu_i^{oc})^2)n^2)^{\frac{1}{2}}$ (25)

Now we shall select the quadrature mapping set suitable for our problem. To select these sets we must take into consideration the following three constraints. The normalization of phase function of the atmosphere must be ensured. The energy must be conserved. The quadrature selected must be capable of normalizing the highly anisotropic scattering phase function for ocean. Now for the continuity of radiation vector striking at the air-water interface from above, the radiation intensity vector just below the water surface will be given by

$$\mathbf{I(z,\mu^{oc})} = \left[\frac{\xi^{at}}{\mu^{oc}}\frac{d\xi^{at}}{d\mu^{oc}}\right]\mathbf{T_{Oc}(n,\mu^{oc},\xi^{at})I(z,\xi^{at})} \quad (26)$$

$$[\frac{\xi^{at}}{\mu^{oc}}\frac{d\xi^{at}}{d\mu^{oc}}]=\begin{bmatrix}\frac{\xi^{at}}{\mu^{oc}}\frac{d\xi^{at}}{d\mu^{oc}} & 0 & 0 & 0\\ 0 & \frac{\xi^{at}}{\mu^{oc}}\frac{d\xi^{at}}{d\mu^{oc}} & 0 & 0\\ 0 & 0 & \frac{\xi^{at}}{\mu^{oc}}\frac{d\xi^{at}}{d\mu^{oc}} & 0\\ 0 & 0 & 0 & \frac{\xi^{at}}{\mu^{oc}}\frac{d\xi^{at}}{d\mu^{oc}}\end{bmatrix} \quad (26a)$$

Using equation (25), equation (27) can be written in a form which is the generalized form of scalar result first obtained by Gershun [27].

$$I(z,\mu^{oc})=T_{Oc}(n,\mu^{oc},\xi^{at})I(z,\xi^{at})n^2 \quad (27)$$

Now in discrete form equation (26) can be written with help of diagonal matrix having elements as the ratios of the weights of atmosphere and ocean respectively

$$I(z,\mu_i^{oc})=[\frac{A_i^{at}}{A_i^{oc}}]T_{Oc}(n,\mu_i^{oc},\xi_i^{at})I(z,\xi_i^{at}) \quad (28)$$

Where we have defined

$$[\frac{A_i^{at}}{A_i^{oc}}]=\begin{bmatrix}\frac{A_i^{at}}{A_i^{oc}} & 0 & 0 & 0\\ 0 & \frac{A_i^{at}}{A_i^{oc}} & 0 & 0\\ 0 & 0 & \frac{A_i^{at}}{A_i^{oc}} & 0\\ 0 & 0 & 0 & \frac{A_i^{at}}{A_i^{oc}}\end{bmatrix} \quad (29)$$

Comparing (27) and (28) we get $[\frac{A_i^{at}}{A_i^{oc}}]=\frac{1}{n^2}[\frac{\xi_i^{at}}{\mu_i^{oc}}]$ (30)

Equation (30) gives us the ocean quadrature weights for the corresponding atmosphere quadrature weights. A detailed analysis for such quadrature evaluation were carried out in[16] to show the superiority of this quadrature scheme over if one uses Gaussian points within the critical angle for the ocean and maps them into the atmosphere which was the technique used by Tanaka and Nakajima [36]. However for layered media there exist better quadrature schemes in the literature [37]. There are certain criteria to use these quadrature schemes regarding the maximum number of quadratures to be used within the total internal reflection region and outside refracting this region [32].

It should be taken into account that the reflectivity ($\mathbf{r}$) and transmissivity ($\mathbf{t}$) given by the following expressions are related to

$$r+t=1 \quad (31)$$

Where we have defined

$$r=\frac{1}{2}(r_p^2+r_L^2) \quad (31a) \qquad t=\frac{1}{2}|n|\left[\frac{\mu^{oc}}{\mu^{at}}\right](t_p^2+t_L^2) \quad (31b)$$

We shall use following **Fourier series** representation of the **source matrix** [28]

$$S_{At/Oc}(z,\mu,\varphi)=\begin{bmatrix}S_{AtL/OcL}(z,\mu,\varphi)\\ S_{AtQ/OcQ}(z,\mu,\varphi)\\ S_{AtU/OcU}(z,\mu,\varphi)\\ S_{AtV/OcV}(z,\mu,\varphi)\end{bmatrix}=\sum_{S=0}^{\infty}(2-\delta_{0,S})\begin{bmatrix}S^S_{AtL/OcL}(z,\mu)\ \cos s(\varphi-\varphi_0)\\ S^S_{AtQ/OcQ}(z,\mu)\ \cos s(\varphi-\varphi_0)\\ S^S_{AtU/OcU}(z,\mu)\ \sin s(\varphi-\varphi_0)\\ S^S_{AtV/OcV}(z,\mu)\ \sin s(\varphi-\varphi_0)\end{bmatrix} \quad (32)$$

Incorporation of the form (32) in equation (15) and (16) with appropriate form of phase matrix functions given for atmosphere and ocean results in the expressions for individual stokes component of the source matrix. We have not shown the results as these derivations are space consuming, straightforward although tedious and cumbersome.

III. Interface and boundary Conditions at flat ocean surface

3.1: In this article we are interested in the solution of equation (2) subject to the following four coupled boundary conditions. This is rather simplified situation. The ocean surface is absolutely calm. The coupling between the two media is described by well known laws of reflection and refraction that apply at the interface as expressed by Snells law and Fresnel's law. The practical complications that arise are due to multiple scattering and total internal reflections. The downward radiation distributed over 2π steredian in the atmosphere will be restricted to an angular cone less than 2π steredian after being refracted across the interface into the ocean. Beams outside the refractive region in the ocean are in the total reflection region. The demarcation between the refractive and total reflective region in the ocean is given by the critical angle. [Region I (Total internal reflection) and region II (Refractive region) in Fig (1)].Upward-traveling beams in total internal reflection region in ocean will be reflected back into the ocean upon reaching in interface. Thus, beams in total internal reflection region cannot reach the atmosphere directly; they must be scattered into other region in order to be returned to the atmosphere.

At the top of the atmosphere there is some prescribed incident radiation. The continuity conditions at each interface between layers in the atmosphere and ocean are omitted in this work as we have not considered layered atmosphere or ocean. Finally the **specular reflection and refraction**, occurring at the atmosphere-ocean interface where we require Fresnel's equation to be satisfied ,are assumed to be equal both ways i.e. air to water and water to air as is evident from equations (18) and (21). The reduced sets of boundary conditions for each '**S**'are given by, dropping '**S**' [11].

(A) $$\mathbf{I_{At}(0,-\mu^{at}) = I_{At\infty} f(\mu^{at})} \qquad (33)$$

(B) $$\mathbf{I_{At}(z_{\omega},\mu^{at}) = R_{At}(n,-\mu^{at},\mu^{oc}) I_{At}(z_{\omega},-\mu^{at}) + T_{OC}(n,+\mu^{oc},\mu^{at}) \frac{I_{Oc}(z_{\omega},\mu^{oc})}{n^2}}$$

(34)

(C)

$$\mathbf{\frac{I_{Oc}(z_{\omega},-\mu^{oc})}{n^2} = R_{Oc}(n,+\mu^{oc},\mu^{oc}) \left\{ \frac{I_{Oc}(z_{\omega},\mu^{oc})}{n^2} \right\} + T_{AT}(n,-\mu^{at},\mu^{oc}) I_{At}(z_{\omega},-\mu^{at})} \qquad (35)$$

(D) $$\mathbf{I_{Oc}(z_b,\mu^{oc}) = I_{Oc\infty} g(\mu^{oc})} \qquad (36).$$

Where $\mathbf{f(\mu)}$ and $\mathbf{g(\mu)}$ are some given function of polar angle with $\mathbf{I_{At\infty}}$ and $\mathbf{I_{Oc\infty}}$ as constants. In section () we have shown one example each of the above mentioned functions that can be used in case of atmosphere and ocean respectively.

IV. The homogeneous solution in terms of eigenvectors, eigenvalues and orthogonality relations

4.1: In this section we shall solve the homogeneous version of equation (10) by a new version of Chandrasekhar's discrete ordinate method developed by Siewert [12]. We use following ansatz (suppressing '**S**') for intensity vector in the discretised version of homogeneous part of the equation (10).

$$\mathbf{I_{At/Oc}(z,\pm\mu_i^{at/oc}) = H^{AT/OC}(\gamma,\pm\mu_i^{at/oc}) \exp(-\frac{z}{\gamma})} \qquad (37)$$

Where the two Gaussian polar quadratures for two different media are related by equation (17) translated here as

$$\mathbf{\mu_i^{oc} = \sqrt{1 - \frac{(1-\mu_i^{at^2})}{n^2}}} \qquad (17)$$

We now find following four expressions for two signed directions $\pm\mu_i^{at/oc}$ and corresponding Gaussian quadrature weights $\omega_k^{at/oc}$ given by the formula (29) for atmosphere and ocean respectively and to ensure simplified look henceforth we have decided to **"at/oc"** as superscript from polar angle quadratures and corresponding Gaussian weights. We get after employing a little algebra

$$(1-\frac{\mu_i}{\gamma})\mathbf{H}^{AT}(\gamma,\mu_i)=\frac{\omega_{AT}}{2}\sum_{J=S}^{M}\mathbf{P}_J^S(\mu_i)B_J^{AT}\sum_{k=1}^{N}\omega_k\mathbf{\Psi}_{j,k}^{AT} \quad (38)$$

$$(1+\frac{\mu_i}{\gamma})D\mathbf{H}^{AT}(\gamma,-\mu_i)=\frac{\omega_{AT}}{2}\sum_{J=S}^{M}(-1)^{J-S}\mathbf{P}_J^S(\mu_i)DB_J^{AT}\sum_{k=1}^{N}\omega_k\mathbf{\Psi}_{j,k}^{AT} \quad (39)$$

$$(1-\frac{\mu_i}{\gamma})\mathbf{H}^{OC}(\gamma,\mu_i)=\frac{\omega_{OC}}{2}\sum_{J=S}^{M}\mathbf{P}_J^S(\mu_i)DB_J^{OC}\sum_{K=1}^{N}\omega_k\mathbf{\Psi}_{j,k}^{OC} \quad (40)$$

$$(1+\frac{\mu_i}{\gamma})D\mathbf{H}^{OC}(\gamma,-\mu_i)=\frac{\omega_{OC}}{2}\sum_{J=S}^{M}(-1)^{J-S}\mathbf{P}_J^S(\mu_i)DB_J^{OC}\sum_{K=1}^{N}\omega_k\mathbf{\Psi}_{j,k}^{OC} \quad (41)$$

Here we have defined

$$\mathbf{D}=\mathbf{diag\{1,1,-1,-1\}} \quad (42)$$

$$\mathbf{\Psi}_{j,k}^{AT}=\left[\mathbf{P}_J^S(\mu_k)\mathbf{H}^{AT}(\gamma,\mu_i)+\mathbf{P}_J^S(-\mu_k)\mathbf{H}^{AT}(\gamma,-\mu_i)\right] \quad (43)$$

$$\mathbf{\Psi}_{j,k}^{OC}=\left[\mathbf{P}_J^S(\mu_k)\mathbf{H}^{OC}(\gamma,\mu_i)+\mathbf{P}_J^S(-\mu_k)\mathbf{H}^{OC}(\gamma,-\mu_i)\right] \quad (44)$$

