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ON $M_{(M,m)}$/M/C/N: Interdependent Queueing Model

R. John Mathew[1], Varaprasad B Sabbithi[2] and J. Lakshinarayana[3]
[1]*Department of Computer Science and Engineering, Srinivasa Engineering College, Amalapuram, India.*
[2]*Department of Computer Science and Engineering, Srinivasa Engineering College, Amalapuram, India.*
[3]*Department of Statistics, Andhra University, Visakhapatnam, india.*

Abstract: *This paper deals with a multiple server Queueing system in which arrivals and services are intredependent and follow a bivariate Poisson process and having startup delay. Using the Supplementary Varible Techniue these models are analyzed. The expected length of the Dorment period, the busy period, expected number of units in the queue are derived and analyzed in the light of the dependence paramater.*
Keywords: *interdependent queue, bivariate Poisson process, the length of the Dorment period, the busy period, expected number of units in the queue*

I. Introduction

Intermittent operation of some production lines is common in factories that manufacture numerous different products for inventory. For such a product, if the finished goods in inventory has dropped sufficiently low, then workers are transffered form there activities to manufacture the goods. Production continues until the inventory is sufficiently high, at which time, the line is shutdown and the workers are allocated to other activities. For these sort of production lines Sobel M.J.[10] and Simha P.S.[9] have developed and analyzed the queueing models known as startup delay and shutdown queueingmodels and denoted them as M_n/M/C models. Inorder to have much more closure approximation for these sort of situations it is reasonable to assume that the arrival and service processes are correlated. Aqueueing model in which arrivals and services are correlated is known as interdependent queueing model (U.N.Bhat[2]).

Much work has been currently reported in literature regarding interdependent queueing models conolly&Hadidi[3],MathewR.J.[4],RaoK.S.[8],Aftabbegum[1],Prasad Reddy[7],Mishra.S.S[6],Maurya. V.N [5]. However very Little work has been reported regarding the startup delay and shtdown queueing models with interdependence, which are much useful in analyzing the situations arising at computer communications systems, neurophysioloical problems, Transportation systems, poduction processes etc., where the arrivals and service processes are to be made interdependent inorder to have optimal operation policies. In this paper an attempt is made to develop and analyze a $M_{\mathbf{M,m}}$/M/C/N queueing model with interdependence, which is more appropriate in approximating the production process much close to reality. Here we assume that there are 'C' servers in the sysstem, each work independent to the other. The arrivals of the system are from a finite source having capacity N. Also assume that the arrivels and services are interependent and follows a Bivariate Poisson process of the form given by Teicher [11]. We further assume that the server is made idle when the number of customers in the system falls to m less than n=M.This model is known as the finite source interdependent Poisson queueing modelwith (M, m) policy. Using the supplementary variable technique the system characters like, the average length of the dormant period, the busy period etc., are derived and analyzed.

II. Notation

M(t)	:	The number of units at time t at the service facility.
I	:	Length of the dormant period.
E(I)	:	Expected length of the dormant period.
$M^I(t)$	:	The number of units waiting at time t at the service facility during dormant period.
$P^I(m,t)$	:	Prob$\{m^I(t)=M, m^I(t_1)< n \forall t_{1,} (0)<t1<t)/ m_1^I(o) =0\}$.
α (t) dt	:	Prob$\{t<I<t+dt/m^I(o)=0\}$
$P^I(m)$	:	Probability that there are m units during the general process, conditional on the system being in the dormant period.
P_I	:	The probalitity that the service facility is idle under steady state.
$M^B t$	:	The number of units that are waiting of being serviced at time t during busy period.
B	:	The lingth of the busy period.
E(B)	:	The expected length of the busy period.
β (t)dt	:	Prob$\{t<b<t+dt\}$

On M_n/M/C/N	:	Interdependent queueing Model
$P^B(m,t)$	:	$Prob(m^B(t)=m, m^B(t_1)>0\ \forall t_1 (0<t_1<t)\ \vert m^B(0)=n\}$
P(m)	:	The steady state probability that there are m units in the system
E(m)	:	The expected number of customers in the system that are waiting or being serviced at the service facility.
$\overline{P}^B(m)$	:	$\lim_{s\to 0} \overline{P}^B(m,s)$

