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A Convergence Theorem Associated With a Pair of Second Order Differential Equations

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Abstract: *We consider the second order matrix differential equation*

$$(M+\lambda)\Phi = 0,\ 0 \le x < \infty .$$

Where M is a second-order matrix differential operator and Φ is a vector having two components. In this paper we prove a convergence theorem for the vector function $f(x)=\begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}$ which is continuous in $0 \le x < \infty$ and of bounded variation in $0 \le x < \infty$, when $p(x)$ and $q(x)$ tend to $-\infty$ as x tend to $+\infty$.
Key Words: *Matrix differential operator, convergence theorem, bounded variation.*

§1. Let M denote the matrix operator

$$M = \begin{bmatrix} \dfrac{d^2}{dx^2} - p(x) & r(x) \\ r(x) & \dfrac{d^2}{dx^2} - q(x) \end{bmatrix} \tag{1.1}$$

and $\Phi = \Phi(x)$ a vector having two components $u = u(x)$ and $v = v(x)$ represented as a column matrix

$$\Phi = \begin{bmatrix} u \\ v \end{bmatrix}.$$

Consider the homogenous system

$$(M+\lambda)\Phi = 0,\quad 0 \le x < \infty . \tag{1.2}$$

where λ is a parameter, real or complex.
We assume the following conditions to be satisfied:

(i) $p(x), q(x)$ tend to $-\infty$ as x tend to $+\infty$.

(ii) $p'(x) \le 0, q'(x) \le 0, p''(x) \le 0, q''(x) \le 0$.

(iii) $p'(x) = \circ\left[\left(p(x)\right)^{c}\right],\ 0 < c < \dfrac{3}{2}$.

(iv) $q'(x) = \circ\left[\left(q(x)\right)^{c_1}\right],\ 0 < c_1 < \dfrac{3}{2}$.

(v) $r(x)$ is bounded or $r(x) = \circ\left[\left(p(x)q(x)\right)^{d}\right],\ 0 < d < \dfrac{1}{4}$.

(vi) $\int_0^\infty \left(p(x)\right)^{-\frac{1}{2}} dx$ and $\int_0^\infty \left(q(x)\right)^{-\frac{1}{2}} dx$ are divergent.

(vii) $\int_0^\infty \left(p(x)\right)^{-\frac{1}{2}} dx \,\Box\, \int_0^\infty \left(q(x)\right)^{-\frac{1}{2}} dx$ is convergent as $x \to \infty$.

Following Bhagat [2], the bilinear concomitant $[\Phi\theta]$ of two vectors

$$\Phi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \text{ and } \theta = \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix}$$

is defined by

$$[\Phi\theta] = \varphi_1'\vartheta_1 - \varphi_1\vartheta_1' + \varphi_2'\vartheta_2 - \varphi_2\vartheta_2' .$$

If Φ and θ are any two solutions of the system (1.2) for the same value of λ, then $\left[\Phi\theta\right]$ is a function of λ alone. It is an integral function of λ, real for real λ (see Bhagat [1]).

Let $\varphi_j(x,\lambda) \equiv \varphi_j(0/x;\lambda) = \begin{bmatrix} u_j(0/x;\lambda) \\ v_j(0/x;\lambda) \end{bmatrix}$, $(j=1,2)$ be the boundary condition vectors at $x=0$ by

$$\left.\begin{matrix} u_j(0/x;\lambda) = a_{j2} & u_j'(0/x;\lambda) = -a_{j1} \\ v_j(0/x;\lambda) = a_{j4} & v_j'(0/x;\lambda) = -a_{j3} \end{matrix}\right\}, \quad (j=1,2),$$

so that the boundary conditions to be satisfied by any solution $\varphi(x,\lambda) = \begin{bmatrix} u(x,\lambda) \\ v(x,\lambda) \end{bmatrix}$ of (1.2) at $x=0$ are given by

$$\left[\varphi(x,\lambda)\varphi_j(x,\lambda)\right] = 0, \quad (j=1,2) \tag{1.3}$$

and

$$\left[\varphi_1 \ \ \varphi_2\right] = 0 \tag{1.4}$$

The vectors $\vartheta_k(x,\lambda) \equiv \vartheta_k(0/x;\lambda) = \begin{bmatrix} x_k(0/x;\lambda) \\ y_k(0/x;\lambda) \end{bmatrix}$, $(k=1,2)$ which take real constant values (independent of λ) at $x=0$ are defined by the relations

$$\left[\varphi_j \ \ \vartheta_k\right] = \delta_{jk}, \ \left[\vartheta_1 \ \ \vartheta_2\right] = 0 \qquad (1 \le j,k \le 2) \tag{1.5}$$

§2. Green Matrix:

