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Solution of some stochastic differential equation

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Abstract: *In this paper we study the method of solution of some stochastic differential equations of first order by using the ito integral and ito formula*

الخــلاصـــة:
في هذه الورقة تمت دراسة طرق حلول بعض المعادلات التفاضلية التصادفية ذات الرتبة الاولي باستخدام تكامل وصيغة اتو

Key words: *stochastic differential equations, ito integral ito formula*

I. Introduction:

Consider the simple population growth model
$\frac{dN}{dt} = a(t)N(t) \quad , N(0) = N_0(constant)$ *(1)*

Where $N(t)$, *is the size of population at time t and* $a(t)$ *is the rate of growth at time t.* $a(t)$ *Is not completely known*
$a(t) = r(t) + noise\ term\ where$ *We don't know the exact behavior of the noise term, the function* $r(t)$ *is assumed to be non random*
How do we solve (1) in this case??
In this paper we discuss the method for solving similar of the above $example$.
(2) The basic concept of stochastic differential equations

<u>1-probability space:</u>
The triple $(\Omega, \mathcal{F}, P)$is called probability space.

<u>2-stochastic process</u>:
A stochastic process is a collection of random variables
$\{X_t\}_{t\in T}, X_t: \Omega \to R^n$ Defined on $(\Omega, \mathcal{F}, P)$.

3-<u>Brownian motion</u>:
A Brownian motion is a random variable satisfies the following:
1-$B(0) = 0$
2-$B(t)$is continuous functions of t
3-$B(t)$ Has independent normally distributed increments

<u>2-Ito integral</u>:

Consider the example

$$\frac{dN}{dt} = (r(t) + noise\ term\ )N(t)$$

Or
$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t).noise\ term$ (2)
Where w_t the white noise $X_t$is the stochastic process
By integrating example (2) from $k$into $k + 1$ we get

$$\int_k^{k+1} \frac{dX}{dt} = \int_k^{k+1} b(t, X_t)dt + \sigma(t, X_t).(noise)\ d\boldsymbol{t}$$

$$X_{k+1} - X_k = b(t_k, X_k)\Delta t_k + \sigma(t_k, X_k)w_k\Delta t_k \qquad \textbf{(3)}$$

$$\ni \Delta t_k = t_{k+1} - t_k$$

$$X_j = X(t_j), w_k = w_{t_k}$$

We replace $w_k\Delta t_k$in (3) byΔV_k such that $E(V_t)_{t\geq 0}$is stochastic process .let $V_t = B_t$
3→ $X_k = X_0 = \sum_{j=0}^{k-1} b(t_j, X_j)\Delta t_j + \sum_{j=0}^{k-1} \sigma(t_j, X_j)\Delta B_j$ (4)
Then apply the usual integration notation we should obtain
$X_t = X_0 = \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s) dB_s$ (5)

The last formula (5) is the ito integral.

<u>The other definition of ito integral</u>:
For function $f \in V$, $V = V(s,t)$ be the class of functions $f(t,w)$: $[0,\infty[\times \Omega \to \mathbb{R}$ we define the ito integral
$I(f)(w) = \int_s^T f(t,w)\, dB_t(w)$ (6)
<u>Lemma1-2</u> :((the ito isometry):
If $\emptyset(t,w)$ is bounded and elementary (if $\emptyset \epsilon V$ is called elementary if it has the form $\emptyset(t,w) = \sum_j e_j(w) . X_{[t_j,t_{j+1}](t)}$)
Then $E\left[\left(\int_S^T \emptyset(t,w)\, dB_t(w)\right)^2\right] = E\left[\int_S^T \emptyset(t,w)^2 dt\right]$ (7)
<u>Proof:</u>
See page 26 in[1]

