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Duality in Multiobjective Variational Problem With Higher Order Derivatives

[1]S.K.Pradhan, [2] D.K.Dalai, [3]R.B.Dash.

[1] *Department of Mathematics, D.A.V.College, Koraput-764001, Odisha, India*
[2] *Department of Mathematics, S.B. Women's College, Cuttack, Odisha, India*
3. Department of Mathematics, Ravenshaw University, Cuttack-753001.Odisha, India.

Abstract: *In this paper two types of duals are considered for a class of variational problems involving 3rd order derivatives. The duality results are derived without any use of optimality conditions. One set of results is based on Mond weir type dual that has the same objective functional as the primal problem but different constraints. The second set of results is based on a dual of an auxiliary primal with single objective function. Under various convexity and generalized convexity assumptions, duality relationships between primal and its various duals are established. Problems with natural boundary values are considered and the analogs of our results in nonlinear programmings are also indicated.*

Our results are generalizations of those presented by I.Husain, B.Ahmad and Z.Jaben.

Keywords: *Variational Problems, Multiobjective duality, Concavity, Generalized convexity*

I. Introduction

Calculus of variations offers a powerful technique for the solution of various important problems appearing in dynamics of rigid bodies, optimization of orbits, theory of vibrations and many areas of science and engineering. The subject of calculus of variation primarily concerns with finding optimal value of a definite integral involving a certain function subject to fixed point boundary conditions.Mond and Hanson [1] were the first to present the problem of calculus of variation as a mathematical programming problem in infinite dimensional space. Since that time many researches contributed to this subject extensively.

In this paper, we consider a vector valued function for the primal problem and its minimality in the Pareto sense. Both equality and inequality constraints are considered in the formulations. In establishing duality results we consider two types of dual problems to the primal problem. The first one has vector valued objective whereas the second set of results based on the duality relations between an auxiliary problems and its associated dual as defined in Mond and Weir[2]. Duality theorems, unlike in case of classical mathematical programming, are not based on optimality criteria but on certain types of convexity and generalized convexity requirements. Finally multiobjective variational problems with natural boundary values rather than fixed end points are mentioned and the analogs of our results in nonlinear programming are pointed out.

We further generalized the work of I.Husain, B.Ahmad and Z.Jaben [5] taking into account of 3rd ordered derivatives instead of 2nd ordered derivatives.

II. Pre-Requisites

In the following variational Problem (VP), by minimally we mean Pareto minimally. Now we consider the following multiobjective variational problem by generalizing the definition in Bector and Husain [6].

(VP) Minimize

$$\left(\int_I f^1(t,x,x',x'',x''')dt,\ldots,\int_I f^p(t,x,x',x'',x''')dt,\right)$$

Subject to

$$x(a)=x(b)=0 \qquad (1)$$

$$x'(a)=x'(b)=0 \qquad (2)$$

$$g(t,x,x',x'',x''')\leq 0 \quad ,t\in I \qquad (3)$$

$$h(t,x,x',x'',x''')=0 \quad ,t\in I \qquad (4)$$

Where **(i)** for I = [a, b] $\subseteq$ R, f: $I \times R^n \times R^n \times R^n \times R^n \to R$, $g: I \times R^n \times R^n \times R^n \times R^n \to R^m$ and $h: I \times R^n \times R^n \times R^n \times R^n \to R^k$ are continuously differentiable functions, and

(ii) X denotes the space of piecewise smooth functions $x: I \to R^n$ having its first, second and third order derivatives x', x'', x''' respectively having norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty + \|D^2x\|_\infty + \|D^3x\|_\infty$, where the differentiation operator D is given by $\omega = Dx \Leftrightarrow x(t) = \int_a^t \omega(s)ds$, Thus D = $\frac{d}{dt}$ except at discontinuities.

We denote the set of feasible solutions of the problem (VP) by K_p i.e.

$$K_p = \left\{ x \in X \middle| \begin{array}{l} x(a) = x(b) = 0, \quad g(t, x, x', x'', x''') \leq 0 \ \ , t \in I \\ x'(a) = x'(b) = 0, h(t, x, x', x'', x''') = 0, \ t \in I \end{array} \right.$$

The following convention for inequalities for vectors in R^n given in Mangasarian [4] will be used throughout the development of the theory.

If x, y $\in R^n$, then $x \geq y \Leftrightarrow x_i \geq y_i \qquad ,i = 1,2,.....n$

$$x \geq y \Leftrightarrow x_i \geq y_i \quad and \ \ x \neq y$$

$$x > y \Leftrightarrow x_i > y_i \ , \ i = 1,2,3,..........n$$

Definition 2.1:
A feasible solution of the problem (VP) i.e. $x \in K_p$ is said to be Pareto minimum if there exists no $y \in K_p$ such that

$$\left(\int_I f^1(t, x, x', x'', x''')dt, ..., \int_I f^p(t, x, x', x'', x''')dt \right)$$

$$\leq \left(\int_I f^1(t, y, y', y'', y''')dt, ..., \int_I f^p(t, y, y', y'', y''')dt \right)$$

Pareto maximality can be defined in the same way except that the inequality in the above definition is reversed.

In the subsequent analysis the following result plays a significant role.

PROPOSITION 2.1: Suppose $\lambda > 0, \lambda \in R^p$ such that Let $y(t) \in K_p$ is an optimal solution of the problem.

$$(P_\lambda): \underset{x(t) \in K_p}{Min} \int_I \lambda^T f(t, x, x', x'', x''')dt$$

Then y (t) is an optimal solution of (MP) in the Pareto sense.