The Green matrix $G(x,y;\lambda) = \begin{bmatrix} G_{11} & G_{21} \\ G_{12} & G_{22} \end{bmatrix}$ for the system (1.2) is given by

$$G_1(x,y;\lambda) = \begin{bmatrix} \psi_{11}(x,\lambda) & \psi_{21}(x,\lambda) \\ \psi_{12}(x,\lambda) & \psi_{22}(x,\lambda) \end{bmatrix} \begin{bmatrix} u_1(y,\lambda) & v_1(y,\lambda) \\ u_2(y,\lambda) & v_2(y,\lambda) \end{bmatrix}; \ y \in [0,x)$$

$$= \begin{bmatrix} u_1(x,\lambda) & v_1(x,\lambda) \\ u_2(x,\lambda) & v_2(x,\lambda) \end{bmatrix} \begin{bmatrix} \psi_{11}(y,\lambda) & \psi_{21}(y,\lambda) \\ \psi_{12}(y,\lambda) & \psi_{22}(y,\lambda) \end{bmatrix}; \ y \in (x,\infty)$$

We shall use the notations and results of Bhagat [3] and Pandey and Kumar [6]. The method of Titchmarsh [8] will be used to obtain the results analogous to [7].

§3. Let $A_j(x,\lambda) = \begin{bmatrix} S_j(x,\lambda) \\ T_j(x,\lambda) \end{bmatrix}$, $(j=1,2)$. It can be verified following Titehmarsh [8,§ 5.4] that $A_j(x,\lambda)$ satisfies the system of integral equations.

$$\left.\begin{aligned} S_j(x,\lambda) &= S_j(0)\cos w(x) + \frac{1}{\mu}S_j'(0)\sin w(x) - \int_0^x [P(t)S_j(t,\lambda) + R(t)T_j(t,\lambda)]\sin\left(w(x)-w(t)\right)dt \\ T_j(x,\lambda) &= T_j(0)\cos z(x) + \frac{1}{\mu}T_j'(0)\sin z(x) - \int_0^x [Q(t)T_j(t,\lambda) + R(t)S_j(t,\lambda)]\sin\left(z(x)-z(t)\right)dt \end{aligned}\right\}, (j=1,2) \tag{3.1}$$

When $\lambda = \mu^2$, then

$$S_j(x,\lambda) = \left(\lambda - p(x)\right)^{\frac{1}{4}} u_j(x,\lambda), \quad (j=1,2) \tag{3.2}$$

$$T_j(x,\lambda) = \left(\lambda - q(x)\right)^{\frac{1}{4}} v_j(x,\lambda), \quad (j=1,2) \tag{3.3}$$

$$w(x) = \int_0^x \left(\lambda - p(t)\right)^{\frac{1}{2}} dt \tag{3.4}$$

$$z(x) = \int_0^x \left(\lambda - q(t)\right)^{\frac{1}{2}} dt \tag{3.5}$$

$$P(x) = \frac{1}{4}\frac{p''(x)}{\left(\lambda - p(x)\right)^{\frac{3}{2}}} + \frac{5}{16}\frac{\left(p'(x)\right)^2}{\left(\lambda - p(x)\right)^{\frac{5}{2}}} \tag{3.6}$$

$$Q(x) = \frac{1}{4}\frac{q''(x)}{\left(\lambda - q(x)\right)^{\frac{3}{2}}} + \frac{5}{16}\frac{\left(q'(x)\right)^2}{\left(\lambda - q(x)\right)^{\frac{5}{2}}} \tag{3.7}$$

$$R(x) = \frac{r(x)}{(\lambda - p(x))^{\frac{1}{4}}(\lambda - q(x))^{\frac{1}{4}}} \tag{3.8}$$

We assume that $p(x)$ and $q(x)$ are bounded for all finite x and $p(0) = q(0) = 0$. So for a fixed x and large $|\lambda|$, we have from (3.2) – (3.8)

$$S_j(0) = (\lambda)^{\frac{1}{4}} u_j(0), \quad (j = 1,2) \tag{3.9}$$

$$T_j(0) = (\lambda)^{\frac{1}{4}} v_j(0), \quad (j = 1,2) \tag{3.10}$$

$$S_j'(0) = (\lambda)^{\frac{1}{4}} u_j'(0) + \circ\left(|\lambda|^{-\frac{3}{4}}\right), \quad (j = 1,2) \tag{3.11}$$

$$T_j'(0) = (\lambda)^{\frac{1}{4}} v_j'(0) + \circ\left(|\lambda|^{-\frac{3}{4}}\right), \quad (j = 1,2) \tag{3.12}$$

$$w(x) = \lambda^{\frac{1}{2}} + \circ\left(|\lambda|^{-\frac{1}{2}}\right) \tag{3.13}$$

$$z(x) = \lambda^{\frac{1}{2}} + \circ\left(|\lambda|^{-\frac{1}{2}}\right) \tag{3.14}$$

