IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728. Volume 5, Issue 1 (Jan. - Feb. 2013), PP 31-35
www.iosrjournals.org

A Study of Double Integral Transformation of Generalized Hypergeometric Function

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Abstract: *In the present paper the authors will establish a double integral transform of Fox's H-function which leads to yet another interesting process of augmenting the parameters in the H-function. The result is of general character and on specializing the parameters suitably, yields several interesting results as particular cases.*

Key words: *H-function, Euler Transformation, Hypergeometric Function, Integral Transformation.(2000 Mathematics subject classification: 33C99)*

I. Introduction

Rainville [7, p.104], Abdul Halim and Al-Salam [1] have shown that the single and double Euler transformations of the hypergeometric function ${}_pF_q$ are effective tools for augmenting its parameters. Srivastava and Singhal [10] and Srivastava and Joshi [11] have discussed some similar interesting properties of ${}_pF_q$ in double H-function and double Whittaker transforms respectively.

In what follows for the sake of brevity, we have used the symbols $(a_r,\alpha_r),\Delta(r,a),\Delta(r,\pm a),\Delta((r,a_p))$ to denote the set of parameters $(a_1,\alpha_1),...,(a_r,\alpha_r);\frac{a}{r},\frac{a+1}{r},...,\frac{a+r-1}{r};\Delta(r,a),\Delta(r,-a)$ and $\Delta(r,a_1),\Delta(r,a_2),...,\Delta(r,a_p)$ respectively.

The A-function is defined by Gautam and Goyal [5] and represented as

$$A_{p,q}^{m,n}\left[x\left|\begin{matrix}{}_1(a_j,\alpha_j)_p\\ {}_1(b_j,\beta_j)_q\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L f(s)x^s ds \tag{1.1}$$

Where

$$f(s)=\frac{\prod_{j=1}^{m}\Gamma(a_j+\alpha_j s)\prod_{j=1}^{n}\Gamma(1-b_j-\beta_j s)}{\prod_{j=m+1}^{p}\Gamma(1-a_j-\alpha_j s)\prod_{j=n+1}^{q}\Gamma(b_j+\beta_j s)} \tag{1.2}$$

The integral on the right hand side of (1.1) is convergent when $f>0$ and $|\arg(ux)|<\frac{f\pi}{2}$, where

$$f=\mathrm{Re}(\sum_{j=1}^{m}\alpha_j-\sum_{j=m+1}^{p}\alpha_j+\sum_{j=1}^{n}\beta_j-\sum_{j=n+1}^{q}\beta_j$$

$$u=\prod_{j=1}^{p}\alpha_j^{\alpha_j}\prod_{j=1}^{q}\beta_j^{-\beta_j} \tag{1.3}$$

(1.1) reduces to H-function given by Fox the following relation

$$A_{p,q}^{n,m}\left[x\left|\begin{matrix}{}_1(1-a_j,\alpha_j)_p\\ {}_1(1-b_j,\beta_j)_q\end{matrix}\right.\right]=H_{p,q}^{m,n}\left[x\left|\begin{matrix}{}_1(a_j,\alpha_j)_p\\ {}_1(b_j,\beta_j)_q\end{matrix}\right.\right]$$

We shall use the following notations:

$$A^*=(1-a_p,\alpha_p),B^*=(1-b_q,\beta_q)$$

$$A^{**} = (1-c_u, \gamma_u), D^{**} = (1-d_v, \delta_v)$$

II. Main Result

In this section, we have established the following double integral transform of H -function:
If s, k and r are positive integers, then

$$\int_0^\infty\int_0^\infty x^{\alpha-1} y^{\beta-1} (x+y)^\sigma A_{u,v}^{f,g}\left[\lambda(x+y)\middle|{}_{D^*}^{C^*}\right] A_{p,q}^{m,n}\left[tx^s y^k (x+y)^r \middle|{}_{B^*}^{A^*}\right] dxdy =$$

$$(2\pi)^{(1-D)\left(f+g-\frac{1}{2}u-\frac{1}{2}v\right)+\frac{1}{2}} D^{\sum_1^v d_j - \sum_1^u c_j + \left(A-\frac{1}{2}\right)(u-v)} \frac{s^{\alpha-\frac{1}{2}} k^{\beta-\frac{1}{2}}}{\lambda^{\alpha+\beta+\sigma}(s+k)^{\alpha+\beta-\frac{1}{2}}}$$