However, the set of equations can be written in an equivalent but more elegant form by introducing following vectors

$$\mathbf{H}_+^{AT/OC}(\gamma)=\left[\mathbf{H}^{AT/OC}(\gamma,\mu_1)^T,\mathbf{H}^{AT/OC}(\gamma,\mu_2)^T,\mathbf{H}^{AT/OC}(\gamma,\mu_3)^T,.....\mathbf{H}^{AT/OC}(\gamma,\mu_N)^T\right]^T \quad (45)$$

$$\mathbf{H}_-^{AT/OC}(\gamma)=\left[\mathbf{H}^{AT/OC}(\gamma,-\mu_1)^T,\mathbf{H}^{AT/OC}(\gamma,-\mu_2)^T,\mathbf{H}^{AT/OC}(\gamma,-\mu_3)^T,....\mathbf{H}^{AT/OC}(\gamma,-\mu_N)^T\right]^T \quad (46)$$

Here each $\mathbf{H}^{AT/OC}(\gamma,\mu_i)^T$ is (4x1) vector. Now let us define for both atmosphere and ocean

$\mathbf{\Sigma}=\mathbf{diag}(\omega_1\mathbf{\Gamma},\omega_2\mathbf{\Gamma},\omega_3\mathbf{\Gamma},.......\omega_N\mathbf{\Gamma})$ (47); $\mathbf{X}=\mathbf{diag}(\mu_1\mathbf{\Gamma},\mu_2\mathbf{\Gamma},\mu_3\mathbf{\Gamma},.......\mu_N\mathbf{\Gamma})$ (48);

$$\mathbf{\Gamma}=\mathbf{diagonal(1,1,1,1)} \quad (49);$$

Using this formalism one can easily verify that the set (38-41) for each 'S' may be written compactly as follows.

$$\left(\mathbf{I}-\frac{1}{\gamma}\mathbf{X}\right)\mathbf{H}_+^{AT}(\gamma)=\frac{\omega^{AT}}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{AT}\mathbf{T}_J^S(AT,\gamma) \quad (50)$$

$$\left(\mathbf{I}+\frac{1}{\gamma}\mathbf{X}\right)\mathbf{H}_-^{AT}(\gamma)=\frac{\omega^{AT}}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)(-1)^{J-S}DB_J^{AT}\mathbf{T}_J^S(AT,\gamma) \quad (51)$$

$$\left(\mathbf{I}-\frac{1}{\gamma}\mathbf{X}\right)\mathbf{H}_+^{OC}(\gamma)=\frac{\omega^{OC}}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{OC}\mathbf{T}_J^S(OC,\gamma) \quad (52)$$

$$\left(\mathbf{I}+\frac{1}{\gamma}\mathbf{X}\right)\mathbf{H}_-^{OC}(\gamma)=\frac{\omega^{OC}}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)(-1)^{J-S}DB_J^{OC}\mathbf{T}_J^S(OC,\gamma) \quad (53)$$

Here $\mathbf{I}$ is a (4Nx4N) identity matrix and (4Nx4) matrix $\mathbf{\Pi}(J,S)$ is given below.

$$\mathbf{\Pi}(J,S)=\left[\mathbf{P}_J^S(\mu_1),\mathbf{P}_J^S(\mu_2),\mathbf{P}_J^S(\mu_3),....\mathbf{P}_J^S(\mu_N)\right]^T \quad (54)$$

$$\mathbf{T}_J^S(AT/OC,\gamma)=\mathbf{\Pi}^T(J,S)\mathbf{\Sigma}\mathbf{H}_+^{AT/OC}(\gamma)+(-1)^{J-S}D\mathbf{\Pi}^T(J,S)\mathbf{\Sigma}\mathbf{H}_-^{AT/OC}(\gamma) \quad (55)$$

In the above expressions the entries of matrix (54) are given by (4x4) matrices (11) and (12)

Continuing Siewert's [12] approach we shall now derive equivalent set of relations by defining the following vectors for atmosphere and ocean respectively

$$\mathbf{NA}^{AT}=\mathbf{H}_+^{AT}(\gamma)+\mathbf{H}_-^{AT}(\gamma)\ ;\ \mathbf{NB}^{AT}=\mathbf{H}_+^{AT}(\gamma)-\mathbf{H}_-^{AT}(\gamma) \quad (56)$$

$$\mathbf{NA}^{OC}=\mathbf{H}_+^{OC}(\gamma)+\mathbf{H}_-^{OC}(\gamma)\ ;\ \mathbf{NB}^{OC}=\mathbf{H}_+^{OC}(\gamma)-\mathbf{H}_-^{OC}(\gamma) \quad (57)$$

Taking sum and difference of (50) & (51) and (52) & (53) respectively for atmosphere and ocean and using (56) and (57) one can derive

$$\mathbf{A}^{AT}\mathbf{X}^{AT}=\frac{1}{\gamma}\mathbf{Y}^{AT}; \quad (58) \quad \mathbf{B}^{AT}\mathbf{Y}^{AT}=\frac{1}{\gamma}\mathbf{X}^{AT}. \quad (59)$$

$$\mathbf{A}^{OC}\mathbf{X}^{OC} = \frac{1}{\gamma}\mathbf{Y}^{OC}; \quad (60) \quad \mathbf{B}^{OC}\mathbf{Y}^{OC} = \frac{1}{\gamma}\mathbf{X}^{OC}. \quad (61)$$

Where

$$\mathbf{A}^{AT} = \left(\mathbf{I} - \frac{\omega^{AT}}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{AT}[\mathbf{\Gamma} + (-1)^{J-S}D]\mathbf{\Pi}(J,S)^T\mathbf{\Sigma}\right)\mathbf{X}^{-1}, \quad (62)$$

$$\mathbf{B}^{AT} = \left(\mathbf{I} - \frac{\omega^{AT}}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{AT}[I - (-1)^{J-S}D]\mathbf{\Pi}(J,S)^T\mathbf{\Sigma}\right)\mathbf{X}^{-1}, \quad (63)$$

$$\mathbf{A}^{OC} = \left(\mathbf{I} - \frac{\omega^{OC}}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{OC}[I + (-1)^{J-S}D]\mathbf{\Pi}(J,S)^T\mathbf{\Sigma}\right)\mathbf{X}^{-1}, \quad (64)$$

$$\mathbf{B}^{OC} = \left(\mathbf{I} - \frac{\omega^{OC}}{2}\sum_{J=S}^{M}\mathbf{\Pi}(J,S)B_J^{OC}[I - (-1)^{J-S}D]\mathbf{\Pi}(J,S)^T\mathbf{\Sigma}\right)\mathbf{X}^{-1}. \quad (65)$$

We also have $\mathbf{X}^{AT} = \mathbf{X}\left[\mathbf{NA}^{AT}\right]$(66) $\mathbf{Y}^{AT} = \mathbf{X}\left[\mathbf{NB}^{AT}\right]$(67)

$\mathbf{X}^{OC} = \mathbf{X}\left[\mathbf{NA}^{OC}\right]$(68) $\mathbf{Y}^{OC} = \mathbf{X}\left[\mathbf{NB}^{OC}\right]$(69)

In general matrix $\mathbf{A}^{AT/OC}$ and $\mathbf{B}^{AT/OC}$ are real asymmetric. We now use equations (58) & (59) and (60) & (61) to get

$$(\mathbf{B}^{AT}\mathbf{A}^{AT})\mathbf{X}^{AT} = \mathfrak{I}^{AT}\mathbf{X}^{AT}; (70) \quad (\mathbf{B}^{OC}\mathbf{A}^{OC})\mathbf{X}^{OC} = \mathfrak{I}^{OC}\mathbf{X}^{OC} \quad (72)$$

$$(\mathbf{A}^{AT}\mathbf{B}^{AT})\mathbf{Y}^{AT} = \mathfrak{I}_{AT}\mathbf{Y}^{AT}; (71) \quad (\mathbf{A}^{OC}\mathbf{B}^{OC})\mathbf{Y}^{OC} = \mathfrak{I}_{OC}\mathbf{Y}^{OC} \quad (73)$$

Our task is now to evaluate the eigenvalues $\mathfrak{I}^{AT}, \mathfrak{I}^{OC}$ which will determine the separation constants γ^{AT}, γ^{OC} for atmosphere and ocean respectively. However it is of general opinion that in such cases the eigenvalues and eigenvectors are highly unstable. Separation constants occur in plus-minus pairs. It is to be noted that the eigenvalues may be complex. We note that in general

$$\mathfrak{I} = \frac{1}{\gamma^2} \quad (74)$$

The eigenfunctions can be expressed either from equations (70-71) or (72-73). Using equations (58-61), (62-65) with (70-73) we can easily deduce

$$\mathbf{H}_{\pm}^{AT/OC}(\gamma_j^{AT/OC}) = \frac{1}{2}\mathbf{X}^{-1}\left(I \pm \gamma_j^{AT/OC}\mathbf{A}^{AT/OC}\right)\mathbf{X}^{AT/OC}(\lambda_j^{AT/OC}) \quad (75)$$

It can be shown that $\mathbf{H}_{+}^{AT/OC}(-\gamma_j^{AT/OC}) = \mathbf{H}_{-}^{AT/OC}(\gamma_j^{AT/OC})$ for $\mathbf{j = 1,2,3,.....,4N}$.

We now have all that we require to write the solution of the homogeneous RTE (10) for both atmosphere and ocean. We define a (4Nx1) matrix as

$$:\mathbf{H}_{AT/OC}(\pm) = \left[I_{AT/OC}(z,\pm\mu_1)^T, I_{AT/OC}(z,\pm\mu_2)^T, I_{AT/OC}(z,\pm\mu_3)^T, ...I_{AT/OC}(z,\pm\mu_N)^T\right]^T \quad (76)$$

The homogeneous solutions for atmosphere ($\mathbf{z} \in (0, \mathbf{z_w})$) and Ocean ($\mathbf{z} \in (\mathbf{z_w}, \mathbf{z_b})$) can now be written as

$$\mathbf{H}_{AT}(+) = \sum_{j=1}^{4N} A_j^{AT}\mathbf{H}_{+}^{AT}(\gamma_j^{AT})\exp\left(-\frac{z}{\gamma_j^{AT}}\right) + B_j^{AT}\mathbf{H}_{-}^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_w - z}{\gamma_j^{AT}}\right) \quad (77)$$

$$\mathbf{H}_{AT}(-) = \mathbf{\Delta}\sum_{j=1}^{4N} A_j^{AT}\mathbf{H}_{-}^{AT}(\gamma_j^{AT})\exp\left(-\frac{z}{\gamma_j^{AT}}\right) + B_j^{AT}\mathbf{H}_{+}^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_\omega - z}{\gamma_j^{AT}}\right) \quad (78)$$

$$\mathbf{H}_{OC}(+) = \sum_{j=1}^{4N} A_j^{OC}\mathbf{H}_{+}^{OC}(\gamma_j^{OC})\exp\left(-\frac{z}{\gamma_j^{OC}}\right) + B_j^{OC}\mathbf{H}_{-}^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_1 - z}{\gamma_j^{OC}}\right) \quad (79)$$

$$\mathbf{H}_{OC}(-) = \mathbf{\Delta}\sum_{j=1}^{4N} A_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC})\exp\left(-\frac{z}{\gamma_j^{OC}}\right) + B_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_1 - z}{\gamma_j^{OC}}\right) \quad (80)$$