$$(\lambda - p(x))^{\frac{1}{4}} = (\lambda)^{\frac{1}{4}} + \circ\left(|\lambda|^{-\frac{3}{4}}\right) \tag{3.15}$$

$$(\lambda - q(x))^{\frac{1}{4}} = (\lambda)^{\frac{1}{4}} + \circ\left(|\lambda|^{-\frac{3}{4}}\right) \tag{3.16}$$

$$P(x) = \circ\left(|\lambda|^{-\frac{3}{4}}\right) \tag{3.17}$$

$$Q(x) = \circ\left(|\lambda|^{-\frac{3}{4}}\right) \tag{3.18}$$

$$R(x) = \circ\left(|\lambda|^{-\frac{1}{2}}\right) \tag{3.19}$$

Let $\mu = s + it, \ t > 0$. Therefore,

$$\left.\begin{aligned} S_j(x,\lambda) &= H_{j1} \cdot e^{t \cdot x} \\ T_j(x,\lambda) &= H_{j2} \cdot e^{t \cdot x} \end{aligned}\right\}, \quad (j = 1,2) \tag{3.20}$$

Therefore, from (3.1), we have

$$\left.\begin{aligned} H_{j1}(x,\lambda) &= \left[S_j(0)\cos w(x) + \frac{1}{\mu} S_j'(0)\sin w(x)\right] \cdot e^{-tx} - \\ &- \int_0^x e^{-t(x-y)}[P(y)H_{j1}(y,\lambda) + R(y)H_{j2}(y,\lambda)]\sin\left(w(x) - w(y)\right)dy \\ H_{j2}(x,\lambda) &= \left[T_j(0)\cos z(x) + \frac{1}{\mu} T_j'(0)\sin z(x)\right] \cdot e^{-tx} - \\ &- \int_0^x e^{-t(x-y)}[Q(y)H_{j2}(y,\lambda) + R(y)H_{j1}(y,\lambda)]\sin\left(z(x) - z(y)\right)dy \end{aligned}\right\}, (j = 1,2) \tag{3.21}$$

Let

$$\left.\begin{aligned} M &= \max\left[S_j(0), T_j(0), S_j'(0), T_j'(0)\right] \\ N(y) &= \max\left[|P(y)|, |Q(y)|, |R(y)|\right] \end{aligned}\right\} \tag{3.22}$$

Now we have

$$\left.\begin{aligned} &|\cos w(x)|, |\sin w(x)| \le e^{tx} \\ \text{and} \\ &|\sin z(x)|, |\cos z(x)| \le e^{tx} \end{aligned}\right\} \text{for large } |\lambda| \tag{3.23}$$

Therefore, using (3.22) and (3.23), (3.21) gives

$$H_{j1}(x,\lambda), H_{j2}(x,\lambda) \le M\left(1+\frac{1}{|\mu|}\right)\times\int_0^x \left\{|H_{j1}(y,\lambda)|, |H_{j2}(y,\lambda)|\right\}\cdot N(y)dy \text{ for large} |\lambda|.$$

Therefore from Conte and Sangren Lemma of [14], we have

$$|H_{j1}(y,\lambda)|, |H_{j2}(y,\lambda)| \le M\left(1+\frac{1}{|\mu|}\right)\cdot \exp\left\{2\int_0^x N(y)dy\right\}, \quad (j=1,2) \tag{3.24}$$

Thus, we see that H_{j1} and H_{j2} are bounded for all x and large $|\lambda|$. It follows from (3.20) that

$$S_j(x,\lambda), T_j(x,\lambda) = o(e^{tx}), \quad (j=1,2) \tag{3.25}$$

for all x and large $|\lambda|$.

From (3.1), using (3.25)

$$\left.\begin{aligned} S_j(x,\lambda) &= S_j(0)\cos w(x) + \circ\left(e^{tx}\cdot|\lambda|^{-\frac{1}{2}}\right) \\ T_j(x,\lambda) &= T_j(0)\cos z(x) + \circ\left(e^{tx}\cdot|\lambda|^{-\frac{1}{2}}\right) \end{aligned}\right\}, \quad (j=1,2) \tag{3.26}$$

Using (3.2) and (3.3), we get from (3.26)

$$\left.\begin{aligned} u_j(x,\lambda) &= u_j(0)\cos w(x) + \circ\left(e^{tx}\cdot|\lambda|^{-\frac{3}{4}}\right) \\ v_j(x,\lambda) &= v_j(0)\cos z(x) + \circ\left(e^{tx}\cdot|\lambda|^{-\frac{3}{4}}\right) \end{aligned}\right\}, \quad (j=1,2) \tag{3.27}$$

Also, we have from [5, Chap.3, §4], for large x.