III. $M_{M,m}$ /M/C/N Interdependent Queueing Model

Consider a multiple server finitet capacity queueing system having 'C' servers, each server independent of the other. Also assume that the source size is finite, say, 'N', and the idleness of the server will be interrupeted only when there are n (n≥C) customers at the service facility, the server becomes idle onlly when the system is empty. We further assume that the arrivals and service completion are interdependent and follow a Bivariate Poission Process. Since the calling population is finite,the mean arrival rate is (N-m)λm the mean service rate is Cμ when m≥ C and m μ when m <C, the mean dependence rate is Cε when m ≥ C and mε when m < C. Following the heuristic arguments given by P.S. Simha [9] and K.S.Rao [4], we have,
The Probability that the system being in dormant Period is

$$p^I(n+m,t) = \binom{N-m}{n}\left[1-e^{-\lambda t}\right]^n e^{(N-m-n)\lambda t} \qquad (M-m)\geq n\geq 0 \qquad (3.1)$$

The expected length of the dormant Period is

$$E(I)= \sum_{n=0}^{M-m-1} \frac{1}{(N-m-n)\lambda} \qquad (3.2)$$

The difference-differential equations satisfying $P^B(m,t)$ for vrious values of m are

$$\frac{d}{dt} p^B(n+m,t) = -\left[(n+m)(\mu-\varepsilon)+(N-m-n)[\lambda-(m+n)\varepsilon]\right]P^B(n+m,t)+(n+m+1)(\mu-\varepsilon)P^B(n+m+1,t)$$

$$+(1-\delta_{n,1})(n-m-n+1)[\lambda-(m+n-1)\varepsilon]P^B(m+n-1,t) \quad , (c-m) > n \geq 1 \qquad (3.3)$$

And

$$\frac{d}{dt} p^B(n+m,t) = -\left[c(\mu-\varepsilon)+(N-m-n)(\mu-\varepsilon)\right]P^B(n+m,t)+c(\mu-\varepsilon)P^B(n+m+1,t)$$

$$+(N-m-n+1)(\lambda-c\varepsilon)P^B(m+n-1,t) \quad , (N-m) > n > (c-m) \qquad (3.4)$$

For solving (3.3) and (3.4) for various P^B (m,t) consider the Laplace transformations $\overline{P}^B$ (m,s) of P^B(m,t).
Then equations (3.3) and (3.4) become

$$\overline{\beta}^B(s) = (m+1)\ (\mu-\varepsilon)\ \overline{P}^B\ (m+1, s)$$
(3.5)

$$\frac{(m+1)(\mu-\varepsilon)\overline{p}^B(m+1,s)}{s} + \sum_{i=m+1}^{N} \overline{p}^B(i,s) = \frac{1}{s} \qquad (3.6)$$

At s=0, the Laplace transformations of the equations (3.5) and (3.6) become

$$(n+m)\ (\mu-\varepsilon)\ \overline{P}^B\ (n+m)\ -\ (N-m-n+1)(\lambda-(n+m-1)c)\ \overline{P}^B\ (n-m-1)-1 = 0 \quad , (c-m)>m>1 \qquad (3.7)$$

$$c\ (\mu-\varepsilon)\ \overline{P}^B\ (n+m)\ -\ (N-m-n+1)(\lambda-\varepsilon c)\ \overline{P}^B\ (n+m-1)-1 = 0 \quad , (M-m)>n>c \qquad (3.8)$$

$$c(\mu-\varepsilon)\ \overline{P}^B\ (n+m)\ -\ (N-m-n+1)(\lambda-(n+m-1)c)\ \overline{P}^B\ (n+m-1) = 0 \quad , (N-m)>n>(M-m) \qquad (3.9)$$