<u>Definition (ito process)</u>:
The 1-dimention ito process:
Let B_t be 1- dimensional Brownian motion on $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \boldsymbol{P})$ An ito processes is stochastic process X_t in of the form
$X_t = X_0 + \int_0^t u(S,w)dS + \int_0^t v(S,w)\, dB_S$ (8)
Where $v \in w_H$, so that

$$P\left[\int_0^t v(S,w)^2\, dS < \infty \quad \forall t \geq 0\right] = 1$$

<u>Example1-2</u>:
Prove directly from the definition of ito integrals that:
$\int_0^t S\, dB_s = t\, B_t - \int_0^t B_s ds$
<u>Solution:</u>
Using the notation $\Delta x_j = x_{j+1} - x_j$

$$\sum_j \Delta(S_j B_j) = \sum_j S_j \Delta B_j + \sum_j B_{j+1} \Delta S_j$$

There fore

$$\sum_j \Delta(t_j B_j) = \sum_j t_j \Delta B_j + \sum_j B_{j+1} \Delta t_j$$

By integrating

$$\int_0^t \Delta(t_j B_j) = \int_0^t S dB_s + \int_0^t B_s ds$$

$$tB_t = \int_0^t S dB_S + \int_0^t B_S\, dS$$

$$\int_0^t S dB_S = t\, B_t - \int_0^t B_S\ dS$$

<u>Example2-2:</u>
Prove from the definition of ito integral that

$$\int_0^t B_S^2 dB_S = \frac{1}{3} B_t^3 - \int_0^t B_S dS$$

<u>Solution</u>:

$$B_t^3 = \sum_j \Delta B_j^3 = \sum_j B_{j+1}^3 - B_j^3$$

$$= \sum_j (B_{j+1} - B_j)^3 + 3B_{j+1}^2 B_j - 3B_j^2 B_{j+1}$$

$$= \sum_j (B_{j+1} - B_j)^3 + 3 \sum_j B_{j+1} B_j (B_{j+1} - B_j)$$

$$B_t^3 = \sum_j (\Delta B_j)^3 + 3 \sum_j B_{j+1} B_j \Delta B_j$$

$$\sum_j B_j B_j \Delta B_j = \frac{1}{3} B_t^3 - \frac{1}{3} \sum_j (\Delta B_j)^3$$

$$\sum_j B_j B_j \Delta B_j = \frac{1}{3} B_t^3 - \sum_j (\Delta B_j)^2 dt_j$$

Where $\sum_j (\Delta B_j)^2 \to B_j$
There fore $\sum_j B_j B_j \Delta B_j = \frac{1}{3} B_t^3 - \sum_j B_j\, dt_j$
By integrating the last equation from 0 into t we get

$$\int_0^t B_S^2 dB_S = \frac{1}{3} B_t^3 - \int_0^t B_S \, dS$$

(3)ito formula:

Theorem1-3:
The 1-dimensional ito formula
Let X_t be an ito process given by

$$dX_t = udt + vdB_t$$

Let $g(t,x) \in C^2([0,\infty[\times R)$, then
$Y_t = g(t, X_t)$ is also an ito process and
$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial X}(t, X_t)dX_t + \frac{1}{2\,\partial}\frac{\partial^2 g}{X^2}(t, X_t)(dX_t)^2$ (9)
Where $(dX_t)^2 = dX_t \times dX_t$

$$dt \times dt = dt\, dB_t = dB_t \times dt = 0$$
$$dB_t \times dB_t = dt$$

Proof:
See page 44in[1]
Example1-3:
Use ito formula to write the stochastic process

$$Y_t = B_t^2$$

On the standard form

$$dY_t = u(t,w)dt + v(t,w)dB_t$$

Solution:
By the 1-dimentional ito formula:
Since $Y_t = B_t^2$
There fore $d(B_t^2) = 2B_t\, dB_t + dt$
$d(B_t^2) = dt + 2B_t\, dB_t$
Where $u(t,w) = 1,$ $v(t,w) = 2B_t$
Example2-3:
Use the ito formula to write the stochastic process
$Y_t = (2 + t + e^{B_t})$
Solution:
By the 1-dimensional ito formula:

$$dY_t = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial X}dB_t + \frac{1}{2}\frac{\partial^2 g}{\partial X^2}(dB_t)^2$$
$$dY_t = dt + e^{B_t}\, dB_t + \frac{1}{2}e^{B_t}\, dt$$
$$dY_t = \left(1 + \frac{1}{2}e^{B_t}\right)dt + e^{B_t}dB_t$$