$$u_j(x,\lambda) = \frac{e^{-iw(x)}\left[M_{j1}(\lambda)+\circ(1)\right]}{\left(\lambda-p(x)\right)^{\frac{1}{4}}}, \quad (j=1,2) \tag{3.28}$$

$$v_j(x,\lambda) = \frac{e^{-iz(x)}\left[M_{j2}(\lambda)+\circ(1)\right]}{\left(\lambda-q(x)\right)^{\frac{1}{4}}}, \quad (j=1,2) \tag{3.29}$$

where

$$M_{j1}(\lambda) = \frac{1}{2}\lambda^{\frac{1}{4}}u_j(0) - \frac{1}{2i}\left(\frac{u_j{}'(0)}{\lambda^{\frac{1}{4}}} - \frac{u_j(0)p'(0)}{4\lambda^{\frac{5}{4}}}\right) + \\ +\frac{1}{2i}\int_0^\infty e^{iw(t)}\left\{P(t)\left(\lambda-p(t)\right)^{\frac{1}{4}}u_j(t,\lambda) + R(t)\left(\lambda-q(t)\right)^{\frac{1}{4}}v_j(t,\lambda)\right\}dt \tag{3.30}$$

$$M_{j2}(\lambda) = \frac{1}{2}\lambda^{\frac{1}{4}}v_j(0) - \frac{1}{2i}\left(\frac{v_j{}'(0)}{\lambda^{\frac{1}{4}}} - \frac{v_j(0)q'(0)}{4\lambda^{\frac{5}{4}}}\right) + \\ +\frac{1}{2i}\int_0^\infty e^{iz(t)}\left\{Q(t)\left(\lambda-q(t)\right)^{\frac{1}{4}}v_j(t,\lambda) + R(t)\left(\lambda-p(t)\right)^{\frac{1}{4}}u_j(t,\lambda)\right\}dt \tag{3.31}$$

under the condition $im(w(x) \square z(x)) = \circ(1)$.

§4. In this section we obtain a solution of the system (1.2) which is small when imaginary part of λ is large and positive and x is large. To find such a solution we consider the system of integral equations

$$\left.\begin{aligned} X_j(x,\lambda) &= e^{iw(x)} - \frac{1}{2i}\int_0^x e^{i(w(x)-w(t))}\left\{P(t)X_j(t,\lambda)+R(t)Y_j(t,\lambda)\right\}dt - \\ &\quad -\frac{1}{2i}\int_x^\infty e^{i(w(x)-w(t))}\left\{P(t)X_j(t,\lambda)+R(t)Y_j(t,\lambda)\right\}dt \\ Y_j(x,\lambda) &= e^{iz(x)} - \frac{1}{2i}\int_0^x e^{i(z(x)-z(t))}\left\{Q(t)Y_j(t,\lambda)+R(t)X_j(t,\lambda)\right\}dt - \\ &\quad -\frac{1}{2i}\int_x^\infty e^{i(z(x)-z(t))}\left\{Q(t)Y_j(t,\lambda)+R(t)X_j(t,\lambda)\right\}dt \end{aligned}\right\}, \quad (j=1,2) \tag{4.1}$$

Exactly following Titchmarsh [8, §6.2] and using (vii) of §3 it can be verified that the solutions of the system of integral equations (4.1) satisfying (1.2). Also we have

$$\left.\begin{aligned} \left|X_j(x,\lambda)\right| &\le \frac{e^{-iw(x)}}{(1-J)} \\ \left|Y_j(x,\lambda)\right| &\le \frac{e^{-iz(x)}}{(1-J)} \end{aligned}\right\}, \quad (j=1,2) \tag{4.2}$$

where

$$J=\max\left[\int_0^\infty |P(y)|\,dy, \int_0^\infty |Q(y)|\,dy, \int_0^\infty |R(y)|\,e^{im(w(y)-z(y))}dy, \int_0^\infty |R(y)|\,e^{im(z(y)-w(y))}dy\right]$$

Considering (4.1) for a fixed λ or λ in the bounded part of the region $J=\circ\left(|\lambda|^{-\frac{1}{2}}\right)<1$, if $|\lambda|$ is sufficiently large and noting that $im\left(w(x)-z(x)\right)=\circ(1)$, it can be shown following [8, §6.2] that

$$\left.\begin{aligned} X_j(x,\lambda) &= e^{iw(x)}\left[L_{j1}(\lambda)+\circ(1)\right] \\ Y_j(x,\lambda) &= e^{iz(x)}\left[L_{j2}(\lambda)+\circ(1)\right] \end{aligned}\right\}, \quad (j=1,2) \tag{4.3}$$

where

$$\left.\begin{aligned} L_{j1}(\lambda) &= 1-\frac{1}{2i}\int_0^\infty e^{-iw(y)}\left\{P(y)X_j(y,\lambda)+R(y)Y_j(y,\lambda)\right\}dy \\ Y_j(x,\lambda) &= 1-\frac{1}{2i}\int_0^\infty e^{-iz(y)}\left\{Q(y)Y_j(y,\lambda)+R(y)X_j(y,\lambda)\right\}dy \end{aligned}\right\}, \quad (j=1,2) \tag{4.4}$$