$\mathbf{\Delta} = \mathrm{diag}\{D, D, D, \ldots, D\}$ is (**4Nx4N**) matrix. (81a)

Since both the eigenvector and eigenvalues may be complex as reported in [12] we prefer to write equations (77-80) in terms of real and imaginary quantities. Let us represent **Nr** for number of real separation constants and **Nc** for the number of complex pairs of separation constants. Decomposing (77-80) in to real and complex parts we get the following set of equations appropriate foe atmosphere and ocean.

$$RE_+^{AT}(z) = \sum_{j=1}^{Nr} A_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\exp\left(-\frac{z}{\gamma_j^{AT}}\right) + B_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_\omega - z}{\gamma_j^{AT}}\right), \quad (81b)$$

$$RE_-^{AT}(z) = \mathbf{\Delta}\sum_{j=1}^{Nr} A_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\exp\left(-\frac{z}{\gamma_j^{AT}}\right) + B_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\exp\left(-\frac{z_\omega - z}{\gamma_j^{AT}}\right), (81c)$$

$$RE_+^{OC}(z) = \sum_{j=1}^{Nr} A_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})\exp\left(-\frac{z}{\gamma_j^{OC}}\right) + B_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_1 - z}{\gamma_j^{AT/OC}}\right), (81d)$$

$$RE_-^{OC}(z) = \mathbf{\Delta}\sum_{j=1}^{Nr} A_j^{OC}\mathbf{H}_-^{OC}(\gamma_j^{OC})\exp\left(-\frac{z}{\gamma_j^{OC}}\right) + B_j^{OC}\mathbf{H}_+^{OC}(\gamma_j^{OC})\exp\left(-\frac{z_1 - z}{\gamma_j^{OC}}\right), (81e)$$

$$CO_\pm^{AT}(z) = \sum_{j=1}^{Nc}\sum_{\alpha=1}^{2} A_j^{AT(\alpha)}F_\pm^{AT(\alpha)}(z, \gamma_j^{AT}) + B_j^{AT(\alpha)}F_\mp^{AT(\alpha)}(z_\omega - z, \gamma_j^{AT}), (81f)$$

$$CO_\pm^{OC}(z) = \sum_{j=1}^{Nc}\sum_{\alpha=1}^{2} A_j^{OC(\alpha)}F_\pm^{OC(\alpha)}(z, \gamma_j^{OC}) + B_j^{OC(\alpha)}F_\mp^{OC(\alpha)}(z_1 - z, \gamma_j^{OC}), (81g)$$

$$F_\pm^{AT/OC(1)}(z, \gamma_j^{AT/OC}) = \mathrm{Re}\left\{\exp\left(-\frac{z}{\gamma_j^{AT/OC}}\right)\right\} X$$

$$\mathrm{Re}\left\{\mathbf{H}_\pm^{AT/OC}(\gamma_j^{AT/OC})\right\} - \mathrm{Im}\left\{\exp\left(-\frac{z}{\gamma_j^{AT/OC}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_\pm^{AT/OC}(\gamma_j^{AT/OC})\right\}$$

$$= F^{AT/OC}1(\pm 1) - F^{AT/OC}1(\pm 2). \quad (81h)$$

$$F_\pm^{AT/OC(2)}(z, \gamma_j^{AT/OC}) = \mathrm{Im}\left\{\exp\left(-\frac{z}{\gamma_j^{AT/OC}}\right)\right\}$$

$$\mathrm{Re}\left\{\mathbf{H}_\pm^{AT/OC}(\gamma_j^{AT/OC})\right\} - \mathrm{Re}\left\{\exp\left(-\frac{z}{\gamma_j^{AT/OC}}\right)\right\}\mathrm{Im}\left\{\mathbf{H}_\pm^{AT/OC}(\gamma_j^{AT/OC})\right\}$$

$$= F^{AT/OC}2(\pm 1) - F^{AT/OC}2(\pm 2). \quad (81i)$$

4.2: The Infinite Medium Green's Function:

The elementary solutions developed in the last section will now be used to find the particular solution following a method developed in [35, 40] to accommodate for the inhomogeneous source term $\mathbf{S}_{At/Oc}(z,\mu)$ .We shall first construct the infinite-medium Green's function $\mathbf{M}^{AT/OC}(z,\pm\mu_i : x,\mu_k)$ & $\mathbf{M}^{AT/OC}(z,\pm\mu_i : x,-\mu_k)$ for any source location at an arbitrary point $\mathbf{X}$ within Atmosphere or Ocean. Let us assume that the source is located at $\mathbf{x} \in (0, z_w)$ for atmosphere and at $\mathbf{x} \in (z_w, z_b)$ for ocean along with the source direction defined by $\mu_k \in \{\mu_i\}$.For the first problem we have following two equations for $\mathbf{i,k = 1,2,\ldots,N}$

$$\left(\mu_i \frac{d}{dz} + I\right)\mathbf{M}^{AT/OC}(z,\mu_i : x,\mu_k) = \frac{\omega_{At/Oc}}{2}\sum_{J=S}^{M}\mathbf{K}^{j}_{AT/OC}(z,\mu_i : x,\mu_k) + I\delta(z-x)\delta_{i,k} \quad (82a)$$

$$\left(-\mu_i \frac{d}{dz} + I\right)\mathbf{M}^{AT/OC}(z,-\mu_i : x,\mu_k) = \frac{\omega_{At/Oc}}{2}\sum_{J=S}^{M}\mathbf{K}^{j}_{AT/OC}(z,-\mu_i : x,\mu_k) \quad (82b)$$

However the second problem is described by the following two expressions.

$$\left(\mu_i \frac{d}{dz} + I\right)\mathbf{M}^{AT/OC}(z,\mu_i : x,-\mu_k) = \frac{\omega_{At/Oc}}{2}\sum_{J=S}^{M}\mathbf{K}^{j}_{AT/OC}(z,\mu_i : x,-\mu_k) \quad (82c)$$

$$\left(-\mu_i \frac{d}{dz} + I\right)\mathbf{M}^{AT/OC}(z,-\mu_i : x,-\mu_k) = \frac{\omega_{At/Oc}}{2}\sum_{J=S}^{M}\mathbf{K}^{j}_{AT/OC}(z,-\mu_i : x,-\mu_k) + I\delta(z-x)\delta_{i,k} \quad (82d)$$

Where $\mathbf{I}$ is as usual the (4X4) identity matrix and we have defined

$$\mathbf{K}^{J}_{AT/OC}(z,\pm\mu_i : x,\pm\mu_k) = \mathbf{P}^{S}_{J}(\pm\mu_i)B^{AT/OC}_{J}\sum_{\beta=1}^{N} w_\beta \mathbf{M}^{AT/OC}_{S,\beta}(z,x,\pm\mu_k) \quad (83a)$$

$$\mathbf{M}^{AT/OC}_{S,\beta}(z : x,\pm\mu_k) = \mathbf{P}^{S}_{J}(\mu_\beta)\mathbf{M}^{AT/OC}(z,\mu_\beta : x,\pm\mu_k) + \mathbf{P}^{S}_{J}(-\mu_\beta)\mathbf{M}^{AT/OC}(z,-\mu_\beta : x,\pm\mu_k) \quad (83b)$$

We have defined $\delta(z-x)$ as the Dirac delta "function" and $\delta_{i,k}$ as the Kronecker delta. Each of the two Green's functions now comes out as a (4X4) matrix. Following Case and Zweifel [39] we now proceed to develop the solution for $\mathbf{M}^{AT}(z,\xi_\beta : x,\pm\mu_k)$.We can find solution bounded as $z \to \infty$ and valid in the region $z > x$ for the homogeneous equation and another solution valid for $z < x$ and bounded as $z \to -\infty$. For matching up these two solutions with the notion of "jump" condition, valid when $i,k = 1,2,\ldots,N$, we can write, for atmosphere and ocean respectively

1. **(+, +) equations:**

$$\mu_i \lim_{\varepsilon\to 0}[\mathbf{M}^{AT}(z+\varepsilon,\mu_i : x,\mu_k) - \mathbf{M}^{AT}(z-\varepsilon,\mu_i : x,\mu_k)] = I\delta_{i,k} \quad (84a)$$

$$\mu_i \lim_{\varepsilon\to 0}[\mathbf{M}^{OC}(z+\varepsilon,\mu_i : x,\mu_k) - \mathbf{M}^{OC}(z-\varepsilon,\mu_i : x,\mu_k)] = I\delta_{i,k} \quad (84b)$$

2. **(–, +) equations:**

$$-\mu_i \lim_{\varepsilon\to 0}[\mathbf{M}^{AT}(z+\varepsilon,-\mu_i : x,\mu_k) - \mathbf{M}^{AT}(z-\varepsilon,-\mu_i : x,\mu_k)] = 0 \quad (85a)$$

$$-\mu_i \lim_{\varepsilon\to 0}[\mathbf{M}^{OC}(z+\varepsilon,-\mu_i : x,\mu_k) - \mathbf{M}^{OC}(z-\varepsilon,-\mu_i : x,\mu_k)] = 0 \quad (85b)$$

For $\mathbf{M}^{AT}(z,\pm\mu_i : x,\mu_k)$ & $\mathbf{M}^{OC}(z,\pm\mu_i : x,\mu_k)$

3. **(+,–) equations:**

$$\mu_i \lim_{\varepsilon\to 0}[\mathbf{M}^{AT}(z+\varepsilon,\mu_i : x,-\mu_k) - \mathbf{M}^{AT}(z-\varepsilon,\mu_i : x,-\mu_k)] = 0 \quad (86a)$$

$$\mu_i \lim_{\varepsilon\to 0}[\mathbf{M}^{OC}(z+\varepsilon,\mu_i : x,-\mu_k) - \mathbf{M}^{OC}(z-\varepsilon,\mu_i : x,-\mu_k)] = 0 \quad (86b)$$

4. **(–,–) equations**:

$$-\mu_i \lim_{\varepsilon\to 0}[\mathbf{M}^{AT}(z+\varepsilon,-\mu_i : x,-\mu_k) - \mathbf{M}^{AT}(z-\varepsilon,-\mu_i : x,-\mu_k)] = I\delta_{i,k} \quad (87a)$$

$$-\mu_i \lim_{\varepsilon\to 0}[\mathbf{M}^{OC}(z+\varepsilon,-\mu_i : x,-\mu_k) - \mathbf{M}^{OC}(z-\varepsilon,-\mu_i : x,-\mu_k)] = I\delta_{i,k} \quad (87b)$$

For $\mathbf{M}^{AT}(z,\pm\mu_i : x,-\mu_k)$ & $\mathbf{M}^{OC}(z,\pm\mu_i : x,-\mu_k)$.

We now have following two sets of solutions for green functions for case A (corresponding to $\mathbf{Z} > \mathbf{X}$) for atmosphere and Ocean and for Case B (corresponding to $\mathbf{Z} < \mathbf{X}$) for atmosphere and ocean respectively.

