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Parametric sensitivity analysis of a mathematical model of facultative mutualism

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Abstract: *The complex dynamics of facultative mutualism is best described by a system of continuous non-linear first order ordinary differential equations. The methods of 1-norm, 2-norm, and infinity-norm will be used to quantify and differentiate the different forms of the sensitivity of model parameters. These contributions will be presented and discussed.*

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I. Introduction

According to Morin (2002), competition and predation have been observed as two dominant types of interaction between species. In the literatures of mathematical ecology or mathematical biology, competition is the most recognised type of interaction while mutualism despite its relevance seems to be not frequently mentioned in most research agenda. Having conducted a detailed literature review on the topic of the dynamics of transitions between population interactions, using a non-linear α-function, we found that the application of parametric sensitivity analysis of the model parameters for this important model is rarely introduced as a method of studying the importance of the model parameters of such model. We are motivated to implement the tool of parametric sensitivity analysis because it is the first stage in the process of developing a deterministic model which describes the dynamics of two interacting populations. To the best of our knowledge, we are yet to see any reported mathematical analysis which involves the utilization of a numerical parametric sensitivity analysis. This method is expected to estimate and classify the sensitivity of model parameters which were proposed by Hernandez (1998). In the next section, we will present the mathematical formulation of the model on which our analysis id based.

II. Mathematical Formulation

Following Hernandez (1998), we consider the following complex system of two continuous nonlinear first order ordinary differential equations of the following mathematical structure:

$$\frac{dN_1}{dt} = r_1 N_1 \left[1 - \frac{N_1}{K_1} + \left(\frac{b_1 N_2 - c_1 N_2^2}{1 + d_1 N_2^2}\right)\frac{N_2}{K_1}\right]$$
$$\frac{dN_2}{dt} = r_2 N_2 \left[1 - \frac{N_2}{K_2} + \left(\frac{b_2 N_1 - c_2 N_1^2}{1 + d_2 N_1^2}\right)\frac{N_1}{K_2}\right]$$

Here, N_1 and N_2 are populations of specie 1 and specie 2 respectively, r_1 and $r_2$are growth rates of specie 1 and specie 2 respectively, K_1 and $K_2$are carrying capacities of specie 1 and specie 2 respectively.The parameters $b_1, b_2, c_1, c_2, d_1$and$d_2$modify the general shape and represent the environmentalinfluence on α_{ij}that is, the "exogenous effect".

This facultative model (Hernandez (1998)) is driven by thirteen (13) parameters. To conduct a realistic and systematic parametric sensitivity analysis of this model, we have proposed to study the sensitivity analysis of the two intrinsic growth rates (r_1 and r_2), carrying capacities (K_1 and K_2) and the initial conditions ($N_1(0)$ and $N_2(0)$) for obvious ecological or biological reasons. In the next section, we will describe the key relevant materials and methods which will enable us to solve this problem numerically.

III. Materials and Methods

Following Ford et al (2010) and Hernandez (1998), these researchers in mathematical biology have identified similar types of interactions between species such as mutualism, commensalism, parasitism or predation and competition. The method of Ford et al (2010) was based on using the biological ideas to construct a model for two competing plant species in a harsh climate whereas the model of Hernandez (1998) simply describes in great detail the undergoing transitions between different types of partially positive interaction hereby called facultative and obligate mutualism which is consistent with the classification of Morin (2002, 187 – 189).

In our recent literature review, the current state of the spread of knowledge clearly shows that the work of Hernandez (1998) has been popularly cited by 32 other researchers. However, none of these researchers has attempted to study the parametric sensitivity analysis of the facultative model.Hernandez (1998) considered an interaction where both α-functions are of the α(+, -) type and reported that at low densities, the association is mutualistic but at higher densities the same association was considered to be competitive. It was also reported in the same paper that depending on the threshold value where α_{12} and α_{21} are equal to zero, immediate density values can be experimentally determined for a situation where a parasitic interaction can also occur. Part of developing this model involves assuming that both species can exist either alone or in association. The approximate model formulation was constructed on the basis of these simplifying assumptions. The method of sensitivity analysis which we have used in this study has been adapted from recent research reports of Ekaka-a (2009), Ekaka-a et al (2012) and Nwachukwu and Ekaka-a (2013). A brief sketch of this method is as follows:

STEP I: Code the given system of continuous non-linear first order ordinary differential equation in a Matlab programing language.

STEP II: Modify and code a similar program which is used for a variation of a single parameter one-at-a-time while other model parameters are fixed.