Proof: Assume y (t) is not a Pareto optimal of (MP).Then there exist an x (t) $\in K_p$ such that $\int_I f^i(t, x, x', x'', x''')dt \leq \int_I f^i(t, y, y', y'', y''')dt \ , i = 1,2,... p$.

$$\int_I f^j(t, x, x', x'', x''')dt < \int_I f^j(t, y, y', y'', y''')dt \ , i \neq j.$$

Hence $\int_I \lambda^T f(t, x, x', x'', x''')dt < \int_I \lambda^T f(t, y, y', y'', y''')dt$.

This contradicts the assumption that y (t) is an optimal solution.

In the following sections some duality results (two types of duals to VP) will be established.

III. Mond-Weir Type Multiobjective Duality

Consider the following Mond-Weir [2] dual to (VP)

(M-WD): Maximize

$$\left(\int_I f^1(t,u,u',u'',u''')dt,\dots,\int_I f^p(t,u,u',u'',u''')dt,\right)$$

Subject to

$$u(a)=u(b)=0 \tag{5}$$

$$u'(a)=u'(b)=0 \tag{6}$$

$$\begin{aligned}&\lambda^T f_u(t,u,u',u'',u''')+s(t)^T g_u(t,u,u',u'',u''')+r(t)^T h_u(t,u,u',u'',u''')\\&-D\left(\lambda^T f_{u'}(t,u,u',u'',u''')+s(t)^T g_{u'}(t,u,u',u'',u''')+r(t)^T h_{u'}(t,u,u',u'',u''')\right)\\&+D^2\left(\lambda^T f_{u''}(t,u,u',u'',u''')+s(t)^T g_{u''}(t,u,u',u'',u''')+r(t)^T h_{u''}(t,u,u',u'',u''')\right)\\&-D^3\left(\lambda^T f_{u'''}(t,u,u',u'',u''')+s(t)^T g_{u'''}(t,u,u',u'',u''')+r(t)^T h_{u'''}(t,u,u',u'',u''')\right)=0\end{aligned} \tag{7}$$

$$\int_I \left(r(t)^T g(t,u',u'',u''')+s(t)^T h(t,u',u'',u''')\right)\geq 0 \tag{8}$$

$$\lambda>0,\lambda\in R^n \ ,\ s(t)\geq 0 \ ,\quad t\in I \tag{9}$$

Let K_D be the set of the feasible solutions of (M-WD).

THEOREM 3.1: Suppose

(A_1): $y(t)\in K_p$

(A_2): $(\lambda,u,s(t),r(t))\in K_D$

(A_3): $\int_I \lambda^T f(t,\dots,..)dt$ is pseudo-convex

(A_4) $\int_I \{s(t)^T g(t,.,.)+r(t)^T h(t,.,.)\}dt$ is quasiconvex.

Then $\int_I \lambda^T f(t,x,x',x'',x''')dt \geq \int_I \lambda^T f(t,u,u',u'',u''')dt$

Proof: Since s (t)≥ 0, $t\in I$, $g(t,x,x',x'',x''')\leq 0$, $t\in I$ and $h(t,x,x',x'',x''')\leq 0$, $t\in I$,we have

$$\int_I \left(s(t)^T g(t,x,x',x'',x''')+r(t)^T h(t,x,x',x'',x''')\right)dt\leq 0 \tag{10}$$

Combining this with (8) we, have

$$\int_I \left(s(t)^T g(t,x,x',x'',x''')+r(t)^T h(t,x,x',x'',x''')\right)dt\leq \int_I \left(s(t)^T g(t,u',u'',u''')+r(t)^T h(t,u',u'',u''')\right)$$

By the hypothesis (A_4), this yields

$$\begin{aligned}0\geq \int_I\Big[&(x-u)^T\left(s(t)^T g_u(t,u,u',u'',u''')+r(t)^T h_u(t,u,u',u'',u''')\right)+\\&(x'-u')^T\left(s(t)^T g_{u'}(t,u,u',u'',u''')+r(t)^T h_{u'}(t,u,u',u'',u''')\right)+\\&(x''-u'')^T\left(s(t)^T g_{u''}(t,u,u',u'',u''')+r(t)^T h_{u''}(t,u,u',u'',u''')\right)\Big]dt\end{aligned}$$

By using integration by parts, we get

$$=\int_I (x-u)^T\left(s(t)^T g_u(t,u,u',u'',u''')+r(t)^T h_u(t,u,u',u'',u''')\right)dt +$$

$$(x-u)^T\left(s(t)^T g_u(t,u,u',u'',u''')+r(t)^T h_u(t,u,u',u'',u''')\right)\Big|_{t=a}^{t=b} -$$

$$\int_I (x-u)^T D\left(s(t)^T g_{u'}(t,u,u',u'',u''')+r(t)^T h_{u'}(t,u,u',u'',u''')\right)dt +$$

$$(x'-u')^T\left(s(t)^T g_{u'}(t,u,u',u'',u''')+r(t)^T h_{u'}(t,u,u',u'',u''')\right)\Big|_{t=a}^{t=b} -$$

$$(x'-u')^T D\left(s(t)^T g_{u''}(t,u,u',u'',u''')+r(t)^T h_{u''}(t,u,u',u'',u''')\right)$$

$$=\int_I (x-u)^T\Big[\left(s(t)^T g_u(t,u,u',u'',u''')+r(t)^T h_u(t,u,u',u'',u''')\right) -$$

$$-D\left(s(t)^T g_{u'}(t,u,u',u'',u''')+r(t)^T h_{u'}(t,u,u',u'',u''')\right)\Big]dt -$$

$$(x-u)^T D\left(s(t)^T g_{u''}(t,u,u',u'',u''')+r(t)^T h_{u''}(t,u,u',u'',u''')\right) +$$

$$\int_I (x-u)^T D^2\left(s(t)^T g_{u''}(t,u,u',u'',u''')+r(t)^T h_{u''}(t,u,u',u'',u''')\right)dt \text{ (by integration by parts)}$$