$$A_{p+\rho+Dv,q+\rho+Du}^{m+Dg,n+\rho+Df}\left[\frac{t\delta D^{D(v-u)}}{\lambda^D}\middle|{}_{\Delta((1-D,1-A-C_g)),B^*,\Delta(1-k-s,1-\alpha-\beta),\Delta(1-D,1-A-c_{g+1}),\ldots,\Delta(1-D,1-A-c_v)}^{\Delta((1-D,1-A-D_f)),\Delta(1-s,1-\alpha),\Delta(1-k,1-\beta),A^*,\Delta(1-D,1-A-d_{f+1}),\ldots,\Delta(1-D,1-A-d_u)}\right], \quad (2.1)$$

Where $\gamma = \frac{s^s k^k}{(s+k)^{s+k)}}, \rho = s+k, D = s+k+r, A = \alpha+\beta+\sigma,$

$$0 \le Dg \le Du \le Dv < Du + q - p, u + v - 2g \le 2f \le 2v, 0 \le n \le p, p + q - 2n < 2m \le 2q,$$

$$\mathrm{Re}\left(\min\frac{1-d_i}{\delta_i} + D\min\frac{1-b_j}{\beta_j}\right) > \mathrm{Re}(-A) > \mathrm{Re}\left[D\left(\frac{s-\alpha}{s}, \frac{k-\beta}{k}, a_l\right) + C_t - D - 1\right]$$

$$i = 1,2,\ldots,f; j = 1,2,\ldots,m; l = 1,2,\ldots n; t = 1,2,\ldots,g; u, \mathrm{Re}(\min C_i + A) - v,$$

$$\mathrm{Re}\left(\max\frac{1-d_j}{\delta_j} + A\right) - uD + v + \frac{1}{2}D\left(Dv - Du + 1\right) > D(Dv - Du), \mathrm{Re}\max\left(\frac{s-\alpha}{s}, \frac{k-\beta}{k}, a_l\right),$$

$$i = 1,2,\ldots,u; j = 1,2,\ldots,v; l = 1,2,\ldots,u; |\arg\lambda| \le \left(f + g - \frac{1}{2}u - \frac{1}{2}v\right)\pi,$$

$$|\arg t| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right)\pi, \mathrm{Re}\left(\alpha + s\frac{1-b_j}{\beta_j}\right) > 0, \mathrm{Re}\left(\beta + k\frac{1-b_j}{\beta_j}\right) > 0, j = 1,2,\ldots,m$$

And the double integral converges.
Proof: To prove (2.1), we start with the following known result [2, p. 177]

$$\int_0^\infty\int_0^\infty \phi(x+y) x^{\alpha-1} y^{\beta-1} dxdy = B(\alpha,\beta)\int_0^\infty \phi(z) z^{\alpha+\beta-1} dz \quad (2.2)$$

Which is valid for $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$.
It is easy to prove by following the technique of reversing the order of integrations, that

$$\int_0^\infty\int_0^\infty \phi(x+y) x^{\alpha-1} y^{\beta-1} A_{p,q}^{m,n}\left[tx^s y^k (x+y)^r \middle|{}_{B^*}^{A^*}\right] dxdy = \sqrt{2\pi}\,\frac{s^{\alpha-\frac{1}{2}} k^{\beta-\frac{1}{2}}}{(s+k)^{\alpha+\beta-\frac{1}{2}}}$$

$$\int_0^\infty \phi(z) z^{\alpha+\beta-1} A_{p+\rho,q+\rho}^{m,n+\rho}\left[t\delta z^D \middle|{}_{B^*,\Delta(1-k-s,1-\alpha-\beta)}^{\Delta(1-s,1-\alpha),\Delta(1-k,1-\beta),A^*}\right] dz \quad (2.3)$$

Where s, k and r are positive integers,

$$\delta = \frac{s^s k^k}{(s+k)^{s+k}}, \rho = s+k, D = s+k+r, p+q < 2(m+n), |\arg t| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right)\pi,$$

$$\mathrm{Re}\left(\alpha + s\frac{1-b_j}{\beta_j}\right) > 0, \mathrm{Re}\left(\beta + k\frac{1-b_j}{\beta_j}\right) > 0, j = 1,2,\ldots,m.$$

In (2.3), taking

$$\phi(z) = z^{\sigma} A_{u,v}^{f,g}\left[\lambda z\left|{}^{A^{**}}_{B^{**}}\right.\right]$$

And evaluating the integral on the right hand side using [9, p.401] the result (2.1) follows.