Case A: $$\mathbf{M}^{AT}_{+}(z : x,\pm\mu_k) = \sum_{J=1}^{4N}\mathbf{H}^{AT}_{+}(\gamma^{AT}_{J})C^{AT}{}_{j}(\pm\mu_k)\exp\left(-\frac{(z-x)}{\gamma^{AT}_{J}}\right), \quad z > x, 88(a)$$

$$\mathbf{M}^{AT}_{-}(z : x,\pm\mu_k) = \mathbf{\Delta}\sum_{J=1}^{4N}\mathbf{H}^{AT}_{-}(\gamma^{AT}_{J})C^{AT}{}_{j}(\pm\mu_k)\exp\left(-\frac{(z-x)}{\gamma^{AT}_{J}}\right), \quad z > x, 88(b)$$

with

$$\mathbf{M}_{\pm}^{AT}(z:x,\pm\mu_k)=\left[[\mathbf{M}^{AT}(z,\pm\mu_1:x,\pm\mu_k)]^T,\ldots[\mathbf{M}^{AT}(z,\pm\mu_N:x,\pm\mu_k)]^T\right]^T \quad (89)$$

and

$$\mathbf{M}_{+}^{OC}(z:x,\pm\mu_k)=\sum_{J=1}^{4N}\mathbf{H}_{+}^{OC}(\gamma_J^{OC})C^{OC}{}_j(\pm\mu_k)\exp\left(-\frac{(z-x)}{\gamma_j^{OC}}\right),\quad z>x, \quad (90a)$$

$$\mathbf{M}_{-}^{OC}(z:x,\pm\mu_k)=\mathbf{\Delta}\sum_{J=1}^{4N}\mathbf{H}_{-}^{OC}(\gamma_J^{OC})C^{OC}{}_j(\pm\mu_k)\exp\left(-\frac{(z-x)}{\gamma_J^{OC}}\right),\quad z>x, \quad (90b)$$

with

$$\mathbf{M}_{\pm}^{OC}(z:x,\pm\mu_k)=\left[[\mathbf{M}^{OC}(z,\pm\mu_1:x,\pm\mu_k)]^T,\ldots[\mathbf{M}^{OC}(z,\pm\mu_N:x,\pm\mu_k)]^T\right]^T \quad (91)$$

Case B:

$$\mathbf{M}_{+}^{AT}(z:x,\pm\mu_k)=-\sum_{J=1}^{4N}\mathbf{H}_{-}^{AT}(\gamma_J^{AT})D^{AT}{}_j(\pm\mu_k)\exp\left(-\frac{(x-z)}{\gamma_J^{AT}}\right),\quad z<x, \quad (92a)$$

$$\mathbf{M}_{-}^{AT}(z:x,\pm\mu_k)=-\mathbf{\Delta}\sum_{J=1}^{4N}\mathbf{H}_{+}^{AT}(\gamma_J^{AT})D^{AT}{}_j(\pm\mu_k)\exp\left(-\frac{(x-z)}{\gamma_J^{AT}}\right), z<x, \quad (92b)$$

$$\mathbf{M}_{+}^{OC}(z:x,\pm\mu_k)=-\sum_{J=1}^{4N}\mathbf{H}_{-}^{OC}(\gamma_J^{OC})D^{OC}{}_j(\pm\mu_k)\exp\left(-\frac{(x-z)}{\gamma_J^{OC}}\right),\quad z<x, \quad (93a)$$

$$\mathbf{M}_{-}^{OC}(z:x,\pm\mu_k)=-\mathbf{\Delta}\sum_{J=1}^{4N}\mathbf{H}_{+}^{OC}(\gamma_J^{OC})D^{OC}{}_j(\pm\mu_k)\exp\left(-\frac{(x-z)}{\gamma_J^{OC}}\right),\quad z<x, \quad (93b)$$

Using discrete ordinate solutions for homogenous equations (88a-93b) for Green functions in the limiting equations (84a-87b), for $\mathbf{z}>\mathbf{x}$ and $\mathbf{z}<\mathbf{x}$ respectively, we get after straightforward algebraic manipulations following solutions keeping appropriate consideration for positive and negative μ_i and μ_k, ($\mathbf{x}$ within atmosphere or ocean).

$$\mathbf{X}\sum_{J=1}^{4N}[\mathbf{H}_{+}^{AT}(\gamma_j^{AT})C_j^{AT}(\mu_k)+\mathbf{H}_{-}^{AT}(\gamma_j^{AT})D_j^{AT}(\mu_k)]=R_k \quad (94a)$$

When similar procedures are applied on (85a) with (78) we get

$$-\mathbf{X}\mathbf{\Delta}\sum_{J=1}^{4N}[\mathbf{H}_{-}^{AT}(\gamma_j^{AT})C_j^{AT}(\mu_k)+\mathbf{H}_{+}^{AT}(\gamma_j^{AT})D_j^{AT}(\mu_k)]=\mathbf{0} \quad (94b)$$

This constitutes the first set of solutions. For the second set of solutions we need equations (86a) and (87a) to give

$$\mathbf{X}\sum_{J=1}^{4N}[\mathbf{H}_{+}^{AT}(\gamma_j^{AT})C_j^{AT}(-\mu_k)+\mathbf{H}_{-}^{AT}(\gamma_j^{AT})D_j^{AT}(-\mu_k)]=\mathbf{0} \quad (95a)$$

$$-\mathbf{X}\mathbf{\Delta}\sum_{J=1}^{4N}[\mathbf{H}_{-}^{AT}(\gamma_j^{AT})C_j^{AT}(-\mu_k)+\mathbf{H}_{+}^{AT}(\gamma_j^{AT})D_j^{AT}(-\mu_k)]=R_k. \quad (95b)$$

For Ocean the corresponding solution pairs are given below.

$$\mathbf{X}\sum_{J=1}^{4N}[\mathbf{H}_{+}^{OC}(\gamma_j^{OC})C_j^{OC}(\mu_k)+\mathbf{H}_{-}^{OC}(\gamma_j^{OC})D_j^{OC}(\mu_k)]=R_k \quad (96a)$$

$$-\mathbf{X}\mathbf{\Delta}\sum_{J=1}^{4N}[\mathbf{H}_{-}^{OC}(\gamma_j^{OC})C_j^{OC}(\mu_k)+\mathbf{H}_{+}^{OC}(\gamma_j^{OC})D_j^{OC}(\mu_k)]=\mathbf{0} \quad (96b)$$

$$\mathbf{X}\sum_{J=1}^{4N}[\mathbf{H}_{+}^{OC}(\gamma_j^{OC})C_j^{OC}(-\mu_k)+\mathbf{H}_{-}^{OC}(\gamma_j^{OC})D_j^{OC}(-\mu_k)]=\mathbf{0} \quad (97a)$$

$$-\mathbf{X}\mathbf{\Delta}\sum_{J=1}^{4N}[\mathbf{H}_{-}^{OC}(\gamma_j^{OC})C_j^{OC}(-\mu_k)+\mathbf{H}_{+}^{OC}(\gamma_j^{OC})D_j^{OC}(-\mu_k)]=R_k \quad (97b)$$

$$R_k=[I\delta_{1,k},I\delta_{2,k},\ldots,I\delta_{N,k}]^T \quad (98)$$

We shall now solve the set of equations given by (94a) and (94b) to get the unknown coefficients for atmosphere. Similar considerations can also be applied on equations (95a) and (95b) and for Ocean also. However complete solution requires certain **orthogonality** relations satisfied by the eigenvectors.

4.3: Orthogonality Relations: Atmospheric/Oceanic adjoint problem is defined by replacing $\mathbf{B_J^{AT/OC}}$ in (50-53) with $[\mathbf{B_J^{AT/OC}}]^{\mathbf{T}}$ and writing the equations in the following form

$$\left(\mathbf{I}-\frac{1}{\gamma}\mathbf{X}\right)\mathbf{AH}_{+}^{\mathbf{AT/OC}}(\gamma)=\frac{\omega^{\mathbf{AT/OC}}}{2}\sum_{\mathbf{J=S}}^{\mathbf{M}}\mathbf{\Pi}(\mathbf{J,S})[\mathbf{B_J^{AT/OC}}]^{\mathbf{T}}\mathbf{AT_J^S}(\mathbf{AT/OC},\gamma) \quad (99a)$$

$$\left(\mathbf{I}+\frac{1}{\gamma}\mathbf{X}\right)\mathbf{AH}_{-}^{\mathbf{AT/OC}}(\gamma)=\frac{\omega^{\mathbf{AT/OC}}}{2}\sum_{\mathbf{J=S}}^{\mathbf{M}}(-1)^{\mathbf{l-J}}\mathbf{\Pi}(\mathbf{J,S})\mathbf{D}[\mathbf{B_J^{AT/OC}}]^{\mathbf{T}}\mathbf{AT_J^S}(\mathbf{AT/OC},\gamma) \quad (99b)$$

$$\mathbf{AT_J^S}(\mathbf{AT/OC},\gamma)=\mathbf{\Pi^T}(\mathbf{J,S})\mathbf{\Sigma AH}_{+}^{\mathbf{AT/OC}}(\gamma)+(-1)^{\mathbf{l-J}}\mathbf{D\Pi^T}(\mathbf{J,S})\mathbf{\Sigma AH}_{-}^{\mathbf{AT/OC}}(\gamma) \quad (100)$$

The eigenvalues defined by the equations (38-41), where asymmetric matrices **A** and **B** are involved, will not change if the above mentioned changes are introduced in (38-41). This means that the adjoint vectors $\mathbf{AH}_{\pm}^{\mathbf{AT/OC}}(\gamma_{\mathbf{J}}^{\mathbf{AT/OC}})$ are defined over the same spectrum as the vectors $\mathbf{H}_{\pm}^{\mathbf{AT/OC}}(\gamma_{\mathbf{J}}^{\mathbf{AT/OC}})$.To evaluate the orthogonality relations following we employ following steps as described below.

Step1: We evaluate equation (99a) and (99b) at $\gamma=\gamma_{\mathbf{J}}^{\mathbf{AT}}$ and premultiply the resulting equations with $[\mathbf{AH}_{+}^{\mathbf{AT}}(\gamma_{\mathbf{k}})]^{\mathbf{T}}\mathbf{W}$ and $[\mathbf{AH}_{-}^{\mathbf{AT}}(\gamma_{\mathbf{k}})]^{\mathbf{T}}\mathbf{W}$ respectively for $\mathbf{k}\neq\mathbf{j}$.We next add the last two equations to get the **first** equation.

Step2: Interchanging **j** and **k** in the **first** equations we get the **second** equation.

Step3: We premultiply now **the first** equation by $[\mathbf{H}_{+}^{\mathbf{AT}}(\gamma_{\mathbf{j}}^{\mathbf{AT}})]^{\mathbf{T}}$ and **second** equation by $[\mathbf{H}_{-}^{\mathbf{AT}}(\gamma_{\mathbf{j}}^{\mathbf{AT}})]^{\mathbf{T}}$, adding the two resulting equations and then interchanging **j** and **k** to get **third** equation.

Step4: Next we set $\mathbf{B_J^{AT}}$ as $[\mathbf{B_J^{AT}}]^{\mathbf{T}}$ in the **third** equation and take transpose of this equation and subtract this equation from the **third** equation to get the following equation

$$\mathbf{AH}_{+}^{\mathbf{AT}}(\gamma_{\mathbf{k}})^{\mathbf{T}}\mathbf{\Sigma XH}_{+}^{\mathbf{AT}}(\gamma_{\mathbf{j}}^{\mathbf{AT}})-\mathbf{AH}_{-}^{\mathbf{AT}}(\gamma_{\mathbf{k}})^{\mathbf{T}}\mathbf{\Sigma XH}_{-}^{\mathbf{AT}}(\gamma_{\mathbf{j}}^{\mathbf{AT}})=\mathbf{0},\quad \gamma_{\mathbf{j}}\neq\gamma_{\mathbf{k}}. \quad (101)$$

Step5: Apply the same above set of four procedures to the same set of equation but interchanging the order of the multiplier eigenvectors leads to the following equation

$$[\mathbf{AH}_{-}^{\mathbf{AT}}(\gamma_{\mathbf{k}})]^{\mathbf{T}}\mathbf{\Sigma XH}_{-}^{\mathbf{AT}}(\gamma_{\mathbf{j}}^{\mathbf{AT}})-[\mathbf{AH}_{-}^{\mathbf{AT}}(\gamma_{\mathbf{k}})]^{\mathbf{T}}\mathbf{\Sigma XH}_{+}^{\mathbf{AT}}(\gamma_{\mathbf{j}}^{\mathbf{AT}})=\mathbf{0}. \quad (102)$$

Equation (101) and (102) are called **orthogonality relations** in case of atmosphere.