Using (7) we get

$$0\le \int_I (x-u)^T\left\{\lambda^T f_u(t,u,u',u'',u''')-D\lambda^T f_{u'}(t,u,u',u'',u''')+D^2\lambda^T f_{u''}(t,u,u',u'',u''')\right\}dt$$

$$0\le \int_I \left\{(x-u)^T\lambda^T f_u(t,u,u',u'',u''')+(x'-u')^T\lambda^T f_{u'}(t,u,u',u'',u''')\right\}dt -$$

$$(x-u)^T\lambda^T f_{u'}(t,u,u',u'',u''')\Big|_{t=a}^{t=b} - (x'-u')^T D\lambda^T f_{u''}(t,u,u',u'',u''')dt +$$

$$(x'-u')^T\lambda^T f_{u''}(t,u,u',u'',u''')\Big|_{t=a}^{t=b}$$

$$=\int_I \left\{(x-u)^T\lambda^T f_u(t,u,u',u'',u'')+(x'-u')^T D\lambda^T f_{u''}(t,u,u',u'',u''')dt +\right.$$

$$(x''-u'')^T D\lambda^T f_{u''}(t,u,u',u'',u''')dt\ge 0$$

Thus by integration by parts using the boundary conditions, we have

By using A_3 we, get $\int_I \lambda^T f(t,x,x',x'',x''')dt \ge \int_I \lambda^T f(t,u,u',u'',u''')dt$

Theorem 3.2: Assume

(B_1): $y(t)\in K_p$

(B_2): $\left(\lambda,\bar{u}(t),\bar{s}(t),\bar{r}(t)\right)\in K_p$

(B_3): $\int_I \left(y(t)^T g(t,..,..)+r(t)^T h(t,..,..)\right)dt$ is quasi-convex

(B_4): $\int_I \lambda^T f(t,..,..)dt$ is pseudo convex

(B_5): $\int_I \bar{\lambda}^T f(t,y,y',y'',y''')dt = \int_I \bar{\lambda}^T f(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')dt$

Then y (t) is an optimal solution of (VP) and $\left(\lambda,\bar{u}(t),\bar{s}(t),\bar{r}(t)\right)$ is an optimal solution of the problem (M-WD).

Proof:
Assume that y (t) is not Pareto-optimal of (VP).Then there exist an x (t) $\in K_p$ such that
$\int_I f^i(t,x,x',x'',x''')dt \leq \int_I f^i(t,y,y',y'',y''')dt$ for all i
and $\int_I f^j(t,x,x',x'',x''')dt \leq \int_I f^j(t,y,y',y'',y''')dt$ for some j, $1 \leq j \leq p$.
Since $\overline{\lambda} > 0$,this implies,
$$\int_I \overline{\lambda}^T(t,x,x',x'',x''')dt < \int_I \overline{\lambda}^T(t,y,y',y'',y''')dt$$

By hypothesis $(\mathbf{B}_5)$,this inequality implies,

$$\int_I \overline{\lambda}^T(t,x,x',x'',x''')dt < \int_I \overline{\lambda}^T(t,\overline{u},\overline{u}',\overline{u}'',\overline{u}''')dt$$

This contradicts the conclusion of Theorem 3.1 thus establishing the Pareto optimality of y (t) for (VP).Similarly we can show that $(\lambda,\overline{u}(t),\overline{s}(t),\overline{r}(t))$ is Pareto optimal for (M-WD).

Theorem 3.3: Assume,
(C_1): $y(t) \in K_p$
(C_2): $(\lambda,\overline{u}(t),\overline{s}(t),\overline{r}(t)) \in K_p$
(C_3): $\int_I \left(y(t)^T g(t,..,..) + r(t)^T h(t,..,..)\right)dt$ is convex;
(C_4): $\int_I \lambda^T f(t,..,..)dt$ is quasiconvex.
(C_5): $\int_I \overline{\lambda}^T f(t,y,y',y'',y''')dt = \int_I \overline{\lambda}^T f(t,\overline{u},\overline{u}',\overline{u}'',\overline{u}''')dt$
Then y (t) = $\overline{u}(t)$,t $\in I$.
Proof: Proof is similar to 3.2.