III. Particular Cases

On choosing the parameters suitably in (2.1), several known and unknown results are obtained as particular cases. However, we mention some of the interesting results here.

(a) Taking $f = v = 2, g = 0, u = 1, c_1 = \frac{1}{2}, 1-d_1 = v, 1-d_2 = -v, \sigma = \mu + \frac{1}{2}, \alpha_j = \beta_j = \delta_j = \gamma_j = 1$ in (2.1), and using [3, p.216, (5)]

$$H_{1,2}^{2,0}\left[x\left|{}^{\left(\frac{1}{2},1\right)}_{(b,1),(-b,1)}\right.\right] = \pi^{-\frac{1}{2}} e^{-\frac{1}{2}x} K_b\left(-\frac{1}{2}x\right),$$

We obtain

$$\int_0^\infty\int_0^\infty x^{\alpha-1} y^{\beta-1}(x+y)^{\mu+\frac{1}{2}} a^{-\frac{1}{2}\lambda(x+y)} K_v\left\{-\frac{1}{2}\lambda(x+y)\right\} H_{p,q}^{m,n}\left[tx^s y^k (x+y)^r\left|{}^{A^*}_{B^*}\right.\right] dxdy =$$

$$(2\pi)^{-\frac{1}{2}(2-D)\left(f+g-\frac{1}{2}u-\frac{1}{2}v\right)+\frac{1}{2}} D^{A-1}\sqrt{\pi}\,\frac{s^{\alpha-\frac{1}{2}} k^{\beta-\frac{1}{2}}}{\lambda^{\alpha+\beta+\sigma}(s+k)^{\alpha+\beta-\frac{1}{2}}}$$

$$H_{p+\rho+2D,q+\rho+D}^{m,n+\rho+2D}\left[\frac{t\delta D^D}{\lambda^D}\left|{}^{\Delta((D,1-A\mp v)),\Delta(s,1-\alpha),\Delta(k,1-\beta),A^*}_{B^*,\Delta((D,\frac{1}{2}-A)),\Delta(k+s,1-\alpha-\beta)}\right.\right], \quad (3.1)$$

Where δ, D and λ have the same value as (2.1) and

$$A = \mu+\alpha+\beta+\frac{1}{2}; p+q < 2(m+n), \mathrm{Re}(\alpha + s\frac{b_j}{\beta_j} \pm v) > 0, \mathrm{Re}(\beta + s\frac{b_j}{\beta_j} \pm v) > 0,$$

$$\mathrm{Re}\left(\alpha+\beta+\mu\pm v + D\frac{b_j}{\beta_j}+\frac{1}{2}\right) > 0, j = 1,2,...,m; \mathrm{Re}(\lambda) > 0, |\arg t| < \left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi.$$

(b) Further, replacing q, t and $\left(1-a_p, \alpha_p\right)$ by $q+1, -t$ and $\left(a_p, \alpha_p\right)$ respectively and then putting $m = 1, n = p, b_1 = 0, b_{j+1} = 1-b_j (j = 1,2,...q)$, using the result [3,p. 215, (1)] and [3,p. 4, (11)], we obtain an interesting result obtained by Srivastava and Singhal [10]:

$$\int_0^\infty\int_0^\infty x^{\alpha-1} y^{\beta-1}(x+y)^{\mu+\frac{1}{2}} a^{-\frac{1}{2}\lambda(x+y)} K_v\left\{-\frac{1}{2}\lambda(x+y)\right\} {}_pF_q\left[{}^{(a_p,\alpha_p)}_{(b_q,\beta_q)} : tx^s y^k (x+y)^r\right] dxdy =$$

$$\frac{\sqrt{\pi}\,\Gamma\left(\alpha+\beta+\mu\pm v+\frac{1}{2}\right)}{\lambda^{\alpha+\beta+\mu+\frac{1}{2}}\Gamma\left(\alpha+\beta+\mu+1\right)} B(\alpha,\beta)$$

$${}_{p+3s+3k+2r}F_{q+2s+2k+r}\left[t\delta\left(\frac{s+k+r}{\lambda}\right)^{s+k+r}\left|{}^{\Delta(s+k+r,\alpha+\beta+\mu\pm v+\frac{1}{2},),\Delta(s,\alpha),\Delta(k,\beta),\left(a_p,1\right)}_{\left(b_q,1\right),\Delta(s+k,\alpha+\beta),\Delta(k+s,\alpha+\beta+\mu+1)}\right.\right], \quad (3.2)$$

provided $\mathrm{Re}(\mu+\alpha+\beta\pm v+\frac{1}{2}) > 0, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0.$

