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Generalised Statistical Convergence For Double Sequences

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Abstract: *Recently, the concept of β-statistical Convergence was introduced considering a sequence of infinite matrices $\beta = (b_{nk}(i))$. Later, it was used to define and study β-statistical limit point, β-statistical cluster point, $st_\beta - limit$ inferior and $st_\beta - limit$ superior. In this paper we analogously define and study 2β-statistical limit, 2β-statistical cluster point, $st_{2\beta} - limit$ inferior and $st_{2\beta} - limit$ superior for double sequences.*

***Keywords**: Double sequences, Statistical convergence, β-statistical Convergence Regular matrices, RH-regular matrices.*

I. Introduction

A double sequence $x = [x_{jk}]_{j,k=0}^{\infty}$ is said to be convergent in the pringsheim sense or p-convergent if for every $\varepsilon > 0$ there exist $N \in \mathbb{N}$ such that $|x_{jk} - L| < \varepsilon$ whenever j, k > N and L is called the Pringsheim limit [1], denoted $P\text{-}lim x = L$.

A double sequence x is bounded if there exist a positive number M such that $|x_{jk}| < M$ for all j, k i.e if $\|x\| = sup_{jk}|x_{jk}| < \infty$. Note that in contrast to the case for single sequences, a convergent double sequence need not be bounded.

Let $A = (a_{nk})_{n,k=1}^{\infty}$ be a non-negative regular matrix. Then A-density of a set $K \subseteq \mathbb{N}$ is defined if $\delta_A(K) = lim_n \sum_{k\in K} a_{nk}$ exists [2].

A sequence $x = (x_k)$ is said to be A-statistically convergent to L if for every $\varepsilon > 0$ the set $K(\varepsilon) = \{k \in K : |x_k - L| \geq \varepsilon\}$ has A-density zero [3], (see also [4] and [5]).

Recently, Kolk [6] generalized the idea of A-statistical convergent to β-statistical Convergence by using the idea of β-summability or F_β-convergence due to Stieglitz [7].

Let $A = [a_{jk}^{mn}]_{j,k=0}^{\infty}$ be a regular doubly infinite matrix of real numbers for all m, n = 0, 1, … . In the similar manner as in [2], we define 2A-density of the set $K = \{(j, k) \in \mathbb{N} \times \mathbb{N}\}$ *if* for m, n = 0, 1, 2, …

$$\delta_{2A}(K) = \lim_{pq\to\infty} \sum_{j=0}^{p} \sum_{k=0}^{q} a_{jk}^{mn} \quad (1.1)$$

exists and a double sequence $x = (x_{jk})$ is said to be 2A-statistically convergent to L if for every $\varepsilon > 0$ the set $K(\varepsilon)$ has 2A-density zero.

Let $\beta = (\beta_i)$ be a sequence of infinite matrices with $\beta_i = \left(b_{jk}(i)\right)$. Then $x = (x_{jk}) \in \ell_\infty^2$, the space of bounded double sequences, is said to be F-convergent (2β-summable) to the value $2\beta - limx$ (denotes the generalized limit) if

$$\lim_n (\beta_i x)_n = \lim_{p,q\to\infty} \sum_{j=0}^{p} \sum_{k=0}^{q} b_{jk}^{mn}(i) = 2\beta\text{-}lim\, x \quad (1.2)$$

uniformly for i > 0, m, n = 0, 1, … . The method 2β is regular if $\beta = [b_{jk}^{mn}]_{j,k=0}^{\infty}$ is RH-Regular. (see [8]).

Kolk [6] introduced the following:

An index set K is said to have $\beta\text{-}density\ \delta_\beta(K)$ equal to *d*, if the characteristic sequence of K is β-summable to *d*, i.e.

$$\lim_n \sum_{k\in K} b_{nk}(i) = d, \quad (1.3)$$

uniformly in *i*, where by index set we mean a set K= $\{k_i\} \subset \mathbb{N},\ k_i < k_{1+i}$ for all i.