Premultiplying (94a) with $[\mathbf{AH}_{+}^{\mathbf{AT}}(\gamma_{\mathbf{k}})]^{\mathbf{T}}\mathbf{\Sigma}$ and (94b) with $[\mathbf{AH}_{-}^{\mathbf{AT}}(\gamma_{\mathbf{k}})]^{\mathbf{T}}\mathbf{\Sigma\Delta}$ and adding these two resulting equations we get

$$\begin{aligned}&\sum_{\mathbf{j=1}}^{\mathbf{4N}}\mathbf{AH}_{+}^{\mathbf{AT}}(\gamma_{\mathbf{k}})\mathbf{\Sigma X}\left[\mathbf{H}_{+}^{\mathbf{AT}}(\gamma_{\mathbf{j}}^{\mathbf{AT}})\mathbf{C_j^{AT}}+\mathbf{H}_{-}^{\mathbf{AT}}(\gamma_{\mathbf{j}}^{\mathbf{AT}})\mathbf{D_j^{AT}}\right]+\\&\mathbf{AH}_{-}^{\mathbf{AT}}(\gamma_{\mathbf{k}})\mathbf{\Sigma\Delta}\left[-\mathbf{X\Delta}\left[\mathbf{H}_{-}^{\mathbf{AT}}(\gamma_{\mathbf{j}}^{\mathbf{AT}})\mathbf{C_j^{AT}}+\mathbf{H}_{+}^{\mathbf{AT}}(\gamma_{\mathbf{j}}^{\mathbf{AT}})\mathbf{D_j^{AT}}\right]\right]\\&=\mathbf{AH}_{+}^{\mathbf{AT}}(\gamma_{\mathbf{k}})\mathbf{\Sigma R_\alpha}\end{aligned} \quad (103)$$

For we have $\mathbf{j}=\mathbf{k}$ after noting the two orthogonality relations (101) and (102),

$$\mathbf{C_j^{AT}}(\mu_{\mathbf{k}})=\frac{\mathbf{1}}{\mathbf{NAT}(\gamma_{\mathbf{j}})}[\mathbf{AH}_{+}^{\mathbf{AT}}(\gamma_{\mathbf{j}})]^{\mathbf{T}}\mathbf{\Sigma R_k} \quad (104)$$

Where

$$\mathbf{NAT}(\gamma_{\mathbf{j}}^{\mathbf{AT}})=[\mathbf{AH}_{+}^{\mathbf{AT}}(\gamma_{\mathbf{j}})]^{\mathbf{T}}\mathbf{\Sigma XH}_{+}^{\mathbf{AT}}(\gamma_{\mathbf{j}}^{\mathbf{AT}})-[\mathbf{AH}_{-}^{\mathbf{AT}}(\gamma_{\mathbf{j}}^{\mathbf{AT}})]^{\mathbf{T}}\mathbf{\Sigma XH}_{-}^{\mathbf{AT}}(\gamma_{\mathbf{j}}^{\mathbf{AT}}). \quad (105)$$

Continuing we shall pre-multiply equation (94a) by $[\mathbf{AH}_{-}^{\mathbf{AT}}(\gamma_{\mathbf{k}})]^{\mathbf{T}}\mathbf{W}$ and equation (94b) by $[\mathbf{AH}_{+}^{\mathbf{AT}}(\gamma_{\mathbf{k}})]^{\mathbf{T}}\mathbf{\Sigma\Delta}$, and add the two resulting equations and use equations (101) and (102) to obtain

$$\mathbf{D_j^{AT}}(\mu_{\mathbf{k}})=-\frac{\mathbf{1}}{\mathbf{NAT}(\gamma_{\mathbf{J}}^{\mathbf{AT}})}[\mathbf{AH}_{-}^{\mathbf{AT}}(\gamma_{\mathbf{j}}^{\mathbf{AT}})]^{\mathbf{T}}\mathbf{\Sigma R_k} \quad (106)$$

Similarly we find from the second set of equations (95a) and (95b)

$$\mathbf{C_j^{AT}(-\mu_k) = \frac{1}{NAT(\gamma_j^{AT})}[AH_-^{AT}(\gamma_j^{AT})]^T \Sigma\Delta R_k} \quad (107)$$

and

$$\mathbf{D_j^{AT}(-\mu_k) = -\frac{1}{NAT(\gamma_j^{AT})}[AH_+^{AT}(\gamma_j^{AT})]^T \Sigma\Delta R_k}. \quad (108)$$

The required constants C and D in equations (94) and (95) are defined by equations (104 &106) and (107&108). Similar steps can also be followed in the case of Ocean to get the **orthogonality conditions** as given

$$\mathbf{AH_+^{OC}(\gamma_k)^T \Sigma X H_+^{OC}(\gamma_j^{OC}) - AH_-^{OC}(\gamma_k)^T \Sigma X H_-^{OC}(\gamma_j^{OC}) = 0,}\ \gamma_j \neq \gamma_k. \quad (109)$$

$$\mathbf{[AH_-^{OC}(\gamma_k)]^T \Sigma X H_-^{OC}(\gamma_j^{OC}) - [AH_-^{OC}(\gamma_k)]^T \Sigma X H_+^{OC}(\gamma_j^{OC}) = 0.} \quad (110)$$

Involving equations (90) and (91) for the first and second set of solutions for Ocean in the above derivations and using equations (103) and (104) we get

$$\mathbf{C_j^{OC}(\mu_k) = \frac{1}{NOC(\gamma_j)}[AH_+^{OC}(\gamma_j)]^T \Sigma R_k} \quad (111)$$

$$\mathbf{D_j^{OC}(\mu_k) = -\frac{1}{NOC(\gamma_j)}[AH_-^{OC}(\gamma_j)]^T \Sigma R_k}. \quad (112)$$

$$\mathbf{C_j^{OC}(-\mu_k) = \frac{1}{NOC(\gamma_j)}[AH_-^{OC}(\gamma_j)]^T \Sigma\Delta R_k}. \quad (113)$$

$$\mathbf{D_j^{OC}(-\mu_k) = -\frac{1}{NOC(\gamma_j)}[AH_+^{OC}(\gamma_j)]^T \Sigma\Delta R_k}. \quad (114)$$

$$\mathbf{NOC(\gamma_j) = [AH_+^{OC}(\gamma_j)]^T \Sigma X H_+^{OC}(\gamma_j) - [AH_-^{OC}(\gamma_j)]^T \Sigma X H_-^{OC}(\gamma_j)}. \quad (115)$$

Having found the expansion coefficients for green functions we are now in a position to determine the particular solution for the inhomogeneous source term in the RTE. This is the objective of the next section. It is to be noted from the above set of equations that determination of adjoint vectors is essential. Here we have adopted the standard text book procedure. For a real asymmetric eigenvalue problem, double shift QR or QZ algorithm should be used. Other methods may produce fictious complex eigenvalues. One may use Qz algorithm in our future numerical adventure as this is more stable.

One may use matlab 7 software packages where standard LAPACK algorithm is used to calculate the left eigenvector as well as the right eigenvectors and corresponding eigenvalues. These left eigenvectors may be used to find the adjoint vector. These calculations will be reported in some detail somewhere.

V. The Particular and Complete Solution:

5.1: Particular solution:

To determine particular solution we recall our transfer equations in the following form.

$$\mathbf{\pm\mu_i \frac{d}{dz} I_{AT}(z,\pm\mu_i) + I_{AT}(z,\pm\mu_i) = \frac{\omega^{AT}}{2} \sum_{J=S}^{M} P_J^S(\pm\mu_i) B_J^{AT} \sum_{\beta=1}^{N} \omega_\beta I_{J,\beta}^{AT}(z) + S_{AT}(z,\pm\mu_i),} \quad (116)$$

$$\mathbf{I_{J,\beta}^{AT}(z) = P_J^S(\mu_\beta) I(z,\mu_\beta) + P_J^S(-\mu_\beta) I(z,-\mu_\beta)} \quad (117)$$

$$\mathbf{\pm\mu_i \frac{d}{dz} I_{OC}(z,\pm\mu_i) + I_{OC}(z,\pm\mu_i) = \frac{\omega^{OC}}{2} \sum_{J=S}^{M} P_J^S(\pm\mu_i) B_J^{OC} \sum_{\beta=1}^{N} \omega_\beta I_{J,\beta}^{OC}(z) + S_{OC}(z,\pm\mu_i),} \quad (118)$$

$$\mathbf{I_{J,\beta}^{OC}(z) = P_J^S(\mu_\beta) I(z,\mu_\beta) + P_J^S(-\mu_\beta) I(z,-\mu_\beta)}. \quad (119)$$

The general solution to the homogeneous version of equations (10) is given by (77-80). With the help of green functions developed in the preceding section for $\mathbf{z > x}$ and $\mathbf{z < x}$, we can immediately write one particular solution in the following manner,

$$\mathbf{I^{p}_{AT}(z,\pm\mu_i)=\sum_{\alpha=1}^{N}\int_0^{z_\omega}[M^{AT}(z,\pm\mu_i:x,\mu_\alpha)S_{AT}(x,\mu_\alpha)+M^{AT}(z,\pm\mu_i:x,-\mu_\alpha)S_{AT}(x,-\mu_\alpha)]\,dx}$$
(120)

Or

$$\mathbf{I^{p}_{AT}(\pm;z)=\sum_{\alpha=1}^{N}\int_0^{z_\omega}[M^{AT}_{\pm}(z:x,\mu_\alpha)S_{AT}(x,\mu_\alpha)+M^{AT}_{\pm}(z:x,-\mu_\alpha)S_{AT}(x,-\mu_\alpha)]\,dx} \quad (121)$$

And

$$\mathbf{I^{p}_{OC}(z,\pm\mu_i)=\sum_{\alpha=1}^{N}\int_0^{z_0}[M^{OC}(z,\pm\mu_i:x,\mu_\alpha)S_{OC}(x,\mu_\alpha)+M^{OC}(z,\pm\mu_i:x,-\mu_\alpha)S_{OC}(x,-\mu_\alpha)]\,dx}$$
(122)

Or

$$\mathbf{I^{p}_{OC}(\pm;z)=\sum_{\alpha=1}^{N}\int_0^{z_0}[M^{OC}_{\pm}(z:x,\mu_\alpha)S_{OC}(x,\mu_\alpha)+M^{OC}_{\pm}(z:x,-\mu_\alpha)S_{OC}(x,-\mu_\alpha)]\,dx}$$
(123)