Finally, we get the Expected length of the dormant period is

$$E(B)=\frac{1}{(\mu-\varepsilon)}\sum_{i=m+1}^{C-1}\phi_1\left[\frac{\lambda-i\varepsilon}{\mu-\varepsilon},i\right]+\frac{1}{C(\mu-\varepsilon)}\sum_{i=N-M}^{N-C-1}F\left[\frac{\lambda-c\varepsilon}{c(\mu-\varepsilon)},i\right]+F\left[\frac{\lambda-\varepsilon}{c(\mu-\varepsilon)},N-c\right]\frac{\phi_1\left[\frac{\lambda-c\varepsilon}{\mu-\varepsilon},c\right]}{\mu-\varepsilon}$$

where $\phi_1(x,m+1)=\frac{1}{(m+1)}$ and $\phi_1(x,n)=\frac{1}{n}+\frac{N-n+1}{n}\phi_1(x,n-1)$ (3.10)

Hence the length of thet renewal period is

$$E(T)=\sum_{n=0}^{M-m-1}\frac{1}{(N-m-n)[\lambda-(n+m)\varepsilon]}+\frac{1}{(\mu-\varepsilon)}\sum_{i=m+1}^{C-1}\phi_1\left[\frac{\lambda-i\varepsilon}{\mu-\varepsilon},i\right]+\frac{1}{C(\mu-\varepsilon)}\sum_{i=N-M}^{N-C-1}F\left[\frac{\lambda-i\varepsilon}{c(\mu-\varepsilon)},i\right]$$

$$+\frac{\phi_1\left[\frac{\lambda-c\varepsilon}{\mu-\varepsilon},c\right]F\left[\frac{\lambda-c\varepsilon}{c(\mu-\varepsilon)},N-c\right]}{\mu-\varepsilon} \qquad (3.11)$$

The expected number of units at the service facility E(m) is

$$E(\hat{m})=\sum_{n=1}^{M-n-1}(n+m)\hat{P}^{I}(n+m)+\sum_{n=1}^{N-m}(n+m)\hat{P}^{B}(n+m)$$

$$=mP_I+\frac{1}{E(T)}\left[\sum_{n=0}^{M-m-1}\frac{1}{(N-m-n)[\lambda-(n+m)\varepsilon]}+\frac{1}{(\mu-\varepsilon)}\sum_{i=m+1}^{C-1}i\phi_1\left[\frac{\lambda-i\varepsilon}{\mu-\varepsilon},i\right]\right.$$

$$+\frac{\phi_1\left[\frac{\lambda-c\varepsilon}{\mu-\varepsilon},c\right]}{\mu-\varepsilon}\left\{(c-1)F\left(\frac{\lambda-c\varepsilon}{c(\mu-\varepsilon)},N-c\right)+f\left(\frac{\lambda-c\varepsilon}{c(\mu-\varepsilon)},N-c\right)\right\}$$

$$\left.+\frac{1}{c(\mu-\varepsilon)}\sum_{i=1}^{M-1}\left\{iF\left(\frac{\lambda-c\varepsilon}{c(\mu-\varepsilon)},N-i-1\right)+f\left(\frac{\lambda-c\varepsilon}{c(\mu-\varepsilon)},N-i-1\right)\right\}\right) \qquad (3.12)$$

IV. $M_{M,m}$/M/C/∞ Interdependent Queueing Model

In this section, along with all other assumptions made in section 3, we assume that the system source size is infinite. This model can also be viewed as a limiting case of the earlier model.The expected length of the dormant period consists of n(n≥ C) independently identically distributed interarrival times during which all servers are idle i.e.there is no possiblity of service completion during the dormant period. Since the marginal desity of the interarrival times are exponential with parameter λ and the dormant period is the sum of n interarrival times during which there is no service completion, and is not influenced by the dependence between arrival and service processses and hence the average length of the dormant period for this model is

$E(I)=\frac{M-m}{(\lambda-\varepsilon}$ (4.1)

The differential equations of the model are

$$\frac{d}{dt}p^B(n+m,t)=-\left[(n+m)(\mu-\varepsilon)+[\lambda-(m+n)\varepsilon]\right]P^B(n+m,t)+(n+m+1)(\mu-\varepsilon)P^B(n+m+1,t)$$

$$+(1-\delta_{n,1})[\lambda-(m+n)\varepsilon]P^B(m+n-1,t) \quad ,\ (c-m)>n\geq 1 \qquad (4.2)$$