Now we extend this definition as follows:

An indexed set K= $\{(j,k)\} \subseteq \mathbb{N} \times \mathbb{N},\ j_i < j_{i+1},\ k_i < k_{1+i}$ for all *i*, is said to have $2\beta\text{-}density\ \delta_{2\beta}(K) = d$, if the characteristic sequence of K is 2β-summable to d, i. e., if

$$\lim_{mn} \sum_{j\in K} \sum_{k\in K} b_{jk}^{mn}(i) = d \quad (1.4)$$

uniformly in *i*.

Let $\mathcal{R}^*$ denote the set of all RH-regular methods 2β with $b_{jk}^{mn}(i) \geq 0$ for all *j*, *k* and *i*. let $\beta \in \mathcal{R}^*$, A double sequence $x = (x_{jk})$ is called 2β-statistically convergent to the number L if for every $\varepsilon > 0$ there exist a subset K= $\{(j,k)\} \subseteq \mathbb{N} \times \mathbb{N}\ j,k = 1,2,...$ such that

$$\delta_{2\beta}\{(j,k),\ j \le n,\ k \le m : |x_{jk} - L| \ge \varepsilon\} = 0 \tag{1.5}$$

and we write $st_{2\beta}$-$lim\, x = L$.

We denote by $st(2\beta)$, the space of all 2β-statistically convergent sequences.

In particular, if $\beta = (C_1)$, the Cesaro matrix, the β-statistical convergence is reduced to C_{11}-statistical convergence.

II. 2β-Statistical Cluster And Limit Points

We use the following examples to show that neither of the two methods, statistical convergence and 2β-statistical convergence, implies the other.

Example 2.1: Consider the sequence of infinite matrices $\beta = (\beta_i)$ with

$$b_{jk}^{mn}(i) = \begin{cases} \frac{1}{i} + \frac{1}{ij}, & if\ k = j^2\ \ \forall\, m,n \\ 1 - \frac{j}{i(j+1)}, & if\ k = j^2 + 1\ \forall m,n \\ 0, & otherwise \end{cases}$$

It is clear that $\beta\, \epsilon\, \mathcal{R}^*$,

Now let the sequence $x = (x_{jk})\ \ and\ \ y = (y_{jk})$ be defined by

$$x_{jk} = \begin{cases} 0, & if\ k = n^2, for\ all\ j\ and\ some\ n \in \mathbb{N} \\ \frac{1}{k}, & if\ k = n^2 + 1, for\ all\ j\ and\ some\ n \in \mathbb{N} \\ k, & otherwise \end{cases}$$

and

$$y_{jk} = \begin{cases} k, & if\ k = n^2, for\ all\ j\ and\ some\ n \in \mathbb{N} \\ 0, & if\ k = n^2 + 1, for\ all\ j\ and\ some\ n \in \mathbb{N} \\ 1, & otherwise \end{cases}$$

Then x is not statistically convergent to zero as $\delta\{(i,j): |x_{ij}| \ge \varepsilon\} \ne 0$ but it is 2β -statistically convergent to zero; and on the other hand y is statistically convergent but not 2β-statistically convergent.

We now give some definitions for the method 2β.

Definition 2.2: Let $\beta\, \epsilon\, \mathcal{R}^*$. Then number γ is said to be 2β −statistical cluster point of a sequence x_{jk} if for given $\varepsilon > 0$, the set $\{(j,k); |x_{jk} - j| < \varepsilon\}$ does not have $2\beta - density\ zero$.

Definition2.3: Let $\beta\, \epsilon\, \mathcal{R}^*$. The number λ is said to be 2β −statistical limit point of a sequence x_{jk} if there is a subsequence of x_{jk} which convergence to λ such that whose indices do not have $2\beta - density\ zero$.

Denote by $\Gamma_x(2\beta)$, the set of 2β −statistical cluster points and by $\Lambda_x(2\beta)$ the set of 2β −statistical limit points of $x = (x_{jk})$ it is clear from the above examples, that $\Gamma_x(2\beta) = \{0\}$ and $\Lambda_x(2\beta) = \{0\}$, $\Gamma_y(2\beta) = \{0\}$ and $\Lambda_y(2\beta) = \{0\}$. Throughout this paper we will consider $\beta\, \epsilon\, \mathcal{R}^*$.