We can rewrite equation (121) using (88a and 88b)

$$\mathbf{I^{p}_{AT}(+;z)=\sum_{j=1}^{4N}\left[\Re^{AT}_j(z)H^{AT}_{+}(\gamma_j)+\aleph^{AT}_j(z)H^{AT}_{-}(\gamma_j)\right]} \quad (124)$$

$$\mathbf{I^{p}_{AT}(-;z)=\Delta\sum_{j=1}^{4N}[\Re^{AT}_j(z)H^{AT}_{-}(\gamma_j)+\aleph^{AT}_j(z)H^{AT}_{+}(\gamma_j)]} \quad (125)$$

Where we have defined

$$\mathbf{\Re^{AT}_j(z)=\int_0^{z}\sum_{\alpha=1}^{N}C^{AT}_j(\mu_\alpha)S_{AT}(x,\mu_\alpha)+C^{AT}_j(-\mu_\alpha)S_{AT}(x,-\mu_\alpha)]\exp\left(-\frac{z-x}{\gamma_j}\right)dx} \quad (126)$$

$$\mathbf{\aleph^{AT}_j(z)=-\int_z^{z_\omega}\sum_{\alpha=1}^{N}[D^{AT}_j(\mu_\alpha)S_{AT}(x,\mu_\alpha)+D^{AT}_j(-\mu_\alpha)S_{AT}(x,-\mu_\alpha)]\exp\left(-\frac{x-z}{\gamma_j}\right)dx.} \quad (127)$$

Using (104-108) in (126) and (127) we obtain

$$\mathbf{\Re^{AT}_j(z)=\int_0^{z}a^{AT}_j(x)\exp\left(-\frac{z-x}{\gamma_j}\right)dx} \quad (128) \text{ and } \mathbf{\aleph^{AT}_j(z)=\int_z^{z_\omega}b^{AT}_j(x)\exp\left(-\frac{x-z}{\gamma_j}\right)dx.} \quad (129)$$

Where we have defined

$$\mathbf{a^{AT}_j(x)=\frac{1}{NAT(\gamma_j)}\left[[AH^{AT}_{+}(\gamma_j)]^T\Sigma S_{AT(+)}(x)+[AH^{AT}_{-}(\gamma_j)]^T\Sigma\Delta S_{AT(-)}(x)\right]} \quad (130)$$

$$\mathbf{b^{AT}_j(x)=\frac{1}{NAT(-\gamma_j)}\left[[AH^{AT}_{-}(\gamma_j)]^T\Sigma S_{AT(+)}(x)+[AH^{AT}_{+}(\gamma_j)]^T\Sigma\Delta S_{AT(-)}(x)\right]} \quad (131)$$

$$\mathbf{S_{AT(\pm)}(x)=\left[[S_{AT}(x,\pm\mu_1)]^T,[S_{AT}(x,\pm\mu_2)]^T,\ldots,[S_{AT}(x,\pm\mu_N)]^T\right]^T} \quad (132)$$

From equations (15) and (16) it can be shown that our model source function can be written as

$$\mathbf{S_{AT(\pm)}(x)=S_{AT}(\pm)\exp\left(-\frac{x}{\mu_0}\right)} \quad (133) \quad \mathbf{S_{OC\,(\pm)}(x)=S_{OC}(\pm)\exp\left(-\frac{x}{\mu_{0n}}\right)} \quad (134)$$

Substituting the expressions (133) in (130) and (131) we can get following simple expressions from (128) and (129)

$$\mathbf{\Re^{AT}_j(z)=\mu_0\gamma^{AT}_j a^{AT}_j CAT(z:\gamma^{AT}_j,\mu_0)} \quad (135)$$

$$\mathbf{\aleph^{AT}_j(z)=\mu_0\gamma^{AT}_j b^{AT}_j\exp(-z/\mu_0)SAT(z_\omega-z:\gamma^{AT}_j,\mu_0).} \quad (136)$$

$$\mathbf{SAT(z:x,y)=\frac{1-\exp\left(-\frac{z}{x}\right)\exp\left(-\frac{z}{y}\right)}{x+y}} \quad (137); \quad \mathbf{CAT(z:x,y)=\frac{\exp\left(-\frac{z}{x}\right)-\exp\left(-\frac{z}{y}\right)}{x-y}}. \quad (138)$$

Following similar approach for ocean starting from (123) we get

$$\mathbf{a_j^{OC}} = \frac{\mathbf{1}}{\mathbf{NOC(\gamma_j)}}\Big[[\mathbf{AH_+^{OC}}(\gamma_j)]^T\mathbf{\Sigma S_{OC}}(+) + [\mathbf{AH_-^{OC}}(\gamma_j)]^T\mathbf{\Sigma\Delta S_{OC}}(-)\Big] \quad (139)$$

$$\mathbf{b_j^{OC}} = \frac{\mathbf{1}}{\mathbf{NOC(\gamma_j)}}\Big[[\mathbf{AH_-^{OC}}(\gamma_j)]^T\mathbf{\Sigma S_{OC}}(+) + [\mathbf{AH_+^{OC}}(\gamma_j)]^T\mathbf{\Sigma\Delta S_{OC}}(-)\Big] \quad (140)$$

$$\mathfrak{R}_j^{OC}(\mathbf{z}) = \mu_{0n}\gamma_j^{OC}\ \mathbf{a_j^{OC}}\ \mathbf{COC}(\mathbf{z}:\gamma_j,\mu_{0n}) \quad (141)$$

$$\aleph_j^{OC}(\mathbf{z}) = \mu_{0n}\gamma_j^{OC}\ \mathbf{b_j^{OC}}\exp(-z/\mu_{0n})\,\mathbf{SOC}(\mathbf{z_1}-\mathbf{z}:\gamma_j,\mu_{0n}). \quad (142)$$

$$\mathbf{SOC}(\mathbf{z}:\mathbf{x},\mathbf{y}) = \frac{1-\exp\left(-\frac{z}{x}\right)\exp\left(-\frac{z}{y}\right)}{x+y}; \ (143) \quad \mathbf{COC}(\mathbf{z}:\mathbf{x},\mathbf{y}) = \frac{\exp\left(-\frac{z}{x}\right)-\exp\left(-\frac{z}{y}\right)}{x-y}; \ (144)$$

We shall now consider the case of **complex separation constants**. For real quantities if we let quantities with **asterisks** as **complex conjugates**

$$\mathbf{AZ_\pm^{AT/OC}}(\mathbf{z},\gamma_j^{AT/OC*}) = \mathfrak{R}_j^{AT/OC}(\mathbf{z})\mathbf{H}_\pm^{AT/OC}(\gamma_j^{AT/OC}) + \mathfrak{R}_j^{AT/OC\,*}(\mathbf{z})\mathbf{H}_\pm^{AT/OC\,*}(\gamma_j^{AT/OC*}); \quad (145)$$

$$\mathbf{BZ_\pm^{AT/OC}}(\mathbf{z},\lambda_j^{AT/OC}) = \aleph_j^{AT/OC}(\mathbf{z})\mathbf{H}_\pm^{AT/OC}(\gamma_j^{AT/OC}) + \aleph_j^{AT/OC\,*}(\mathbf{z})\mathbf{H}_\pm^{AT/OC\,*}(\gamma_j^{AT/OC}). \quad (146)$$

5.2. The Complete Solution:

The **complete particular solution** can now be written as

$$\mathbf{I_{AT}^p}(+;\mathbf{z}) = \sum_{j=1}^{Nr}[\mathfrak{R}_j^{AT}(\mathbf{z})\mathbf{H}_+^{AT}(\gamma_j^{AT}) + \aleph_j^{AT}(\mathbf{z})\mathbf{H}_-^{AT}(\gamma_j^{AT})] + \sum_{j=1}^{Nc}\mathbf{AZ_+^{AT}}(\mathbf{z},\gamma_j^{AT*}) + \mathbf{BZ_-^{AT}}(\mathbf{z},\gamma_j^{AT*})$$

(147)

$$\mathbf{I_{AT}^p}(-;\mathbf{z}) = \mathbf{\Delta}\sum_{j=1}^{Nr}[\mathfrak{R}_j^{AT}(\mathbf{z})\mathbf{H}_-^{AT}(\gamma_j^{AT}) + \aleph_j^{AT}(\mathbf{z})\mathbf{H}_+^{AT}(\gamma_j^{AT})] + \mathbf{\Delta}\sum_{j=1}^{Nc}\mathbf{AZ_-^{AT}}(\mathbf{z},\gamma_j^{AT*}) + \mathbf{BZ_+^{AT}}(\mathbf{z},\gamma_j^{AT*}) \quad (148)$$

$$\mathbf{I_{OC}^p}(+;\mathbf{z}) = \sum_{j=1}^{Nr}[\mathfrak{R}_j^{OC}(\mathbf{z})\mathbf{H}_+^{OC}(\gamma_j^{OC}) + \aleph_j^{OC}(\mathbf{z})\mathbf{H}_-^{OC}(\gamma_j^{OC})] + \sum_{j=1}^{Nc}\mathbf{AZ_+^{OC}}(\mathbf{z},\gamma_j^{OC*}) + \mathbf{BZ_-^{OC}}(\mathbf{z},\gamma_j^{OC*})$$

(149)

$$\mathbf{I_{OC}^p}(-;\mathbf{z}) = \mathbf{\Delta}\sum_{j=1}^{Nr}[\mathfrak{R}_j^{OC}(\mathbf{z})\mathbf{H}_-^{OC}(\gamma_j^{OC}) + \aleph_j^{OC}(\mathbf{z})\mathbf{H}_+^{OC}(\gamma_j^{OC})] + \mathbf{\Delta}\sum_{j=1}^{Nc}[\mathbf{AZ_-^{OC}}(\mathbf{z},\gamma_j^{OC*}) + \mathbf{BZ_+^{OC}}(\mathbf{z},\gamma_j^{OC*})]$$

(150)

We now find out appropriate expressions of the particular solutions suitable for application in the boundary conditions. These will be used to find the unknown coefficients.