Where $u(t,w) = \left(1 + \frac{1}{2}e^{B_t}\right)$, $v(t,w) = e^{B_t}$
The multi-dimensional ito formula:
Let $B_1(t,w) = \big(B_1(t,w), \dots, B_m(t,w)\big)$
Denotes m-dimensional Brownian motion.
If each of the processes $u_i(t,w)$, $v_{ij}(t,w)$ satisfies equation (9) then we can form the following n-ito processes

$$\begin{cases} dX_1 = u_1 dt + v_{11} dB_1 + \cdots + v_{1m} dB_m \\ \vdots \\ dX_n = u_n dt + v_{n1} dB_1 + \cdots + v_{nm} dB_m \end{cases} \quad \textbf{(10)}$$

Or in matrix notation simply

$$dX(t) = udt + vdB_t$$

Where $X(t) = \begin{bmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{bmatrix}$, $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $v = \begin{bmatrix} v_{11} & \cdots & v_{1m} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nm} \end{bmatrix}$, $dB_t = \begin{bmatrix} dB_1 \\ \vdots \\ dB_2 \end{bmatrix}$

Theorem2-3 (the general ito formula):
Let $dX(t) = udt + vdB_t$
Let $g(t,X) = \big(g_1(t,X), g_2(t,X), \cdots, g_p(t,X)\big)$
Be $C^2: [0,\infty[\times \mathbb{R}^n \to \mathbb{R}^p$
Then $Y(t,w) = g\big(t, X(t)\big)$

The general ito formula is

$$dY_k = \frac{\partial g_k}{\partial t}(t,X)dt + \sum_i \frac{\partial g_k}{\partial X_i}(t,X)\,dX_i + \frac{1}{2}\sum_{i,j}\frac{\partial^2 g_k}{\partial X_i \partial X_j}(t,X)dX_i dX_j$$

Where $dB_i dB_j = \delta_{ij}\,dt\,, dB_i dt = dtdB_i = 0$

Proof:

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Example3-3:

Use the general ito formula to write the stochastic process Y_t on the standard form

$dY_t = u(t,w)dt + v(t,w)dB_t$

If $Y_t = \left(B_1(t) + B_2(t) + B_3(t), B_2^2(t) - B_1(t) - B_3(t)\right)$

Solution:

$$dX_1(t) = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial X_t}dB_1(t) + \frac{\partial g}{\partial X_t}dB_2(t) + \frac{\partial g}{\partial X_t}dB_3(t)$$

$$= dB_1(t) + dB_2(t) + dB_3(t)$$

$$dX_2(t) = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial X_t}dB_t(t) + \frac{1}{2}\frac{\partial^2 g}{\partial {X_t}^2}(dB_t)^2 + \frac{\partial g}{\partial X_t}dB_3(t)$$

$$= 2B_2 dB_2 + \frac{1}{2}\cdot 2dt - B_3 dB_1(t) - B_1 dB_3(t)$$

Or in matrix form

$$dX_t = \begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt + \begin{bmatrix} 1 & 1 & 1 \\ -B_3(t) & 2B_2(t) & -B_1(t) \end{bmatrix} \begin{bmatrix} dB_1(t) \\ dB_2(t) \\ dB_3(t) \end{bmatrix}$$

4-Solution of stochastic differential equation:

The solution of stochastic differential equations

$\frac{dX_t}{dt} = b(t,X_t) + \sigma(t,X_t)W_t\ \ , b(t,X_t)\,, \sigma(t,X_t) \epsilon \mathbb{R}$ (12)