Definition 2.4: Let us write $G_x = \{g \epsilon \mathcal{R}: \delta_{2\beta}(\{(j,k): x_{jk} > g\}) \ne 0\}$ and $F_x = \{f \epsilon \mathcal{R}: \delta_{2\beta}(\{(j,k): x_{jk} < f\}) \ne 0$ for a double sequence $x = xjk$. Then we define 2β–statistical limit superior and 2β–statistical limit inferior of $x = (x_{jk})$ as follows:

$$st_{2\beta} - \lim Sup x = \begin{cases} Sup G_x, & if\ G_x \ne \emptyset \\ -\infty,, & if\ G_x = \emptyset \end{cases}$$

$$st_{2\beta} - lim\, Inf x = \begin{cases} Inf F_x, & if\ F_x \ne \emptyset \\ +\infty, & if\ F_x = \emptyset. \end{cases}$$

Definition 2.5: The double sequence $x = (x_{jk})$ is said to be 2β −statistically bounded if there is a number d such that $\delta_{2\beta}\{(j,k): |x_{jk}| > d\} = 0$.

Definition 2.6: Consider the same β as defined in example 2.1, define the sequence $z = (z_{jk})$ by

$$z_{jk} = \begin{cases} 0, & if\ k = n^2, for\ all\ k\ and\ some\ n \in \mathbb{N} \\ 1, & if\ k = n^2 + 1, for\ all\ k\ and\ some\ n \in \mathbb{N} \\ k, & otherwise. \end{cases}$$

Here we see that z is not bounded above but it is 2β −statistically bounded for $\delta_{2\beta}\{(j,k): |z_{jk}| > 1\} = 0$. Also z is not statistically bounded. Thus $G_x = (-\infty, 1)\ and\ F_x = (0, \infty)$ so that $st_{2\beta} - \lim Sup z =$

$1 \; and \; st_{2\beta} - limInf\, z$. Moreover $\Gamma_x(2\beta) = \{0,1\} = \Lambda_x(2\beta)$ and z is neither 2β −statistically nor statistically convergent. In this example we see that z is 2β −statistically bounded but not 2β −statistically convergent. On the other hand in example 2.1 y is statistically convergent and not 2β −statistically bounded.

Also note that $st_{2\beta} - \lim Supz$ equals the greatest element of $\Gamma_x(2\beta)$ while $st_{2\beta} - limInf\, z$ is the least element $\Gamma_x(2\beta)$. This observation suggests the following.

Theorem 2.7

(a) If $l_1 = st_{2\beta} - \lim Supx$ is finite, then for every positive number ε

(i) $\delta_{2\beta}\{(j,k): |z_{jk}| > l_1 - \varepsilon\} \neq 0$ and $\delta_{2\beta}\{(j,k): |z_{jk}| > l_1 + \varepsilon\} = 0$.

Conversely, if (i) holds for every $\varepsilon > 0$, then $l_1 = st_{2\beta} - \lim Supx$.

(b) if $l_2 = st_{2\beta} - \lim Infx$ is finite, then from every positive number ε

(ii) $\delta_{2\beta}\{(j,k): |z_{jk}| > l_2 + \varepsilon\} \neq 0$ and $\delta_{2\beta}\{(j,k): |z_{jk}| > l_2 - \varepsilon\} = 0$.

Conversely, if (i) holds for every $\varepsilon > 0$ then $l_2 = st_{2\beta} - \lim Infx$.

From definition 2.2 we see that the above theorem can be interpreted as showing that $st_{2\beta} - \lim Supx$ and $st_{2\beta} - \lim Infx$ are the greatest and the least 2β −statistical cluster points of x.

Note that 2β −statistical boundedness implies that $st_{2\beta} - \lim Supx$ and $st_{2\beta} - \lim Infx$ are finite, so that properties (i) and (ii) of Theorem 2.7 hold.

III. The Main Results

Throughout this paper by $\delta_{2\beta}(K) \neq 0$; $K = \{(j,k) \in N \times \mathbb{N}\}$ we mean that either $\delta_{2\beta}(K) > 0 \; or \; K$ does not have $2\beta - density$.

Theorem 3.1: For every real number sequence x, $st_{2\beta} - \lim Infx \leq st_{2\beta} - \lim Supx$.