For top boundary condition the following particular form will be used by setting **z=0** in (148)

$$\mathbf{I_{AT}^p}(-;\mathbf{0}) = \mathbf{\Delta}\sum_{j=1}^{Nr}[\mathfrak{R}_j^{AT}(\mathbf{0})\mathbf{H}_-^{AT}(\gamma_j) + \aleph_j^{AT}(\mathbf{0})\mathbf{H}_+^{AT}(\gamma_j)] + \sum_{j=1}^{Nc}\mathbf{AZ_+^{AT}}(\mathbf{0},\gamma_j^{AT}) + \mathbf{BZ_-^{AT}}(\mathbf{0},\gamma_j^{AT}) \quad (151)$$

The particular form of solutions for application in the second and third boundary conditions are given by

$$[\mathbf{I_{AT}^p}(+;\mathbf{z_\omega})] = \sum_{j=1}^{Nr}[\mathfrak{R}_j^{AT}(\mathbf{z_\omega})\mathbf{H}_+^{AT}(\gamma_j) + \aleph_j^{AT}(\mathbf{z_\omega})\mathbf{H}_-^{AT}(\gamma_j)] + \sum_{j=1}^{Nc}\mathbf{AZ_+^{AT}}(\mathbf{z_\omega},\gamma_j^{AT}) + \mathbf{BZ_-^{AT}}(\mathbf{z_\omega},\gamma_j^{AT}) \quad (152)$$

$$[I^p_{AT}(-;z_\omega)] = \boldsymbol{\Delta}\sum_{j=1}^{Nr}[\Re^{AT}_j(z_\omega)\mathbf{H}^{AT}_-(\gamma_j) + \aleph^{AT}_j(z_\omega)\mathbf{H}^{AT}_+(\gamma_j)] \ + \ \boldsymbol{\Delta}\sum_{j=1}^{Nc} AZ^{AT}_-(z_\omega,\gamma^{AT}_j) \ + \ BZ^{AT}_+(z_\omega,\lambda^{AT}_j)$$

(153) $$[I^p_{OC}(+;z_\omega)] = \sum_{j=1}^{Nr}[\Re^{OC}_j(z_\omega)\mathbf{H}^{OC}_+(\gamma_j) + \aleph^{OC}_j(z_\omega)\mathbf{H}^{OC}_-(\gamma_j)] \ + \ \sum_{j=1}^{Nc} AZ^{OC}_+(z_\omega,\gamma^{OC}_j) \ + \ BZ^{OC}_-(z_\omega,\lambda^{OC}_j)$$

(154)

$$[I^p_{OC}(-;z_\omega)] = \boldsymbol{\Delta}[\sum_{j=1}^{Nr}\Re^{OC}_j(z_\omega)\mathbf{H}^{OC}_-(\gamma_j) + \aleph^{OC}_j(z_\omega)\mathbf{H}^{OC}_+(\gamma_j)] \ + \ \boldsymbol{\Delta}\sum_{j=1}^{Nc}[AZ^{OC}_-(z_\omega,\gamma^{OC}_j) \ + \ BZ^{OC}_+(z\omega,\lambda^{OC}_j)]$$

(155)

The particular solution that will be used in the bottom boundary condition

$$[I^p_{OC}(+;z_b)] = \sum_{j=1}^{Nr}\Re^{OC}_j(z_1)\mathbf{H}^{OC}_+(\gamma_j) + \aleph^{OC}_j(z_1)\mathbf{H}^{OC}_-(\gamma_j) + \sum_{j=1}^{Nc} AZ^{OC}_+(z_1,\gamma^{OC}_j) + BZ^{OC}_-(z_1,\lambda^{OC}_j) \quad (156)$$

We also have $\Re^{AT}_j(0) = \mu_0\gamma^{AT}_j a^{AT}_j CAT(0:\gamma_j,\mu_0) = 0 \ \forall \ j;$ (157)

$$\aleph^{AT}_j(0) = \mu_0\gamma^{AT}_j \ b^{AT}_j \ SAT(z_\omega:\gamma_j,\mu_0); \qquad (158)$$

$$\Re^{AT}_j(z_\omega) = \mu_0\gamma^{AT}_j a^{AT}_j CAT(z_\omega:\gamma_j,\mu_0); \qquad (159)$$

$$\aleph^{AT}_j(z_\omega) = \mu_0\gamma^{AT}_j \ b^{AT}_j \ SAT(0:\gamma_j,\mu_0) = 0 \ \forall \ j. \qquad (160)$$

We can now write down the **complete solution** in the following form.

$I_{AT+}(z) = RE^{AT}_+(z) + CO^{AT}_+(z) + I^p_{AT}(+;z);$ (161 $\quad I_{AT-}(z) = RE^{AT}_-(z) + CO^{AT}_-(z) + I^p_{AT}(-;z);$ (162)

$I_{OC+}(z) = RE^{OC}_+(z) + CO^{OC}_+(z) + I^p_{OC}(+;z);$ (163) $\quad I_{OC-}(z) = RE^{OC}_-(z) + CO^{OC}_-(z) + I^p_{OC}(-;z);$

(164)

In the next section we shall use these complete solutions for discrete directional quadratures in the boundary and inter face conditions to get a set of four linear algebraic equations, solutions of which give the unknown coefficients. However we need to find solutions for any desired angle.

VI. Evaluation of the Integral

7.1: In equation (218) there are three integrals to be evaluated to get exact expressions for the emergent intensity. Bellow as an example we have calculated the integral present in equation (214) only. This integral contains seven separate integrals over known functions. In order to save space we have omitted the deduction of all the integrals (21 integrals) in (218). However one can easily complete the relevant calculations using the following simple straight forward procedures.

Using expressions (161) in the second term of (214) we can establish the following expressions consisting of seven expressions involving integrations over known integrands suppressing the superscript ‘At” and ‘Oc’ in mu

$$\int_{z(at)}^{z_\omega} RSAT(x,\mu)\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}$$

$$= \sum_{J=S}^{M}\mathbf{P}^S_J(\mu)\mathbf{B}^{AT}_J\left[\int_{z(at)}^{z_\omega}\mathbf{\Pi}(J,S)^T\frac{\omega^{AT}}{2}\mathbf{\Sigma}\left\{RE^{AT}_+(x)\right\}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu} + \right.$$

$$\int_{z(at)}^{z_\omega}\mathbf{\Pi}(J,S)^T\frac{\omega^{AT}}{2}\mathbf{\Sigma}\left\{CO^{AT}_+(x)\right\}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu} +$$

$$\int_{z(at)}^{z_\omega}\mathbf{\Pi}(J,S)^T\mathbf{\Sigma}\left\{I^P_{AT}(+;x)\right\}\frac{\omega^{AT}}{2}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}$$

$$+\int_{z(at)}^{z_\omega}(-1)^{J-S}D\mathbf{\Pi}(J,S)^T\mathbf{\Sigma}\left\{RE^{AT}_-(x)\right\}\frac{\omega^{AT}}{2}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu} +$$

$$\int_{z(at)}^{z_\omega}(-1)^{J-S}D\mathbf{\Pi}(J,S)^T\mathbf{\Sigma}\left\{CO^{AT}_-(x)\right\}\frac{\omega^{AT}}{2}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu} +$$

$$\int_{z(at)}^{z_\omega} (-1)^{J-S} \mathbf{D}\boldsymbol{\Pi}(J,S)^T \boldsymbol{\Sigma}\left\{I_{AT}^{p}(-;x)\right\}\frac{\omega^{AT}}{2}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}\Bigg]$$

$$+\left[\int_{z(at)}^{z_\omega} S_{AT}(x,\mu)\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}\right]. \quad (219$$

Where we have defined earlier $\boldsymbol{\Pi}(J,S) = \left[\mathbf{P}_J^S(\mu_1), \mathbf{P}_J^S(\mu_2), \mathbf{P}_J^S(\mu_3), \ldots, \mathbf{P}_J^S(\mu_N)\right]^T$

$$\boldsymbol{\Sigma} = \mathrm{diag}(\omega_1\boldsymbol{\Gamma}, \omega_2\boldsymbol{\Gamma}, \omega_3\boldsymbol{\Gamma}, \ldots\ldots, \omega_N\boldsymbol{\Gamma})$$

7.2: Keeping **only the relevant** terms within the integral sign of the **first term** in (219) and using expression (81b) we get,

$$\int_{z(at)}^{z_\omega} \left\{RE_+^{AT}(x)\right\}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}$$

$$= \sum_{j=1}^{Nr} A_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})(\frac{1}{\mu})\exp(\frac{z(at)}{\mu})\left[-(\frac{1}{\mu}+\frac{1}{\gamma_j^{At}})\right]\exp\left[(\frac{1}{\mu}+\frac{1}{\gamma_j^{At}})(z_\omega - z(at))\right]$$

$$+\sum_{j=1}^{Nr} B_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})(\frac{1}{\mu})\exp(\frac{z(at)}{\mu})\exp(-\frac{z_\omega}{\gamma_j^{At}})\exp\left[(\frac{1}{\gamma_j^{At}}-\frac{1}{\mu})(z_\omega - z(at))\right] \quad (220)$$

.

Similar approach for the **second integral** in (219) yields using (81f)

$$\int_{z(at)}^{z_\omega} \left\{CO_+^{AT}(x)\right\}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu} = \sum_{j=1}^{Nc} A_j^{AT(1)}\,\mathrm{Re}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\mathrm{Re}(X1) -$$

$$\sum_{j=1}^{Nc} A_j^{AT(1)}\,\mathrm{Im}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\mathrm{Im}(X1) +$$

$$\sum_{j=1}^{Nc} A_j^{AT(2)}\,\mathrm{Re}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\mathrm{Im}(X1) - \sum_{j=1}^{Nc} A_j^{AT(2)}\,\mathrm{Im}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\mathrm{Re}(X1) +$$

$$\sum_{j=1}^{Nc} B_j^{AT(1)}\,\mathrm{Re}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\mathrm{Re}(X1) - \sum_{j=1}^{Nc} B_j^{AT(1)}\,\mathrm{Im}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\mathrm{Im}(X1) +$$

$$\sum_{j=1}^{Nc} B_j^{AT(2)}\,\mathrm{Re}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\mathrm{Im}(X1) - \sum_{j=1}^{Nc} B_j^{AT(2)}\,\mathrm{Im}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\mathrm{Re}(X1). \quad (221)$$

$$X1 = (\frac{1}{\mu})\left[-(\frac{1}{\mu}+\frac{1}{\gamma_j^{At}})\right]\left\{\exp\left(-(\frac{1}{\mu}+\frac{1}{\gamma_j^{AT}})(z(at)-z_\omega)\right)\exp\left(\frac{z(at)}{\gamma_j^{AT}}\right)\right\}. \quad (222)$$

7.3: The **third integral** requires equation (147) together with the pair of equations (135,136), (137,138) and (145,146) to get

$$\int_{z(at)}^{z_\omega} \left\{I_{AT}^{P}(+;x)\right\}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}$$

$$= 2\sum_{j=1}^{Nr} \mu_0\gamma_j^{AT}a_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})\left[\begin{array}{l}(\frac{1}{\mu})\exp(\frac{z(at)}{\mu})\left[-(\frac{1}{\mu}+\frac{1}{\gamma_j^{AT}})\right]\exp\left[\left[-(\frac{1}{\mu}+\frac{1}{\gamma_j^{AT}})\right](z(at)-z_\omega)\right] \\ -(\frac{1}{\mu})\exp(\frac{z(at)}{\mu})\left[-(\frac{1}{\mu}+\frac{1}{\mu_0})\right]\exp\left[\left[-(\frac{1}{\mu}+\frac{1}{\mu_0})\right](z(at)-z_\omega)\right]\end{array}\right]$$

$$+\sum_{j=1}^{N_r} \mu_0\gamma_j^{AT} b_j^{AT} \mathbf{H}_-^{AT}(\gamma_j^{AT})$$

$$\left[\begin{array}{l}\left[(\frac{1}{\mu})\left[-(\frac{1}{\mu}+\frac{1}{\mu_0})\right]\exp(\frac{z(at)}{\mu}).\exp\left[\left[-(\frac{1}{\mu}+\frac{1}{\mu_0})\right](z(at)-z_\omega)\right]\right] \\ -(\frac{1}{\mu})\left[-(\frac{1}{\mu}-\frac{1}{\gamma_j^{AT}})\right]\exp(\frac{z(at)}{\mu})\exp\left[-(\frac{1}{\mu_0}+\frac{1}{\gamma_j^{AT}})z_\omega\right]\exp\left[-(\frac{1}{\mu}-\frac{1}{\gamma_j^{AT}})(z(at)-z_\omega)\right]\end{array}\right]$$

$$+2\sum_{j=1}^{Nc} \mu_0\gamma_j^{AT} a_j^{AT} \mathbf{H}_+^{AT}(\gamma_j^{AT})\left[\begin{array}{l}(\frac{1}{\mu})\exp(\frac{z(at)}{\mu})\left[-(\frac{1}{\mu}+\frac{1}{\gamma_j^{AT}})\right]\exp\left[\left[-(\frac{1}{\mu}+\frac{1}{\gamma_j^{AT}})\right](z(at)-z_\omega)\right] \\ -(\frac{1}{\mu})\exp(\frac{z(at)}{\mu})\left[-(\frac{1}{\mu}+\frac{1}{\mu_0})\right]\exp\left[\left[-(\frac{1}{\mu}+\frac{1}{\mu_0})\right](z(at)-z_\omega)\right]\end{array}\right] +$$

$$\sum_{j=1}^{N_c} \mu_0\gamma_j^{AT} b_j^{AT} \mathbf{H}_-^{AT}(\gamma_j^{AT})$$

$$\left[\begin{array}{l}\left[(\frac{1}{\mu})\exp(\frac{z(at)}{\mu})\left[-(\frac{1}{\mu}+\frac{1}{\mu_0})\right]\exp\left[\left[-(\frac{1}{\mu}+\frac{1}{\mu_0})\right](z(at)-z_\omega)\right]\right] \\ -(\frac{1}{\mu})\exp(\frac{z(at)}{\mu})\left[-(\frac{1}{\mu}-\frac{1}{\gamma_j^{AT}})\right]\exp\left[-(\frac{1}{\mu_0}+\frac{1}{\gamma_j^{AT}})z_\omega\right]\exp\left[-(\frac{1}{\mu}-\frac{1}{\gamma_j^{AT}})(z(at)-z_\omega)\right]\end{array}\right]$$