(c) Setting $v = f = 2, g = 0, u = 1, c_1 = 1-\mu, 1-d_1 = \frac{1}{2}+v, 1-d_2 = \frac{1}{2}-v$ in (2.1) and using the known formula [3,p.216,(6)]

$$H_{1,2}^{2,0}\left[x\left|\begin{matrix}(1-k,1)\\(\frac{1}{2}+m,1),(\frac{1}{2}-m,1)\end{matrix}\right.\right]=e^{-\frac{1}{2}x}W_{k,m}(x),$$

We have

$$\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}(x+y)^{\sigma}e^{-\frac{1}{2}\lambda(x+y)}W_{\mu,v}[\lambda(x+y)]H_{p,q}^{m,n}\left[tx^sy^k(x+y)^r\left|\begin{matrix}A^*\\B^*\end{matrix}\right.\right]dxdy=$$

$$(2\pi)^{\frac{1}{2}(2-D)}D^{\mu+A-\frac{1}{2}}\frac{s^{\alpha-\frac{1}{2}}k^{\beta-\frac{1}{2}}}{\lambda^{\alpha+\beta+\sigma}(s+k)^{\alpha+\beta-\frac{1}{2}}}$$

$$H_{p+\rho+2D,q+\rho+D}^{m,n+\rho+2D}\left[\frac{t\delta D^D}{\lambda^D}\left|\begin{matrix}\Delta(D,\frac{1}{2}-A\pm v),\Delta(s,1-\alpha),\Delta(k,1-\beta),A^*\\B^*,\Delta(k+s,1-\alpha-\beta),\Delta(D,\mu-A)\end{matrix}\right.\right],\qquad(3.3)$$

Where

D,ρ,δ and A are given in (2.1);

$$p+q<2(m+n),|\arg t|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi,\mathrm{Re}(\lambda)>0,\mathrm{Re}(\alpha+sb_j>0,$$

$$\mathrm{Re}(k+sb_j)>0,\mathrm{Re}\left(m+n+\sigma+Db_j\pm v+\frac{1}{2}\right)>0,\ j=1,2,...,m.$$

(d) Further, replacing q,t and $\left(1-a_p,\alpha_p\right)$ by $q+1,-t$ and (a_p,α_p) respectively and then putting $m=1,n=p,b_1=0,b_{j+1}=1-b_j(j=1,2,...,q)$ and using the result [3,p.215,(1)], (3.3) reduces to a result due to Srivastava and Joshi [11,p.19,(2.3)]

$$\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}(x+y)^{\sigma}e^{-\frac{1}{2}\lambda(x+y)}W_{\mu,v}\{\lambda(x+y)\}\,{}_pF_q\left[\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}:tx^sy^k(x+y)^r\right]dxdy=$$

$$\frac{\Gamma\left(\alpha+\beta+\sigma\pm v+\frac{1}{2}\right)}{\lambda^{\alpha+\beta+\sigma}\Gamma\left(\alpha+\beta+\sigma-\mu+1\right)}B(\alpha,\beta)$$

$${}_{p+3s+3k+2r}F_{q+2s+2k+r}\left[t\delta\delta'\left|\begin{matrix}\Delta(s+k+r,\alpha+\beta+\sigma\pm v+\frac{1}{2},),\Delta(s,\alpha),\Delta(k,\beta),\left(a_p,1\right)\\\left(b_q,1\right),\Delta(s+k,\alpha+\beta),\Delta(k+s+r,\alpha+\beta+\sigma-\mu+1)\end{matrix}\right.\right]\qquad(3.4)$$

Where $$\delta=\frac{s^sk^k}{(s+k)^{s+k}},\delta'=\left(\frac{s+k+r}{\lambda}\right)^{s+k+r}$$

$\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(\lambda)>0,\mathrm{Re}\left(\alpha+\beta+\sigma\pm v+\frac{1}{2}\right)>0$ and the resulting hypergeometric series converges.

With $\mu=0,v=\pm\frac{1}{2}$ and $\sigma=-\frac{1}{2}$,(3.4) reduces to the earlier results of Jain [6] and Singh [8].