STEP III: Design an appropriate ODE45 Runge-Kutta scheme which will simulate the program in step I and step II

STEP IV: Use the program in step III to calculate the 1-norm, 2-norm, and infinity norm of three solution trajectories in the same manner, use the same program to calculate the three popular norms of the differences of the solution trajectories.

STEP V: Based on the original parameter in the step I, calculate the cumulative percentage effect on the solution trajectories due to variation of each chosen parameter at a time when other parameters are fixed.

STEP VI: Interprete the result quantitatively. That is, the parameter which when varied a little and produces the biggest cumulative effect on the solution trajectories is called a most sensitive.

IV. Results andDiscussion

In this section, we will present and discuss our results which we have achieved in this study. In this section, the notation CV stands for the popular statistical coefficient of variation.

Table 1 Percentage variations of r_1

variations / mn	**0.015**	**0.03**	**0.045**	**0.06**	**0.075**
1-norm	10.5213	5.3085	3.5391	2.6518	2.1221
2-norm	10.9065	7.5868	6.0936	5.1905	4.5654
∞-norm	32.5020	30.7169	29.1662	27.6902	26.5351

Table 2 Percentage variations of r_2

variations / mn	**0.015**	**0.030**	**0.045**	**0.06**	**0.075**
1-norm	18.5029	9.1881	6.0497	4.4794	3.5403
2-norm	22.8126	15.5895	12.3174	10.3293	8.9562
∞-norm	98.7772	88.7117	81.2441	74.9255	69.5011

Table 3 Percentage variations of K_1

variations / mn	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**
1-norm	65.2987	64.9995	64.6973	64.3887	64.0780
2-norm	47.5824	47.3460	47.1080	46.8654	46.6221
∞-norm	54.2792	54.0124	53.7196	53.4540	53.1737

Table 4 Percentage variations of K_2

variations / mn	0.01	0.02	0.03	0.04	0.05
1-norm	65.4713	65.1745	64.8735	64.5666	64.2571
2-norm	47.5803	47.3472	47.1114	46.8715	46.6304
∞-norm	62.0988	61.9169	61.6954	61.4953	61.2576

Table 5 Percentage variations of IC_1

variations / mn	0.04	0.08	0.12	0.16	0.20
1-norm	0.5121	0.5079	0.5028	0.5022	0.5013
2-norm	5.5154	5.4601	5.4051	5.3500	5.2949
∞-norm	74.1298	73.5982	73.0003	72.3749	71.7193

Table 6 Percentage variations of IC_2

variations / mn	0.1	0.2	0.3	0.4	0.5
1-norm	1.3768	1.3709	1.3681	1.3499	1.3443
2-norm	13.8051	13.6737	13.5407	13.4063	13.2710
∞-norm	117.6685	116.9752	116.1566	115.2819	114.3127

Table 7 results of coefficient of variation for model parameters

parameter r_1	CV
1-norm	0.9640
2-norm	0.4310
∞-norm	0.0838

parameter K_1	CV
1-norm	0.0075
2-norm	0.0081
∞-norm	0.0082

parameter IC_1	CV
1-norm	0.0091
2-norm	0.0162
∞-norm	0.0132

parameter r_2	CV
1-norm	1.0038
2-norm	0.4678
∞-norm	0.1481

parameter K_2	CV
1-norm	0.0074
2-norm	0.0080
∞-norm	0.0054

parameter IC_2	CV
1-norm	0.0104
2-norm	0.0157
∞-norm	0.0115

V. Discussion of results

From our results which have been presented in the previous section, we observe that the two carrying capacities can be classified as the most sensitive parameters using the 1-norm and 2-norm estimated sensitivity values while the two initial conditions can be classified as relatively least sensitive parameters. However, the ∞-norm estimated value shows that the second initial condition is clearly a more sensitive parameter than any of the other five model parameters. In the context of measuring the best estimate of sensitivity, we can clearly see that the infinity norm is the best estimate for measuring the sensitivity of model parameters such as r_1, r_2, and K_2 while the 1-norm is the best estimate for measuring the sensitivity of model parameters such as K_1 and the initial conditions.

VI. Conclusion

In terms of further parameter estimation to minimize uncertainty in model prediction, carrying capacities and the growth rates should be targeted for the purpose of model re-validation. In this study, the initial conditions in the main should be considered as rough estimates. In our further research, we propose to study the sensitivity of other seven (7) parameters which we have not considered in this present study.

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