+

$$2\sum_{j=1}^{Nc} \mu_0\gamma_j^{AT*} a_j^{AT*} \mathbf{H}_+^{AT*}(\gamma_j^{AT*})$$

$$\left[\begin{array}{l}(\frac{1}{\mu})\exp(\frac{z(at)}{\mu})\left[-(\frac{1}{\mu}+\frac{1}{\gamma_j^{AT*}})\right]\exp\left[\left[-(\frac{1}{\mu}+\frac{1}{\gamma_j^{AT*}})\right](z(at)-z_\omega)\right] \\ -(\frac{1}{\mu})\exp(\frac{z(at)}{\mu})\left[-(\frac{1}{\mu}+\frac{1}{\mu_0})\right]\exp\left[\left[-(\frac{1}{\mu}+\frac{1}{\mu_0})\right](z(at)-z_\omega)\right]\end{array}\right]$$

$$+ \quad 2\sum_{j=1}^{Nc} \mu_0\gamma_j^{AT*} b_j^{AT*} \mathbf{H}_-^{AT*}(\gamma_j^{AT*}) \qquad (223)$$

The above expression for third integral can be evaluated numerically. The third and forth term of this expression are relevant when complex conjugate pairs of eigenvalues are present in the eigenvalues spectrum.

7.4: The **forth integral** can be evaluated using equation (81e)

$$\int_{z(at)}^{z_\omega} \left\{RE_-^{AT}(x)\right\}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu} =$$

$$\mathbf{\Delta}\sum_{j=1}^{Nr} A_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})\ (\frac{1}{\mu})\exp(\frac{z(at)}{\mu})\exp(-\frac{z_\omega}{\gamma_j^{At}})\exp\left[(\frac{1}{\gamma_j^{At}}-\frac{1}{\mu})(z_\omega-z(at))\right]$$

$$+\mathbf{\Delta}\sum_{j=1}^{Nr} B_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})(\frac{1}{\mu})\exp(\frac{z(at)}{\mu})\exp(-\frac{z_\omega}{\gamma_j^{At}})\exp\left[(\frac{1}{\gamma_j^{At}}-\frac{1}{\mu})(z_\omega-z(at))\right] \qquad (224)$$

7.5: The **fifth integral** can be evaluated with the help of (81f)

$$\int_{z(at)}^{z_\omega}\left\{CO_-^{AT}(x)\right\}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}$$
$$=\sum_{j=1}^{Nc} A_j^{AT(1)}\,\mathrm{Re}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\mathrm{Re}(X2)-\sum_{j=1}^{Nc} A_j^{AT(1)}\,\mathrm{Im}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\mathrm{Im}(X2)+$$
$$\sum_{j=1}^{Nc} A_j^{AT(2)}\,\mathrm{Re}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\mathrm{Im}(X2)-\sum_{j=1}^{Nc} A_j^{AT(2)}\,\mathrm{Im}\{\mathbf{H}_-^{AT}(\gamma_j^{AT})\}\mathrm{Re}(X2)+$$
$$\sum_{j=1}^{Nc} B_j^{AT(1)}\,\mathrm{Re}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\mathrm{Re}(X2)-\sum_{j=1}^{Nc} B_j^{AT(1)}\,\mathrm{Im}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\mathrm{Im}(X2)+$$
$$\sum_{j=1}^{Nc} B_j^{AT(2)}\,\mathrm{Re}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\mathrm{Im}(X2)-\sum_{j=1}^{Nc} B_j^{AT(2)}\,\mathrm{Im}\{\mathbf{H}_+^{AT}(\gamma_j^{AT})\}\mathrm{Re}(X2).\quad(225)$$

$$X2=(\frac{1}{\mu})\left[-(\frac{1}{\mu}+\frac{1}{\gamma_j^{At}})\right]\left\{\exp\left(-(\frac{1}{\mu}+\frac{1}{\gamma_j^{AT}})(z(at)-z_\omega)\right)\exp\left(\frac{z(at)}{\gamma_j^{AT}}\right)\right\}.\quad(226)$$

7.6: The **sixth integral** again requires equation (148) to take the following form

$$\int_{z(at)}^{z_\omega}\left\{I_{AT}^{p}(-;x)\right\}\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}\Bigg]$$
$$=2\sum_{j=1}^{Nr}\mu_0\gamma_j^{AT}a_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})$$
$$\Delta\begin{bmatrix}(\frac{1}{\mu})\exp(\frac{z(at)}{\mu})\left[-(\frac{1}{\mu}+\frac{1}{\gamma_j^{AT}})\right]\exp\left[\left[-(\frac{1}{\mu}+\frac{1}{\gamma_j^{AT}})\right](z(at)-z_\omega)\right]\\-(\frac{1}{\mu})\exp(\frac{z(at)}{\mu})\left[-(\frac{1}{\mu}+\frac{1}{\mu_0})\right]\exp\left[\left[-(\frac{1}{\mu}+\frac{1}{\mu_0})\right](z(at)-z_\omega)\right]\end{bmatrix}+$$
$$2\sum_{j=1}^{Nr}\mu_0\gamma_j^{AT}b_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})$$

$$2\sum_{j=1}^{Nc}\mu_0\gamma_j^{AT}a_j^{AT}\mathbf{H}_-^{AT}(\gamma_j^{AT})$$
$$\Delta\begin{bmatrix}(\frac{1}{\mu})\exp(\frac{z(at)}{\mu})\left[-(\frac{1}{\mu}+\frac{1}{\gamma_j^{AT}})\right]\exp\left[\left[-(\frac{1}{\mu}+\frac{1}{\gamma_j^{AT}})\right](z(at)-z_\omega)\right]\\-(\frac{1}{\mu})\exp(\frac{z(at)}{\mu})\left[-(\frac{1}{\mu}+\frac{1}{\mu_0})\right]\exp\left[\left[-(\frac{1}{\mu}+\frac{1}{\mu_0})\right](z(at)-z_\omega)\right]\end{bmatrix}$$
$$+\qquad 2\sum_{j=1}^{Nc}\mu_0\gamma_j^{AT*}a_j^{AT*}\mathbf{H}_-^{AT*}(\gamma_j^{AT*})\qquad\Delta$$
$$\begin{bmatrix}(\frac{1}{\mu})\exp(\frac{z(at)}{\mu})\left[-(\frac{1}{\mu}+\frac{1}{\gamma_j^{AT*}})\right]\exp\left[\left[-(\frac{1}{\mu}+\frac{1}{\gamma_j^{AT*}})\right](z(at)-z_\omega)\right]\\-(\frac{1}{\mu})\exp(\frac{z(at)}{\mu})\left[-(\frac{1}{\mu}+\frac{1}{\mu_0})\right]\exp\left[\left[-(\frac{1}{\mu}+\frac{1}{\mu_0})\right](z(at)-z_\omega)\right]\end{bmatrix}+$$

$$2\sum_{j=1}^{Nc}\mu_0\gamma_j^{AT}b_j^{AT}\mathbf{H}_+^{AT}(\gamma_j^{AT})$$

$$\Delta\left[\begin{array}{l}\left[(\frac{1}{\mu})\left[-(\frac{1}{\mu}+\frac{1}{\mu_0})\right]\exp(\frac{z(at)}{\mu}).\exp\left[\left[-(\frac{1}{\mu}+\frac{1}{\mu_0})\right](z(at)-z_\omega)\right]\right] \\ -(\frac{1}{\mu})\left[-(\frac{1}{\mu}-\frac{1}{\gamma_j^{AT}})\right]\exp(\frac{z(at)}{\mu})\exp\left[-(\frac{1}{\mu_0}+\frac{1}{\gamma_j^{AT}})z_\omega\right]\exp\left[-(\frac{1}{\mu}-\frac{1}{\gamma_j^{AT}})(z(at)-z_\omega)\right]\end{array}\right]+$$

$$2\sum_{j=1}^{Nc}\mu_0\gamma_j^{AT*}b_j^{AT*}\mathbf{H}_+^{AT*}(\gamma_j^{AT*})$$

$$\Delta\left[\begin{array}{l}\left[(\frac{1}{\mu})\left[-(\frac{1}{\mu}+\frac{1}{\mu_0})\right]\exp(\frac{z(at)}{\mu}).\exp\left[\left[-(\frac{1}{\mu}+\frac{1}{\mu_0})\right](z(at)-z_\omega)\right]\right] \\ -(\frac{1}{\mu})\left[-(\frac{1}{\mu}-\frac{1}{\gamma_j^{AT*}})\right]\exp(\frac{z(at)}{\mu})\exp\left[-(\frac{1}{\mu_0}+\frac{1}{\gamma_j^{AT*}})z_\omega\right]\exp\left[-(\frac{1}{\mu}-\frac{1}{\gamma_j^{AT*}})(z(at)-z_\omega)\right]\end{array}\right]$$

(227)

7.7: The **last integral** is the source integral which takes the following form

$$\left[\int_{z(at)}^{z_\omega} S_{AT}(x,\mu)\exp\left(-\frac{x-z(at)}{\mu}\right)\frac{dx}{\mu}\right]$$

$$=S_{AT}(+)\frac{-\mu_0}{\mu+\mu_0}\left[\exp(-(\frac{1}{\mu}+\frac{1}{\mu_0})z_\omega-\exp(-(\frac{1}{\mu_0})z(at)\right]. \quad (228)$$

The integrals evaluated above can now be calculated numerically once we have calculated the eigenvalues correctly. We have not specified the form of the source function here. This issue will be taken up in some detail in our numerical consideration.

VII. Conclusion

We have applied analytical discrete ordinate method for the radiation transfer in homogeneous composite atmosphere-ocean system and found exact analytical expressions for emergent upward polarized radiation intensity in any direction from the top of the atmosphere. However exit polarized radiation intensity vector in any direction from the air water interfaces and from any point within atmosphere and ocean for any direction can be computed from the foregoing analysis without much effort. In this article we have not given specific mathematical expression for source matrix elements for space constraints. However this will be considered in a subsequent article where we shall report our numerical considerations.

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