From (3.2), (3.3) and (3.4), we have

$$\left.\begin{aligned} u_j(x,) &= \frac{e^{iw(x)}\left[L_{j1}(\lambda)+\circ(1)\right]}{(\lambda-p(x))^{\frac{1}{4}}} \\ v_j(x,\lambda) &= \frac{e^{iz(x)}\left[L_{j2}(\lambda)+\circ(1)\right]}{(\lambda-q(x))^{\frac{1}{4}}} \end{aligned}\right\}, \quad (j=1,2) \tag{4.5}$$

§5. From (3.28) and (3,29) we see that $\varphi_j(x,\lambda),(j=1,2)$ are large when the imaginary part of $w(x)$ and $z(x)$ are large and positive. Therefore $\varphi_j(x,\lambda),(j=1,2)$ are not $L^2[0,\infty)$. But from (4.5) we see that

$$\alpha_j(x,\lambda)=\begin{bmatrix} u_j(x,\lambda) \\ v_j(x,\lambda)\end{bmatrix}, \quad (j=1,2)$$

are small when the imaginary part of $w(x)$ and $z(x)$ are large and positive. Thus $\varphi_j(x,\lambda)$ and $\alpha_j(x,\lambda)$, $(j=1,2)$ are linearly independent. Then

$$\psi_r(x,\lambda)=\sum_{s=1}^{2}K_{rs}(\lambda)\alpha_s(x,\lambda)+\sum_{s=1}^{2}L_{rs}(\lambda)\varphi_s(x,\lambda), \quad (r=1,2) \tag{5.1}$$

Since $\psi_r(x,\lambda)$, $(r=1,2)$ are $L^2[0,\infty)$ but $\varphi_j(x,\lambda)$ are not $L^2[0,\infty)$, therefore $L_{rs}(\lambda)=0,\ (1\le r,s\le 2)$. Hence

$$\psi_r(x,\lambda)=\sum_{s=1}^{2}K_{rs}(\lambda)\alpha_s(x,\lambda),\quad (r=1,2) \tag{5.2}$$

From asymptotic formulae (3.28), (3.29) and (4.3) we have, as x tend to infinity

$$\left.\begin{aligned} u_j{}'(x,\lambda) &\sim -i\left(\lambda-p(x)\right)^{\frac{1}{4}}e^{-iw(x)}M_{j1}(\lambda)\\ v_j{}'(x,\lambda) &\sim -i\left(\lambda-q(x)\right)^{\frac{1}{4}}e^{-iz(x)}M_{j2}(\lambda)\\ u_j{}'(x,\lambda) &\sim -i\left(\lambda-p(x)\right)^{\frac{1}{4}}e^{iw(x)}L_{j1}(\lambda)\\ v_j{}'(x,\lambda) &\sim -i\left(\lambda-q(x)\right)^{\frac{1}{4}}e^{-iz(x)}L_{j1}(\lambda) \end{aligned}\right\},\quad (j=1,2) \tag{5.3}$$

where dashes denote differentiation with respect to x. Using (3.28), (3.29), (4.5), (5.2) and (5.3) we obtain from (1.5)

$$\left.\begin{aligned} K_{11}(\lambda) &= \frac{M_{21}L_{21}+M_{22}L_{22}}{2i\left(M_{11}M_{22}-M_{12}M_{21}\right)\left(L_{12}L_{21}-L_{11}L_{22}\right)}\\ K_{12}(\lambda) &= \frac{M_{21}L_{11}+M_{22}L_{12}}{2i\left(M_{11}M_{22}-M_{12}M_{21}\right)\left(L_{12}L_{21}-L_{11}L_{22}\right)}\\ K_{21}(\lambda) &= \frac{M_{11}L_{21}+M_{12}L_{22}}{2i\left(M_{11}M_{22}-M_{12}M_{21}\right)\left(L_{12}L_{21}-L_{11}L_{22}\right)}\\ K_{22}(\lambda) &= \frac{M_{11}L_{11}+M_{12}L_{12}}{2i\left(M_{11}M_{22}-M_{12}M_{21}\right)\left(L_{12}L_{21}-L_{11}L_{22}\right)} \end{aligned}\right\} \tag{5.4}$$

§6. Convergence Theorem:

If $f(x)=\begin{bmatrix} f_1(x)\\ f_2(x)\end{bmatrix}$ be a real valued continuous vector of bounded variation in $0\le x<\infty$, and $L^2[0,\infty)$ and is such that the integrals

$$\int_0^\infty p(x)f_1(x)dx \qquad ; \int_0^\infty p(x)f_1(x)dx \tag{6.1}$$

are uniformly convergent for large $|\lambda|$, then

$$f(x)=-\frac{1}{\pi i}\lim_{R\to\infty}\int_{-R+i\varepsilon}^{R+i\varepsilon}\varphi(x,\lambda)d\lambda \tag{6.2}$$

uniformly for $0<\varepsilon\le 1$, where

$$\begin{bmatrix}\varphi_1(x,\lambda)\\ \varphi_2(x,\lambda)\end{bmatrix}=\varphi(x,\lambda)=\int_0^\infty G(x,y;\lambda)f(y)dy \tag{6.3}$$

We prove convergence theorem for $\varphi_1(x,\lambda)$ because similar result holds for $\varphi_2(x,\lambda)$.

Now we write $\varphi_1(x,\lambda)$ as

$$\begin{aligned}\varphi_1(x,\lambda)=\ &\psi_{11}(x,\lambda)\int_0^x\varphi_1{}^T\left(y,\lambda\right)f(y)dy+\psi_{21}(x,\lambda)\int_0^x\varphi_2{}^T\left(y,\lambda\right)f(y)dy+\\ &+u_1(x,\lambda)\int_x^\infty\psi_1{}^T\left(y,\lambda\right)f(y)dy+u_2(x,\lambda)\int_x^\infty\psi_2{}^T\left(y,\lambda\right)f(y)dy.\end{aligned}$$

$$\begin{aligned}\varphi_1(x,\lambda)=\ &\psi_{11}(x,\lambda)\int_0^x u_1\left(y,\lambda\right)f_1(y)dy+u_1(x,\lambda)\int_x^\infty\psi_{11}\left(y,\lambda\right)f_1(y)dy+\psi_{21}(x,\lambda)\int_0^x u_2\left(y,\lambda\right)f_1(y)dy+\\ &+u_2(x,\lambda)\int_x^\infty\psi_{21}\left(y,\lambda\right)f_1(y)dy+\psi_{11}(x,\lambda)\int_0^x v_1\left(y,\lambda\right)f_2(y)dy+\psi_{21}(x,\lambda)\int_0^x v_2\left(y,\lambda\right)f_2(y)dy+\\ &+u_1(x,\lambda)\int_x^\infty\psi_{12}\left(y,\lambda\right)f_2(y)dy+u_2(x,\lambda)\int_x^\infty\psi_{22}\left(y,\lambda\right)f_2(y)dy.\\ &=A+B+C+D+E+F\end{aligned} \tag{6.4}$$

where

$$A=\psi_{11}(x,\lambda)\int_0^x u_1(y,\lambda)f_1(y)dy+u_1(x,\lambda)\int_x^\infty \psi_{11}(y,\lambda)f_1(y)dy$$

$$B=\psi_{21}(x,\lambda)\int_0^x u_2(y,\lambda)f_1(y)dy+u_2(x,\lambda)\int_x^\infty \psi_{21}(y,\lambda)f_1(y)dy$$

$$C=\psi_{11}(x,\lambda)\int_0^x v_1(y,\lambda)f_2(y)dy$$

$$D=\psi_{21}(x,\lambda)\int_0^x v_2(y,\lambda)f_2(y)dy$$

$$E=u_1(x,\lambda)\int_x^\infty \psi_{12}(y,\lambda)f_2(y)dy$$

$$F=u_2(x,\lambda)\int_x^\infty \psi_{22}(y,\lambda)f_2(y)dy.$$

We evaluate A , the other term can be evaluated in the same way. Now

$$A=\psi_{11}(x,\lambda)\int_0^x u_1(y,\lambda)f_1(y)dy+u_1(x,\lambda)\int_x^\infty \psi_{11}(y,\lambda)f_1(y)dy$$

$$=\psi_{11}(x,\lambda)\left[\int_0^{x-\delta}+\int_{x-\delta}^{x} u_1(y,\lambda)\, f_1(y)dy\right]+u_1(x,\lambda)\left[\int_x^{x+\delta}\int_{x+\delta}^{\infty}\psi_{11}(y,\lambda)\, f_1(y)dy\right]$$

$= A_1+A_2+A_3+A_4$, say

For $J<1$, if $|\lambda|$ is sufficiently large, we have from (5.2) and (5.4)

$$|\psi_{11}(x,\lambda)|\le\frac{|M_{22}(\lambda)|\left|e^{iw(x)}\right|}{2\{|M_{11}(\lambda)M_{22}(\lambda)-M_{12}(\lambda)M_{21}(\lambda)|\}|\lambda-p(x)|^{\frac{1}{4}}}\times \times\frac{|M_{22}(\lambda)|e^{-im(w(x))}}{2\{|M_{11}(\lambda)M_{22}(\lambda)-M_{12}(\lambda)M_{21}(\lambda)|\}\lambda^{\frac{1}{4}}}\left[1-\frac{p(x)}{\lambda}\right]^{\frac{1}{4}} \tag{6.5}$$