And

$$\frac{d}{dt}p^B(n+m,t)=-\left[c(\mu-\varepsilon)+(N-m-n)(\mu-\varepsilon)\right]P^B(n+m,t)+c(\mu-\varepsilon)P^B(n+m+1,t)+(\lambda-c\varepsilon)P^B(m+n-1,t)$$

$, (N-m) > n > (c-m)$ (4.3)

Solving the above differential equation, the expected length of the busy period E(B) is

$$E(B)=\frac{M-c}{c(\mu-\varepsilon)-(\lambda-c\varepsilon)}+\frac{1}{(\mu-\varepsilon)}\sum_{i=m+1}^{c-1}\phi_1\left(\frac{\lambda-i\varepsilon}{\mu-\varepsilon},i\right)+\frac{1}{(\mu-\varepsilon)(1-\rho)}\phi\left(\frac{\lambda-c\varepsilon}{\mu-\varepsilon},c\right) \tag{4.4}$$

The length of the renewal period is

$$E(T)=\frac{M-m}{\lambda-c\varepsilon}+\frac{M-c}{c(\mu-\varepsilon)-(\lambda-c\varepsilon)}+\frac{1}{(\mu-\varepsilon)}\sum_{i=m+1}^{c-1}\phi_1\left(\frac{\lambda-i\varepsilon}{\mu-\varepsilon},i\right)+\frac{1}{(\mu-\varepsilon)(1-\rho)}\phi_1\left(\frac{\lambda-c\varepsilon}{\mu-\varepsilon},c\right) \tag{4.5}$$

And

$$P_I=\frac{\left[\dfrac{M-n}{\lambda-c\varepsilon}\right]}{\left[\dfrac{M-m}{\lambda-c\varepsilon}+\dfrac{M=c}{c(\mu-\varepsilon-(\lambda-c\varepsilon)}+\dfrac{1}{(\mu-\varepsilon)}\sum_{i=m+1}^{c-1}\phi_1\left[\dfrac{\lambda-i\varepsilon}{\mu-\varepsilon},i\right]+\dfrac{1}{(\mu-\varepsilon)(1-\rho)}\phi_1\left[\dfrac{\lambda-c\varepsilon}{\mu-\varepsilon},c\right]\right]} \tag{4.6}$$

The expected number of units at the service facility for this model is

$$\begin{aligned}E(m)&=\sum_{n=1}^{M-m-1}(n+m)\,\hat{p}^{\,I}(n+m)+\sum_{n=1}^{\infty}(n+m)\,\hat{p}^{\,B}(n+m)\\&=\frac{1}{E(T)}\left[\frac{(M-m)(M+m-1)}{2(\lambda-c\varepsilon)}+\frac{1}{(\mu-\varepsilon)}\sum_{i=m+1}^{c-1}i\phi_1\left[\frac{\lambda-i\varepsilon}{\mu-\varepsilon},i\right]\right.\\&\left.+\frac{(M-c)(M-c+1)\rho}{2(\lambda-c\varepsilon)(1-\rho)}+\frac{(M-c)\rho^2}{(\lambda-c\varepsilon)(1-\rho)^2}+\frac{1}{(\mu-\varepsilon)}\left\{\frac{c\rho}{1-\rho}+\frac{\rho^2}{(1-\rho)^2}\right\}\phi_1\left(\frac{\lambda-c\varepsilon}{\mu-\varepsilon},c\right)\right]\end{aligned} \tag{4.7}$$

V. Conclusions

The busy period is heavily influenced by the dependence parameter and hence by increasing the dependence between arrival and service process, the avarage busy period of the system can be decreased. This feature is very optimal with respect to the production processes because the servers can be utilised on some other secondary job. This also influences the startup number which is of prime concern for taking any policy decisions, when one is concerned with startup cost. The startup number can be increased by increasing the dependence parameter for the same queue length. These models include the earlier queueing models, namely models with independent arrival and service processes as particular case when the dependent parameter ε(the covariance between arrival and service processes) tends to zero.

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