Proof: First consider the case in which $st_{2\beta} - \lim Supx = -\infty$, This implies that $G_x = \emptyset$, Therefore for every $g \in R$, $\delta_{2\beta}\{(j,\ k): x_{jk} > g\} = 0$, which implies that $\delta_{2\beta}\{(j,\ k): x_{jk} \leq g\} = 1$. So that for every $f \in R$, $\delta_{2\beta}\{(j,\ k): x_{jk} < f\} \neq 0$. Hence $st_{2\beta} - \lim Infx = -\infty$,

Now consider $st_{2\beta} - \lim Supx = +\infty$. This implies that for every $\in R$, $\delta_{2\beta}\{(j,\ k): x_{jk} > g\} \neq 0$. This implies that $\delta_{2\beta}\{(j,\ k): x_{jk} \leq g\} = 0$. Therefore for every $f \in R$, $\delta_{2\beta}\{(j,\ k): x_{jk} < f\} = 0$, which implies that $F_x = \emptyset$. Hence $st_{2\beta} - \lim Infx = +\infty$.

Next assume that $l_1 = st_{2\beta} - \lim Supx < +\infty$ and let $l_2 = st_{2\beta} - \lim Infx$. Given $\varepsilon > 0$ we show that $l_1 + \varepsilon \in F_x$, so that $l_2 \leq l_1 + \varepsilon$. By Theorem 2.7(a), $\delta_{2\beta}\left\{(j,k): x_{jk} > l_1 + \frac{\varepsilon}{2}\right\} = 0$, since $l_1 = lub\; G_x$. This implies that $\delta_{2\beta}\left\{(j,k): x_{jk} \leq l_1 + \frac{\varepsilon}{2}\right\} = 1$, which in turn gives $\delta_{2\beta}\{(j,k): x_{jk} < l_1 + \varepsilon\} = 1$. Hence $l_1 + \varepsilon = F_x$ and so that $l_2 \leq l_1 + \varepsilon$ i.e $l_2 \leq l_1$ since ε was arbitrary.

Remark: For any double sequence $x = (x_{jk})$

$$st_2 - \lim Infx \leq st_{2\beta} - \lim Infx \leq st_{2\beta} - lim\, Supx \leq st_2 - lim\, Supx$$

where

$$st_2 - lim\, Supx = p - limSupx$$
$$st_2 - \lim Infx = p - limInfx$$

Theorem 3.2: For any double sequence $x = x_{jk}$, 2β −statistical boundness implies 2β −statistical convergence if and only if

$$st_{2\beta} - \lim Infx = st_{2\beta} - lim\, Supx.$$

Proof: Let $l_1 = st_{2\beta} - \lim Supx$ and $l_2 = st_{2\beta} - \lim Infx$. First assume that $st_{2\beta} - \lim x = L$ and $\varepsilon > 0$. Then $\delta_{2\beta}\{(j,k): |x_{jk} - L| \geq 0\} = 0$. So that

$$\delta_{2\beta}\{(j,k): x_{jk} > L + \varepsilon\} = 0,$$

which implies that $l_1 \leq L$. Also

$$\delta_{2\beta}\{(j,k): x_{jk} > L - \varepsilon\} = 0,$$

which implies that $L \leq l_2$. By theorem 3.1, we finally have $l_1 = l_2$

Conversely, suppose $l_1 = l_2 = L$ and x be 2β −statistically bounded. Then for $\varepsilon > 0$, by Theorem 2.7 we have $\delta_{2\beta}\left\{(j,k): x_{jk} > l_1 + \frac{\varepsilon}{2}\right\} = 0$ and $\delta_{2\beta}\left\{(j,k): x_{jk} < l_1 - \frac{\varepsilon}{2}\right\} = 0$. Hence $st_{2\beta} - \lim x = L$

Theorem 3.3: If a double sequence $x = x_{jk}$ is bounded above and $2\beta - summable$ to the number $L = st_{2\beta} - \lim Supx$, then x is $2\beta - statistically$ convergent to L.