Therefore, using (3.2), (3.25) and (6.5), we have

$$A_4=o\left\{\frac{e^{tx}}{|\lambda|^{\frac{1}{2}}}\int_{x+\delta}^{\infty}e^{-ty}\,|f_1(y)|dy\right\}$$

For a fixed y , $p(y)$ is less than $|\lambda|$. Therefore, using (6.1), we have

$$A_4=o\left\{\frac{e^{-t\delta}}{|\lambda|^{\frac{1}{2}}}\right\} \tag{6.6}$$

The integral of (6.6) round the semicircle tends to zero as R tends to infinity for any fixed $\delta>0$. A similar argument holds for A_1 also. Now, we consider A_3 . For fixed x or in a finite interval, from (4.1), we have

$$\left|X_j(x,\lambda)-e^{iw(x)}\right|=\left|\frac{1}{2i}\int_0^x e^{i(w(x)-w(y))}\{P(y)X_j(y,\lambda)+R(y)Y_j(y,\lambda)\}dy+\frac{1}{2i}\int_x^\infty e^{i(w(y)-w(x))}\{P(y)X_j(y,\lambda)+R(y)Y_j(y,\lambda)\}dy\right|$$

$$\le\frac{e^{-im(w(x))}}{2}(J+---)$$

$$<\alpha\cdot e^{-im(w(x))},\quad (j=1,2)\quad \text{(say)} \tag{6.7}$$

Similarly,

$$\left|Y_j(x,\lambda)-e^{iz(x)}\right|<\alpha\cdot e^{-im(z(x))},\quad (j=1,2)\quad \text{(say)} \tag{6.8}$$

Also from (4.4), we have

$$\left.\begin{aligned} L_{j1}(\lambda) &= 1+\circ\left(|\lambda|^{-\frac{1}{2}}\right) \\ L_{j2}(\lambda) &= 1+\circ\left(|\lambda|^{-\frac{1}{2}}\right) \end{aligned}\right\}, \quad (j=1,2) \tag{6.9}$$

Similarly from (3.30) and (3.31), we have

$$\left.\begin{aligned} M_{j1}(\lambda) &= \frac{1}{2}\lambda^{\frac{1}{4}}u_j(0)+\circ\left(|\lambda|^{-\frac{1}{4}}\right) \\ M_{j2}(\lambda) &= \frac{1}{2}\lambda^{\frac{1}{4}}v_j(0)+\circ\left(|\lambda|^{-\frac{1}{4}}\right) \end{aligned}\right\}, \quad (j=1,2) \tag{6.10}$$

Therefore, by using (6.9), (4.3) can be written as

$$\left.\begin{aligned} X_j(x,\lambda) &= e^{iw(x)}\left[1+\circ\left(|\lambda|^{-\frac{1}{2}}\right)\right] \\ Y_j(x,\lambda) &= e^{iz(x)}\left[1+\circ\left(|\lambda|^{-\frac{1}{2}}\right)\right] \end{aligned}\right\}, \quad (j=1,2) \tag{6.11}$$

Now using (4.5), (6.10) and (6.11), (5.2) and (5.4) give

$$\psi_{11}(x,\lambda) = \frac{v_2(0)\cdot e^{iw(x)}\left[1+\circ\left(|\lambda|\right)^{-\frac{1}{2}}\right]}{i\lambda^{\frac{1}{4}}\left[v_1(0)u_2(0)-u_1(0)v_2(0)\right]\left(\lambda-p(x)\right)^{\frac{1}{4}}}$$

$$= \frac{v_2(0)\cdot e^{iw(x)}\left[1+\circ\left(|\lambda|\right)^{-\frac{1}{2}}\right]}{i\lambda^{\frac{1}{2}}\left[v_1(0)u_2(0)-u_1(0)v_2(0)\right]}\left(1-\frac{p(x)}{\lambda}\right)^{-\frac{1}{4}} \tag{6.12}$$