IV. Wolfe Type Multiobjective Duality

To establish duality results similar to the preceding ones but under different convexity and generalized convexity assumption we formulate the following Wolfe type dual [3] to the problem (P_λ) stated in the preposition 2.1.
We assume that $\overline{\lambda}$ is known and $\overline{\lambda}$ >0.
(WC D_λ): Maximize:
$$\int_I \left(\overline{\lambda}^T f(t,x,x',x'',x''') + s(t)^T g(t,x,x',x'',x''') + r(t)^T h(t,x,x',x'',x''')\right) dt$$
Subject to
$$x(a) = 0 = x(b),\ \ x'(a) = 0 = x'(b) \tag{11}$$
$$\begin{aligned}&\lambda^T f_u(t,u,u',u'',u''') + s(t)^T g_u(t,u,u',u'',u''') + r(t)^T h_u(t,u,u',u'',u''')\\ &-D\left(\lambda^T f_{u'}(t,u,u',u'',u''') + s(t)^T g_{u'}(t,u,u',u'',u''') + r(t)^T h_{u'}(t,u,u',u'',u''')\right)\\ &+D^2\left(\lambda^T f_{u''}(t,u,u',u'',u''') + s(t)^T g_{u''}(t,u,u',u'',u''') + r(t)^T h_{u''}(t,u,u',u'',u''')\right)\\ &-D^3\left(\lambda^T f_{u'''}(t,u,u',u'',u''') + s(t)^T g_{u'''}(t,u,u',u'',u''') + r(t)^T h_{u'''}(t,u,u',u'',u''')\right) = 0\end{aligned} \tag{12}$$

and $\lambda > 0, \lambda \in R^n$, $s(t) \geq 0$, $t \in I$ (13)

In the following L_p represents the set of feasible solutions of (P_λ) and L_D the set of feasible solution of (WC D_λ).

Theorem 4.1: Assume

(H_1) y (t) $\in L_p$: $\left(\bar{u}(t), \bar{s}(t), \bar{r}(t)\right) \in L_D$

$\left(H_2\right)$ $\int_I \bar{\lambda}^T f(t,.,...)$ *and* $\int_I s(t)^T g(t,.,..) + r(t)^T h(t,.,..)dt$ are convex.

Then,

$$\int_I \bar{\lambda}^T f(t,x,x',x'',x''')dt \geq \int_I \bar{\lambda}^T f(t,u,u',u'',u''') + s(t)^T g(t,u,u',u'',u''') + r(t)^T h(t,u,u',u'',u''')$$

Proof: By the convexity of $\int_I \bar{\lambda}^T f(t,.,...)dt$, we have

$$\int_I \bar{\lambda}^T f(t,x,x',x'',x''')dt \geq \int_I \bar{\lambda}^T f(t,u,u',u'',u''')dt + \int_I \Big[(x-u)^T \lambda^T f_u(t,u,u',u'',u''') +$$
$$(x'-u')^T \bar{\lambda}^T f_{u'}(t,u,u',u'',u''') +$$
$$(x''-u'')^T \bar{\lambda}^T f_{u''}(t,u,u',u'',u''') + (x'''-u''')^T \bar{\lambda}^T f_{u'''}(t,u,u',u'',u''') \Big] \quad (14)$$

From the dual constraints (12), we have,

$$\int_I (x-u)^T \Big[(\lambda^T f_u(t,u,u',u'',u''') + s(t)^T g_u(t,u,u',u'',u''') + r(t)^T h_u(t,u,u',u'',u'''))$$
$$- D\Big(\lambda^T f_{u'}(t,u,u',u'',u''') + s(t)^T g_{u'}(t,u,u',u'',u''') + r(t)^T h_{u'}(t,u,u',u'',u'''))$$
$$+ D^2\Big(\lambda^T f_{u''}(t,u,u',u'',u''') + s(t)^T g_{u''}(t,u,u',u'',u''') + r(t)^T h_{u''}(t,u,u',u'',u'''))$$
$$- D^3\Big(\lambda^T f_{u'''}(t,u,u',u'',u''') + s(t)^T g_{u'''}(t,u,u',u'',u''') + r(t)^T h_{u'''}(t,u,u',u'',u''')) = 0$$

This, by integrating by parts and using the boundary conditions as earlier, implies

$$\int_I (x-u)^T \Big[(\lambda^T f_u(t,u,u',u'',u''') + s(t)^T g_u(t,u,u',u'',u''') + r(t)^T h_u(t,u,u',u'',u''')$$
$$+ (x'-u')^T \Big(\lambda^T f_{u'}(t,u,u',u'',u''') + s(t)^T g_{u'}(t,u,u',u'',u''') + r(t)^T h_{u'}(t,u,u',u'',u''')) +$$
$$(x''-u'')^T \Big(\lambda^T f_{u''}(t,u,u',u'',u''') + s(t)^T g_{u''}(t,u,u',u'',u''') + r(t)^T h_{u''}(t,u,u',u'',u''')) \,]\, dt = 0$$

Using this in (14) we have

$$\int_I \bar{\lambda}^T f(t,x,x',x'',x''')dt \geq \int_I \bar{\lambda}^T f(t,u,u',u'',u''')dt$$
$$- \int_I (x-u)^T \Big(s(t)^T g_u(t,u,u',u'',u''') + r(t)^T h_u(t,u,u',u'',u''')\Big)$$
$$+ \int_I (x'-u')^T \Big(s(t)^T g_{u'}(t,u,u',u'',u''') + r(t)^T h_{u'}(t,u,u',u'',u''')\Big)$$
$$+ \int_I (x''-u'')^T \Big(s(t)^T g_{u''}(t,u,u',u'',u''') + r(t)^T h_{u''}(t,u,u',u'',u''')\Big)$$
$$+ \int_I (x'''-u''')^T \Big(s(t)^T g_{u'''}(t,u,u',u'',u''') + r(t)^T h_{u'''}(t,u,u',u'',u''')\Big) \,]\, dt$$