(e) Choosing $f=g=u=1,v=2,c_1=1-k,1-d_1=\frac{1}{2}+M,1-d_2=\frac{1}{2}-M,\alpha_j=\beta_j=\delta_j=\gamma_j=1$ in (2.1) and using the known result $$H_{1,2}^{1,1}\left[x\left|\begin{matrix}(1-k,1)\\(\frac{1}{2}+m,1),(\frac{1}{2}-m,1)\end{matrix}\right.\right]=\frac{\Gamma\left(\frac{1}{2}+k+m\right)}{\Gamma(2m+1)}e^{-\frac{1}{2}x}M_{k,m}(x),$$

We obtain $$\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}(x+y)^{\sigma}e^{-\frac{1}{2}\lambda(x+y)}M_{k,m}[\lambda(x+y)]H_{p,q}^{m,n}\left[tx^sy^k(x+y)^r\left|\begin{matrix}A^*\\B^*\end{matrix}\right.\right]dxdy=$$

$$(2\pi)^{\frac{1}{2}(2-D)} D^{k+A-\frac{1}{2}} \frac{\Gamma(2m+1)}{\Gamma\left(k+m+\frac{1}{2}\right)} \frac{s^{\alpha-\frac{1}{2}} k^{\beta-\frac{1}{2}}}{\lambda^{\alpha+\beta+\sigma}(s+k)^{\alpha+\beta-\frac{1}{2}}}$$

$$H^{m+D,n+\rho+D}_{p+\rho+2D,q+\rho+D}\left[\frac{t\delta D^D}{\lambda^D}\middle|\begin{matrix}\Delta(D,\frac{1}{2}-A-m),\Delta(s,1-\alpha),\Delta(k,1-\beta),\Delta(D,\frac{1}{2}-A+m)\\ \Delta(k+s,1-\alpha-\beta),\Delta(D,k-A)\end{matrix}\right], \qquad (3.5)$$

Where D,ρ,δ and A are given in (2.1);

$$p+q<2(m+n), |\arg t|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi, \mathrm{Re}(\lambda)>0, \mathrm{Re}(\alpha+sb_j)>0,$$

$$\mathrm{Re}(\beta+kb_j)>0, \mathrm{Re}\left(\alpha+\beta+\sigma+Db_j+m+\frac{1}{2}\right)>0, j=1,2,...,m.$$

(f) Substituting $f=1, g=u=0, v=2, 1-d_1=\frac{1}{2}v, 1-d_2=-\frac{1}{2}v$ and using the result [3,p.216,(3)]

$$H^{1,0}_{0,2}\left[x\middle|\begin{matrix}-\\ \left(\frac{1}{2}v,1\right),\left(-\frac{1}{2}v,1\right)\end{matrix}\right]=J_v(2\sqrt{x}),$$

(2.1) reduces to

$$\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}(x+y)^\sigma J_v(2\sqrt{\lambda(x+y)})H^{m,n}_{p,q}\left[tx^s y^k\middle|\begin{matrix}A^*\\ B^*\end{matrix}\right]dxdy$$

$$=\sqrt{2\pi}\frac{D^{2A-1}s^{\alpha-\frac{1}{2}}k^{\beta-\frac{1}{2}}}{\lambda^{\alpha+\beta+\sigma}(s+k)^{\alpha+\beta-\frac{1}{2}}}$$

$$H^{m,n+\rho+D}_{p+\rho+2D,q+p}\left[t\delta\left(\frac{D}{\lambda}\right)^D\middle|\begin{matrix}\Delta\left(D,1-A-\frac{1}{2}v\right),\Delta(s,1-\alpha),\Delta(k,1-\beta),A^*,\Delta\left(D,1-A+\frac{1}{2}v\right)\\ B^*,\Delta(k+s,1-\alpha-\beta)\end{matrix}\right] \qquad (3.6)$$

Where δ, D, ρ and A have the same values given in (2.1);

$$p+q<2(m+n), |\arg t|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi, \mathrm{Re}(\lambda)>0, \mathrm{Re}(\alpha+sb_j)>0,$$

$$\mathrm{Re}(\beta+kb_j)>0, \mathrm{Re}\left(\alpha+\beta+\sigma+\frac{1}{2}v+Db_j\right)>0, j=1,2,...,m;$$

$$\mathrm{Re}\left(\alpha+\beta+\sigma-D+Da_i\right), \frac{1}{4}, i=1,2,...,n.$$

In view of the numerous properties of A-function, on specializing the parameters suitably, a large number of interesting results may be obtained as particular case.

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