Thus from the first result of (3.27) and (6.12), we get

$$A_3 = \frac{u_1(0)v_2(0)\cos w(x)}{i|\lambda|^{\frac{1}{2}}\left[v_1(0)u_2(0)-u_1(0)v_2(0)\right]}\int_x^{x+\delta} e^{iw(y)}f_1(y)\left(1-\frac{P(y)}{\lambda}\right)^{-\frac{1}{4}}dy+$$

$$+\mathrm{o}\left\{\frac{e^{|tx|}}{|\lambda|}\int_x^{x+\delta} e^{-im(w(y))}f_1(y)\left(1-\frac{P(y)}{\lambda}\right)^{-\frac{1}{4}}dy\right\} \tag{6.13}$$

$$= \frac{u_1(0)v_2(0)\cos w(x)}{i|\lambda|^{\frac{1}{2}}\left[v_1(0)u_2(0)-u_1(0)v_2(0)\right]}\int_x^{x+\delta} e^{iw(y)}f_1(y)dy+\mathrm{o}\left\{\frac{e^{|tx|}}{|\lambda|}\int_x^{x+\delta} e^{-im(w(y))}f_1(y)dy\right\}+$$

$$+\mathrm{o}\left\{\frac{e^{|tx|}}{|\lambda|^{\frac{3}{2}}}\int_x^{x+\delta} e^{-im(w(y))}p(y)f_1(y)dy\right\} \tag{6.14}$$

The last two terms of A_3 are

$$\mathrm{o}\left\{\frac{1}{|\lambda|}\int_x^{x+\delta} f_1(y)dy\right\} \text{ and } \mathrm{o}\left\{\frac{1}{|\lambda|^{\frac{3}{2}}}\int_x^{x+\delta} f(y)p(y)dy\right\}. \tag{6.15}$$

The integral of these round the semicircle are $\mathrm{o}\left\{\int_x^{x+\delta} f_1(y)dy\right\}$ and $\mathrm{o}\left\{\frac{1}{|\lambda|^{\frac{1}{2}}}\int_x^{x+\delta} f(y)p(y)dy\right\}$ respectively. These integrals can be made as small as we please by properly choosing δ and using (6.1). The first term in A_3 can be written as

$$\frac{u_1(0)v_2(0)\left[e^{iw(x)}+e^{-iw(x)}\right]}{2i|\lambda|^{\frac{1}{2}}\left[v_1(0)u_2(0)-u_1(0)v_2(0)\right]}\int_x^{x+\delta} e^{iw(y)}f_1(y)dy \tag{6.16}$$

Using (3.13), we have from (6.13), the first term of A_3 is

$$\frac{u_1(0)v_2(0)\left[e^{i\mu(x)}+e^{-i\mu(x)}\right]}{2i\mu\left[v_1(0)u_2(0)-u_1(0)v_2(0)\right]}\int_x^{x+\delta} e^{i\mu(y)} f_1(y)dy \qquad .$$

The term involving $e^{i\mu x}$ also gives a zero limit. The other term is the same as in the case of an ordinary Fourier series, and similarly for A_2. Hence we conclude that in the case of continuous function of bounded variation

$$\lim_{R\to\infty}\int_{-R+i\varepsilon}^{R+i\varepsilon} A\cdot d\lambda = \frac{\pi i u_1(0)v_2(0)f_1(x)}{v_1(0)u_2(0)-u_1(0)v_2(0)}$$

Similarly

$$\lim_{R\to\infty}\int_{-R+i\varepsilon}^{R+i\varepsilon} B\cdot d\lambda = -\frac{\pi i v_1(0)u_2(0)f_1(x)}{v_1(0)u_2(0)-u_1(0)v_2(0)}$$

$$\lim_{R\to\infty}\int_{-R+i\varepsilon}^{R+i\varepsilon} C\cdot d\lambda = \frac{1}{2}\frac{\pi i v_2(0)v_1(0)f_2(x)}{v_1(0)u_2(0)-u_1(0)v_2(0)}$$

$$\lim_{R\to\infty}\int_{-R+i\varepsilon}^{R+i\varepsilon} D\cdot d\lambda = -\frac{1}{2}\frac{\pi i v_2(0)v_1(0)f_2(x)}{v_1(0)u_2(0)-u_1(0)v_2(0)}$$

$$\lim_{R\to\infty}\int_{-R+i\varepsilon}^{R+i\varepsilon} E\cdot d\lambda = -\frac{1}{2}\frac{\pi i u_1(0)u_2(0)f_2(x)}{v_1(0)u_2(0)-u_1(0)v_2(0)}$$

$$\lim_{R\to\infty}\int_{-R+i\varepsilon}^{R+i\varepsilon} F\cdot d\lambda = \frac{1}{2}\frac{\pi i u_1(0)u_2(0)f_2(x)}{v_1(0)u_2(0)-u_1(0)v_2(0)}$$

Thus we have

$$f_1(x) = -\frac{1}{\pi i}\lim_{R\to\infty}\int_{-R+i\varepsilon}^{R+i\varepsilon}\varphi_1(x,\lambda)d\lambda \tag{6.17}$$

Similarly

$$f_2(x) = -\frac{1}{\pi i}\lim_{R\to\infty}\int_{-R+i\varepsilon}^{R+i\varepsilon}\varphi_2(x,\lambda)d\lambda \tag{6.18}$$

The above results are true, uniformly for $0<\varepsilon\le 1$.

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