By hypothesis (H_2), this implies

$$\int_I \bar{\lambda}^T f(t,x,x',x'',x''')dt \geq \int_I \bar{\lambda}^T f(t,u,u',u'',u''')dt$$

$$+ \int_I \left(s(t)^T g(t,u,u',u'',u''') + r(t)^T h(t,u,u',u'',u''')\right)dt$$

$$- \int_I \left(s(t)^T g(t,u,u',u'',u''') + r(t)^T h(t,u,u',u'',u''')\right)dt,$$

Since $x \in L_p$,this implies

$$\int_I \lambda^T f(t,x,x',x'',x''')dt \geq \int_I (\lambda^T f(t,u,u',u'',u''') + s(t)^T g(t,u,u',u'',u''') + r(t)^T g(t,u,u',u'',u'''))dt$$ The following theorem gives a situation in which a Pareto optimal solution of (VP) exists

Theorem 4.2. Suppose

(F_1) : y (t) $\in L_p$ $:(\bar{u}(t),\bar{s}(t),\bar{r}(t)) \in L_D$;

(F_2) : $\int_I \bar{\lambda}^T f(t,.,...)$ *and* $\int_I s(t)^T g(t,.,..) + r(t)^T h(t,.,..)dt$ are convex.

(F_3)

$$\int_I \bar{\lambda}^T f(t,x,x',x'',x''')dt = \int_I (\bar{\lambda}^T f(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''') + s(t)^T g(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''') + r(t)^T h(t,u,\bar{u}',\bar{u}'',\bar{u}'''))dt$$

Then y (t) and $(\bar{s}(t),\bar{r}(t),\bar{u}(t))$ are optimal solutions of (P_λ) and (WCD_λ).Hence y (t) is a Pareto optimal solution of (VP).

Proof: Suppose y (t) does not minimize (P) then there exist, $\bar{y}(t) \in L_p$ such that

$$\int_I \bar{\lambda}^T f(t,\bar{y}(t),\bar{y}'(t),\bar{y}''(t),\bar{y}'''(t))dt < \int_I \bar{\lambda}^T f(t,\bar{x},\bar{x}',\bar{x}'',\bar{x}''')dt$$

$$= \int_I (\bar{\lambda}^T f(t,u,\bar{u}',\bar{u}'',\bar{u}''') + s(t)^T g(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''') + r(t)^T h(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}'''))dt$$

This contradicts the conclusion of Theorem 4.1.Hence y (t) minimizes (P_λ).

We can similarly prove that prove that $(\bar{s}(t),\bar{r}(t),\bar{u}(t))$ maximizes (WC D_λ) and y (t) is a Pareto optimal solution of (VP), follows from proposition 2.1.

Theorem 4.3. Assume

(1) $\bar{x}(t) \in L_p;(\bar{s}(t),\bar{r}(t),\bar{u}(t)) \in L_D$

(2) $\int_I (\bar{\lambda}^T f(t,.,...) + \int_I s(t)^T g(t,.,..) + r(t)^T h(t,.,..))dt$ is convex.

(3)

$$\int_I \bar{\lambda}^T f(t,y,,\bar{y}',\bar{y}'',\bar{y}''')dt = \int_I (\bar{\lambda}^T f(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''') + s(t)^T g(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''') + r(t)^T h(t,u,\bar{u}',\bar{u}'',\bar{u}'''))dt$$

Then $\int_I s(t)^T g(t,\bar{x},\bar{x}',\bar{x}'',\bar{x}''')dt = 0, t \in I$

Proof: By hypothesis (2) and (3), we have

$$\int_I \bar{\lambda}^T f(t,y,,y',y'',y''')dt = \int_I (\bar{\lambda}^T f(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''') + s(t)^T g(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''') + r(t)^T h(t,u,\bar{u}',\bar{u}'',\bar{u}'''))dt$$

$$\leq \int_I (\bar{\lambda}^T f(t,y,,y',y'',y''') + \bar{s}(t)^T g(t,y,,y',y'',y''') + \bar{r}(t)^T h(t,y,,y',y'',y'''))dt$$

$$-\int_I \Big[(y-\bar{u})^T\Big(\bar{\lambda}^T f_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T g_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\Big)+$$
$$(y'-\bar{u}')^T\Big(\bar{\lambda}^T f_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T g_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\Big)+$$
$$(y''-\bar{u}'')^T\Big(\bar{\lambda}^T f_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T g_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\Big)+$$
$$(y'''-\bar{u}''')^T\Big(\bar{\lambda}^T f_{u'''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T g_{u'''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h_{u'''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\Big)\Big]dt$$
$$=\int_I (\bar{\lambda}^T f(t,y,,y',y'',y''')+\bar{s}(t)^T g(t,y,,y',y'',y''')+\bar{r}(t)^T h(t,y,,y',y'',y'''))dt\ -$$
$$\int_I \Big[(y-\bar{u})^T\Big(\bar{\lambda}^T f_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T g_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\Big)-$$
$$(y-\bar{u})^T\Big(\bar{\lambda}^T f_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T g_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\Big)\Big|_{t=a}^{t=b}-$$
$$\int_I \Big[(y-u)^T D\Big(\bar{\lambda}^T f_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T g_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\Big)dt+$$
$$(y'-\bar{u}')^T\Big(\bar{\lambda}^T f_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T g_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\Big)\Big|_{t=a}^{t=b}-$$
$$\int_I \Big[(y'-\bar{u}')^T D^2\Big(\bar{\lambda}^T f_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T g_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\Big)dt$$
$$+(y''-\bar{u}'')^T\Big(\bar{\lambda}^T f_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T g_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\Big)dt$$
$$+(y'''-\bar{u}''')^T\Big(\bar{\lambda}^T f_{u'''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T g_{u'''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h_{u'''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\Big)]dt$$