W_t Is the 1-dimentional white noise

Is $X_t = X_0 + \int_0^t b(s,X_s)\,ds + \int_0^t \sigma(s,X_s)\,dB_s$

Or in differential form

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t$$

Example1-4:

Verify that the given processes solve corresponding stochastic differential equation: $X_t = e^{B_t}$ Solves $dX_t = \frac{1}{2}X_t dt + X_t dB_t$

Solution:

Let $X_t = g$

Use the 1-dimentional ito formula

$$dY_t = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial X}dB_t + \frac{1}{2}\frac{\partial^2 g}{\partial X^2}(dB_t)^2$$

We get, $dX_t = e^{B_t}dB_t + \frac{1}{2}e^{B_t}(dB_t)^2$

$\Rightarrow dX_t = X_t dB_t + \frac{1}{2}X_t dt$

$\Rightarrow dX_t = \frac{1}{2}X_t dt + X_t dB_t$

Example 2-4:

Verify that the given processes solve corresponding stochastic differential equation

$(X_1, X_2) = (\cosh(B_t), sinh(B_t))$

Solves $\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} dt + \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} dB_t$

Solution:

We use the general ito formula

$$dY_k = \frac{\partial g_k}{\partial t}(t,X)dt + \sum_i \frac{\partial g_k}{\partial X_i}(t,X)\,dX_i + \frac{1}{2}\sum_{i,j}\frac{\partial^2 g_k}{\partial X_i \partial X_j}(t,X)dX_i dX_j$$

$$dX_1 = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial X}dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial X^2}(dX_t)^2$$

$= \sinh(B_t)\, dB_t + \frac{1}{2} cosh(B_t)(dB_t)^2 =$

$$= X_2 dB_t + \frac{1}{2} X_1 dt$$

$$= \frac{1}{2} X_1 dt + X_2 dB_t$$

$$dX_2 = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial X} dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial X^2}(dX_t)^2 = coshB_t dB_t + \frac{1}{2}\sinh B_t\,(dB_t)^2$$

$$= \frac{1}{2} X_2 dt + X_1 dB_t$$

Therefore, the given processes solves the given stochastic differential equation s

<u>Example3- 4:</u>

<u>(exponential growth with noise)</u>

Solve the following stochastic differential equation

$dX_t = X_t dt + dB_t$

<u>Solution</u>:

Multiply both sides with the integrating factor e^{-t}

Thus,

$$e^{-t} dX_t = e^{-t} X_t dt + e^{-t} dB_t$$

On the other hand applying the stochastic rule

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t$$

We have

$$d(e^{-t} X_t) = X_t de^t + e^{-t} dX_t + de^{-t} dX_t$$

$$= -e^{-t} X_t dt + e^{-t} dX_t - e^{-t}(X_t dt + dB_t)$$

$$= -e^{-t} X_t dt + e^{-t}(X_t dt + dB_t) X_t - e^{-t} dt(X_t dt + dB_t)$$

$$= -e^{-t} X_t dt + e^{-t} X_t dt + e^{-t} dB_t - e^{-t} X_t (dt)^2 + e^{-t} dt\, dB_t$$

$$= e^{-t} dB_t$$

Integrating both sides, we have

$$\int_0^t d(e^{-t} X_t) = \int_0^t e^{-t} dB_t$$

$$e^{-t} X_t - X_0 = \int_0^t e^{-t} dB_t \quad , X_0 = C$$

$$e^{-tX_t} = C + \int_0^t e^{-t} d\, B_t$$

$$X_t = Ce^t + e^t \int_0^t e^{-t} dB_t$$

We have the expected value

$$E[\, X_t] = E[Ce^t] + E\left[e^t \int_0^t e^{-t} dB_t\right]$$

$$= Ce^t$$

II. Conclusion:

The above method of solution of some stochastic differential equations is a good method for the equations which contain the random variable and their solution depends on the given an ito integral and an ito formula which shows above .in the next papers I will discuss the solution of second order stochastic differential equation also discuss the solution of partial stochastic differential equations.

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