Proof: Suppose that $x = x_{jk}$ is not $2\beta - stistically$ convergent to L. Then by Theorem 3.2 $st_{2\beta} - \lim Infx < L$. So there is a number M < L such that $\delta_{2\beta}\{(j,k): x_{jk} < M\} \neq 0$.

Let $K_1 = \{(j,k): x_{jk} < M\}$, then for every $\varepsilon > 0$, $\delta_{2\beta}\{(j,k): x_{jk} > L + \varepsilon\} = 0$.

We write $K_2 = \{(j,k): M \leq x_{jk} \leq L + \varepsilon\}$ and $K_3 = \{(j,k): x_{jk} > L + \varepsilon\}$, and let $G = Sup_{jk}\, x_{jk} < \infty$, since $\delta_{2\beta}(K_1) \neq 0$, there are many n such that

$$\lim Sup_n \sum_{jk=0,0}^{\infty\infty} b_{mnjk}(i) \geq d > 0$$

and for each n, i

$$\sum_{j,k=1,1}^{\infty\infty} \left|b_{mnjk}(i)x_{jk}\right| < \infty.$$

Now

$$\begin{aligned}
\sum_{jk=1,1}^{\infty\infty} b_{mnjk}(i)x_{jk} &= \left(\sum_{jk\in K_1} + \sum_{jk\in K_2} + \sum_{jk\in K_3}\right) b_{mnjk}(i)x_{jk} \\
&\leq M\sum_{jk\in K_1} b_{mnjk}(i) + (L+\varepsilon)\sum_{jk\in K_2} b_{mnjk}(i) + G\sum_{jk\in K_3} b_{mnjk}(i) \\
&= M\sum_{jk\in K_1} b_{mnjk}(i) + (L+\varepsilon)\sum_{jk=1,1}^{\infty\infty} b_{mnjk}(i) - (L+\varepsilon)\sum_{jk\in K_1} b_{mnjk}(i) + O(i) \\
&= -\sum_{jk\in K_1} b_{mnjk}(i)[-M + (L+\varepsilon)] + (L+\varepsilon)\sum_{jk=1,1}^{\infty\infty} b_{mnjk}(i) + O(1) \\
&\leq L\sum_{jk=1,1}^{\infty\infty} b_{mnjk}(i) - d(L-M) + \varepsilon(\sum_{jk=1,1}^{\infty\infty} b_{mnjk}(i) - d) + O(1)
\end{aligned}$$

Since ε is arbitrarily, it follows that

$$\lim Inf\ 2\beta x \leq L - d(l - M) < L.$$

Hence x is not $2\beta -$ summable to L

Theorem 3.4: If the double sequence $x = (x_{jk})$ is bounded below and $2\beta - summable$ to the number $L = st_{2\beta} - lim\, Inf x$, then x is $2\beta -$ statistically convergent to L.

Proof. The proof follows on the same lines as that of Theorem 3.3.

Note: It is easy to observe that in the above Theorems 3.3 and 3.4 the boundedness of $x = (x_{jk})$ can not be omitted or even replaced by the $2\beta -$ statistical boundedness.

For example consider the matrix $\beta = A = (a_{nk})_{n.k=1}^{\infty}$ and define $x = (x_{jk})$ by

$$x = x_{jk} = \begin{cases} 2k-1, & \text{if k is an odd square for all } j \\ 2, & \text{if k is an even square for all } j \\ 1, & \text{if k is an odd nonsquare for all } j \\ 0, & \text{if k is an even nonsquare for all } j \end{cases}$$

Then, $Sup\ x_{jk} = \infty,\ but\ G_x = [2,\infty), F_x = (-\infty, 0]$,

$st_{2\beta} - lim\, Sup x = 2$, and $st_{2\beta} - lim\, Inf x = 0$. [see [9])

This makes it clear that every bounded double sequence is $st_2 -$ bounded and every $st_2 -$ bounded sequence is $st_{2\beta} -$ bounded, but not conversely, in general.

IV. Conclusion

The double sequence which is bounded above and 2β-summable to the number L = $st_{2\beta} - \limsup x$, then it is 2β-statistically convergent to L. Similarly, a double sequence which is bounded below and 2β-summable to the number $\ell = st_{2\beta} - \liminf x$, then it is 2β-statistically convergent to ℓ.

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