Integrating by parts,

$$=\int_I (\bar{\lambda}^T f(t,y,,y',y'',y''')+\bar{s}(t)^T g(t,y,,y',y'',y''')+\bar{r}(t)^T h(t,y,,y',y'',y'''))dt\ +$$
$$\int_I \Big[(y-\bar{u})^T\Big(\bar{\lambda}^T f_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T g_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\Big)-$$
$$(y-\bar{u})^T D\Big(\bar{\lambda}^T f_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T g_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\Big)]dt\ -$$
$$(y-\bar{u})^T D\Big(\bar{\lambda}^T f_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T g_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\Big)\Big|_{t=a}^{t=b}+$$
$$\int_I (y-\bar{u})^T D^2\Big(\bar{\lambda}^T f_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T g_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\Big)dt$$

(Using boundary conditions and integrating by parts)

$$=\int_I (\bar{\lambda}^T f(t,y,,y',y'',y''')+\bar{s}(t)^T g(t,y,,y',y'',y''')+\bar{r}(t)^T h(t,y,,y',y'',y'''))dt\ +$$
$$\int_I \Big[(y-\bar{u})^T((\bar{\lambda}^T f_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T g_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\Big)-$$
$$D\Big(\bar{\lambda}^T f_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T g_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\Big)]dt\ -$$
$$D^2\Big(\bar{\lambda}^T f_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T g_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\Big))]dt$$
$$=\int_I (\bar{\lambda}^T f(t,y,,y',y'',y''')+\bar{s}(t)^T g(t,y,,y',y'',y''')+\bar{r}(t)^T h(t,y,,y',y'',y'''))dt$$

Using (13)

This implies $\int_I \left(\bar{s}(t)\right)^T g(t,y,y',y'',y''') + \bar{r}(t)h(t,y,y',y'',y'''))dt \geq 0$ (15)

But $\bar{s}(t) \geq 0, g(t,y,y',y'',y''') \leq 0 \; and \; h(t,y,y',y'',y''') = 0$, $t \in I$ yield

$$\int_I \left(\bar{s}(t)^T g(t,y,y',y'',y''') + \bar{r}(t)^T h\left(t,y,y',y'',y'''\right)\right)dt \leq 0 \quad (16)$$

From (15) and (16), we have

$$\int_I \left(\bar{s}(t)^T g(t,y,y',y'',y''') + \bar{r}(t)^T h\left(t,y,y',y'',y'''\right)\right)dt = 0$$

This, because of $h(t,y,y',y'',y''') = 0$, $t \in I$,gives $\int_I \bar{s}(t)^T g(t,y,y',y'',y''')dt = 0$

This, together with $\bar{s}(t)^T g(t,y,y',y'',y''') \leq 0$, $t \in I$, implies $\bar{s}(t)^T g(t,y,y',y'',y''') = 0$, $t \in I$

Theorem 4.4. Suppose

(1) $\left(\bar{s}(t)^T, \bar{r}(t)^T, u(t)\right) \in L_D$ and $\bar{u}(t) \in L_P$;

(2) $\bar{s}(t)^T g(t,u,\bar{u}',\bar{u}'',\bar{u}''') = 0$, $t \in I$

(3) $\int_I \bar{\lambda}^T f(t,.,...)dt \quad and$

$\int_I \left(\bar{s}(t)^T g(t,.,..) + \bar{r}(t)^T h(t,.,..)\right)dt$ are convex;

Then $\bar{u}(t)$ is an optimal solution of (P_λ) and hence of (VP).

Proof: If $\bar{u}(t)$ is the only feasible solution of (P_λ), then there is nothing to prove so, assume that y (t) is another feasible solution of (P_λ). Then by hypothesis (1) and (3),

We have $\int_I \bar{\lambda}^T f(t,y,y',y'',y''')dt \geq \int_I \bar{\lambda}^T f_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')dt$

$$\int_I \left[(y-\bar{u})^T\left(\bar{\lambda}^T f_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}'''))+(y'-\bar{u}')^T(\bar{\lambda}^T f_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\right)+(y''-\bar{u}'')^T(\bar{\lambda}^T f_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}'''))+(y'''-\bar{u}''')^T(\bar{\lambda}^T f_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}'''))\right]dt$$

Now integrating by parts, we have, $\int_I \bar{\lambda}^T f(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')dt - \int_I \left[(y-\bar{u})^T \bar{\lambda}^T f_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')dt\right.$ -

$$(y-\bar{u})^T\left(\bar{\lambda}^T f_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\right)\Big|_{t=a}^{t=b} - \int_I (x-u)^T D\left(\bar{\lambda}^T f_{u'}(t,,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\right)]dt +$$

$$(y'-\bar{u}')^T\left(\bar{\lambda}^T f_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\right)\Big|_{t=a}^{t=b} - \int_I \left(y'-\bar{u}'\right)^T D\left(\bar{\lambda}^T f_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\right)dt$$

$$= \int_I \bar{\lambda}^T f(t,u,\bar{u}',\bar{u}'',\bar{u}''')dt + \int_I \left(y-\bar{u}\right)^T D\left(\bar{\lambda}^T f_{u'}\left(t,u,\bar{u}',\bar{u}'',\bar{u}'''\right)\right)dt + \int_I \left(y'-\bar{u}'\right)^T D\left(\left(\bar{\lambda}^T f_{u''}\left(t,u,\bar{u}',\bar{u}'',\bar{u}'''\right)\right)\right)dt$$

$$+\int_I \left(y''-\bar{u}''\right)^T D\left(\bar{\lambda}^T f_{u''}\left(t,u,\bar{u}',\bar{u}'',\bar{u}'''\right)\right) \text{dt.}$$

Using boundary conditions (11)

$$= \int_I \bar{\lambda}^T f\left(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}'''\right)dt + \int_I \left(y-\bar{u}\right)^T\left[\bar{\lambda}^T f\left(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}'''\right) - D\left(\bar{\lambda}^T f_{u'}\left(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}'''\right)\right) + \right.$$

$$\left. +D^2\left(\bar{\lambda}^T f_{u''}\left(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}'''\right)\right) + D^3\left(\bar{\lambda}^T f_{u'''}\left(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}'''\right)\right)\right]dt$$

$$= \int_I \left(\lambda^T f(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\right)dt - \int_I (y-\bar{u})^T\left[\bar{s}(t)^T g_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''') + \bar{r}(t)^T h_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\right.$$

$$- D\left(\bar{s}(t)^T g_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''') + \bar{r}(t)^T h_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\right)$$

$$+D^2\left(\bar{s}(t)^T g_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\right)$$
$$-D^3\left(\bar{s}(t)^T g_{u'''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h_{u'''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\right)$$

Thus, by integrating by parts and using boundary conditions, as earlier, we get

$$=\int_I \bar{\lambda}^T f(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')dt-\int_I\Big[(y-\bar{u})^T\left(\bar{s}(t)^T g_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T h_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\right)$$
$$+(y'-\bar{u}')^T\left(\bar{s}(t)^T g_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T h_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\right)$$
$$+(y''-\bar{u}'')^T\left(\bar{s}(t)^T g_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T h_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\right)$$
$$+(y'''-\bar{u}''')^T\left(\bar{s}(t)^T g_{u'''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{s}(t)^T h_{u'''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\right)\Big]dt$$
$$\geq\int_I\left(\lambda^T f(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\right)dt+\int_I\left[\bar{s}(t)^T g(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\right]dt$$
$$-\int_I\left[\bar{s}(t)^T g(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+\bar{r}(t)^T h(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\right]dt$$
$$\geq\int_I\left(\lambda^T f(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\right)dt$$

Using hypothesis (A_1) and (A_2) and $x\in L_P$

This implies that $\bar{u}$ minimizes $\int_I \bar{\lambda}^T f(t,y,y',y'',y''')dt \quad over\ L_P$.

Remark: In Theorem 4.4, we assume that a part of feasible solution of (WC D_λ) is a feasible solution of (P_λ) .It is a natural question if there is any set of appropriate conditions under which this assumption is true. The following theorem gives one such set of conditions.

Theorem 4.5: Assume

(1) x (t) $\in L_P \quad and\left(\bar{s}(t)^T,\bar{r}(t)^T,u(t)\right)\in L_P$

(2) g (t,.,.,…..) and h (t,.,,,,,) are differential convex functions.

(3) $\int_I\left(g(t,u,u',u'',u''')+h(t,u,u',u'',u''')\right)dt=0$

(4) $(x-u)^T g_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+(x'-u')^T g_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+$
$(x''-u'')^T g_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+(x'''-u''')^T g_{u'''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\geq 0\ ,t\in I$

(5) $(x-u)^T h_u(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+(x'-u')^T h_{u'}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')$
$+(x''-u'')^T h_{u''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')+(x'''-u''')^T h_{u'''}(t,\bar{u},\bar{u}',\bar{u}'',\bar{u}''')\geq 0\ ,t\in I$

Then $\bar{u}\in L_p$.

Proof: By convexity of g (t...,…) and h (t…,…), we have

$$g(t,x,x',x'',x''')\geq g(t,u,u',u'',u''')+(x-u)^T g_u(t,u,u',u'',u''')$$
$$+(x'-u')^T g_{u'}(t,u,u',u'',u''')+(x''-u'')^T g_{u''}(t,u,u',u'',u''') \tag{17}$$
$$+(x'''-u''')^T g_{u'''}(t,u,u',u'',u''')\geq 0$$

$$h(t,x,x',x'',x''') \geq h(t,u,u',u'',u''') + (x-u)^T h_u(t,u,u',u'',u''')$$
$$+(x'-u')^T h_{u'}(t,u,u',u'',u''') + (x''-u'')^T h_{u''}(t,u,u',u'',u''') \qquad (18)$$
$$+(x'''-u''')^T h_{u'''}(t,u,u',u'',u''') \geq 0$$

Using (13) and (14) together with the hypothesis (1),(2) and (3) , we have

$$g(t,u,u',u'',u''') \leq 0 \ \ ,t \in I \qquad (19)$$

and

$$h(t,u,u',u'',u''') \leq 0 \ \ ,t \in I \qquad (20)$$

By (15) and (16), we have

$$g(t,u,u',u'',u''') + h(t,u,u',u'',u''') \leq 0 \ ,t \in I \qquad (21)$$

The hypotheses (2) with (17) implies

$$g(t,u,u',u'',u''') + h(t,u,u',u'',u''') = 0 \ ,t \in I \qquad (22)$$

But $g(t,u,u',u'',u''') \leq 0 \ \ ,t \in I$ .Hence by (22) we have

$$h(t,u,u',u'',u''') \geq 0 \ \ ,t \in I \qquad (23)$$

The inequalities (20) and (21) imply

$$h(t,u,u',u'',u''') = 0 \ \ ,t \in I \qquad (24)$$

The relations (19) and (24) imply that $\bar{u} \in L_p$.

V. Variational Problems With Natural Boundary Values.

It is possible to construct variational problems with natural boundary values rather than the problem with fixed end point considered in the preceding sections. The problems of section 2 can be formulated as follows:

$$(VP)_N: \ \max imize\left(\int_I f'(t,x,x',x'',x''')dt,.......,\int_I f'(t,x,x',x'',x''')dt\right)$$

Subject to

$$g(t,x,x',x'',x''') \leq 0, \ t \in I$$
$$h(t,x,x',x'',x''') = 0, \ \ t \in I$$

$$(M-WD)_N : Maximize\left(\int_I f'(t,u,u',u'',u''')dt,....,\int_I f'(t,u,u',u'',u''')dt\right)$$

$$\left(\lambda^T f_u(t,u,u',u'',u''')\right) + s(t) g_u(t,u,u',u'',u''') + r(t)^T h_u(t,u,u',u'',u''')$$
$$-D\left(\lambda^T f_{u'}(t,u,u',u'',u''')\right) + s(t)^T g_{u'}(t,u,u',u'',u''') + r(t)^T h_{u'}(t,u,u',u'',u''')$$
$$+D^2\left(\lambda^T f_{u''}(t,u,u',u'',u''')\right) + s(t)^T g_{u''}(t,u,u',u'',u''') + r(t)^T h_{u''}(t,u,u',u'',u''')$$
$$-D^3\left(\lambda^T f_{u'''}(t,u,u',u'',u''')\right) + s(t)^T g_{u'''}(t,u,u',u'',u''') + r(t)^T h_{u'''}(t,u,u',u'',u''') = 0$$

$$\int_I \left(s(t)^T g(t,u,u',u'',u''') + r(t)^T h(t,u,u',u'',u''')\right)dt \geq 0 \, , \lambda > 0 \ , s(t) \geq 0, t \in I$$

$$\lambda^T f_{u'}(t,u,u',u'',u''') + s(t)^T g_{u'}(t,u,u',u'',u''') + r(t)^T h_{u'}(t,u,u',u'',u''') = 0, \text{ t=a, t=b}$$
$$\lambda^T f_{u''}(t,u,u',u'',u''') + s(t)^T g_{u''}(t,u,u',u'',u''') + r(t)^T h_{u''}(t,u,u',u'',u''') = 0, \text{ t=a, t=b}$$
$$\lambda^T f_{u'''}(t,u,u',u'',u''') + s(t)^T g_{u'''}(t,u,u',u'',u''') + r(t)^T h_{u'''}(t,u,u',u'',u''') = 0, \text{ t=a, t=b}$$

The above conditions are popularly known as natural boundary conditions in calculus of variation.

Theorem of the section 3 for $(VP)_N$ *and* $(M-WD)_N$ can easily be proved in view of earlier analysis in this research. The problems of the section 4 can be written with natural boundary values as follows;

For given $0 < \lambda < R^n$

$$\left(P_{\lambda}\right)_N : Minimize \int_I \lambda^T f(t, x, x', x'', x''')dt$$

Subject to

$$g(t, x, x', x'', x''') \leq 0, t \in I$$
$$h(t, x, x', x'', x''') = 0, t \in I$$

$$\left(WCD_{\lambda}\right)_N : Maximize \int_I \left(\lambda^T f(t,u,u',u'',u''')\right) + s(t)^T g(t,u,u',u'',u''') + r(t)^T h(t,u,u',u'',u''')$$

$$\left(\lambda^T f_u(t,u,u',u'',u''')\right) + s(t) g_u(t,u,u',u'',u''') + r(t)^T h_u(t,u,u',u'',u''')$$
$$-D\left(\lambda^T f_{u'}(t,u,u',u'',u''')\right) + s(t)^T g_{u'}(t,u,u',u'',u''') + r(t)^T h_{u'}(t,u,u',u'',u''')$$
$$+D^2\left(\lambda^T f_{u''}(t,u,u',u'',u''')\right) + s(t)^T g_{u''}(t,u,u',u'',u''') + r(t)^T h_{u''}(t,u,u',u'',u''')$$
$$-D^3\left(\lambda^T f_{u'''}(t,u,u',u'',u''')\right) + s(t)^T g_{u'''}(t,u,u',u'',u''') + r(t)^T h_{u'''}(t,u,u',u'',u''') = 0$$
$$\lambda^T f_{u'}(t,u,u',u'',u''') + s(t)^T g_{u'}(t,u,u',u'',u''') + r(t)^T h_{u'}(t,u,u',u'',u''') = 0, \text{ t=a, t=b}$$
$$\lambda^T f_{u''}(t,u,u',u'',u''') + s(t)^T g_{u''}(t,u,u',u'',u''') + r(t)^T h_{u''}(t,u,u',u'',u''') = 0, \text{ t=a, t=b}$$
$$\lambda^T f_{u'''}(t,u,u',u'',u''') + s(t)^T g_{u'''}(t,u,u',u'',u''') + r(t)^T h_{u'''}(t,u,u',u'',u''') = 0, \text{ t=a, t=b}$$
$$\lambda > 0, s(t) \geq 0 \ , t \in I \ ./\!/\!/$$

VI. Conclusion:

In this paper, we have generalized duality in (MOVP) with Mond-Weir type & Wolf type duality by considering 3rd order derivative and prove the Propositions and Theorems.

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Conflict of Interest

Declared none

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