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On Certain Class of Analytic Functions Involving Linear Operators

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Abstract: *Invoking the Hadamard product (or convolution) , a class of univalent functions has been introduced. In the present paper we obtain necessary and sufficient conditions and some important properties for the analytic functions for its belongingness to certain class of functions. The distortion inequalities, closer theorems, radii of close-to-convexity, radii of starlikeness and radii of convexity are obtained for the same class of functions. Some properties involving Hadamard product are also obtained.*
Key words: *Univalent function; Hadamard product; Starlike function; convex function; Generalized hypergeometric function; Linear operator; Fractional differential and integral operators.*

I. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m , \tag{1.1}$$

which are analytic in the open unit disk U = $\{z: |z| \leq 1\}$.
For function $f(z) \in A$, given by (1.1), and $g(z) \in A$, given by

$$g(z) = z + \sum_{m=2}^{\infty} b_m z^m , \tag{1.2}$$

we define the Hadamard product of $f(z)$ and $g(z)$ by

$$f * g(z) = z + \sum_{m=2}^{\infty} a_m b_m z^m \quad , \quad z \in U. \tag{1.3}$$

By using the Hadamard product, Chena Ram and Garima [6] studied a linear operator

$$L^k(\alpha_1, \alpha_2, \beta_1) f(z) = z \; {}_2\mathrm{R}_1\big(\alpha_1, \alpha_2; \beta_1; k; z\big) * f(z)$$

$$= z + \sum_{m=2}^{\infty} a_m \Phi(k, m) z^n, \tag{1.4}$$

where

$$\Phi(k, m) = \frac{(\alpha_1)_{m-1} \Gamma(\alpha_2 + k(m-1)) \Gamma\beta_1}{\Gamma(\alpha_2)\Gamma(\beta_1 + k(m-1)) \; (1)_{m-1}},$$

and ${}_2\mathrm{R}_1\big(\alpha_1, \alpha_2; \beta_1; k; z\big)$ is the generalized hypergeometric function, defined by [9]

$${}_2\mathrm{R}_1\big(\alpha_1, \alpha_2; \beta_1; k; z\big) = \frac{\Gamma(\beta_1)}{\Gamma(\alpha_2)} \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \Gamma(\alpha_2 + nk)}{\Gamma(\beta_1 + nk) \, (1)_n} z^n, \qquad (k \in R, k > 0, |z| < 1). \tag{1.5}$$

Let T denote the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{m=2}^{\infty} a_m z^m \tag{1.6}$$

Recently, Aouf et al. [2], [3], [4], Joshi [5], Salagean [7] and others, introduced and studied certain subclasses of analytic functions with negative coefficients.
Now, let $\mathrm{S}(\alpha_1, \alpha_2, \beta_1, \lambda, A, B)$ denote the class of functions $f(z) \in T$ such that

$$(1-\lambda) \frac{L^k(\alpha_1, \alpha_2, \beta_1) f(z)}{z} + \frac{L^k(\alpha_1 + 1, \alpha_2, \beta_1) f(z)}{z} \prec \frac{(1+Az)}{(1+Bz)}, \tag{1.7}$$

for $z \in U$, where $\lambda \geq 0, \; -1 \leq A < B \leq 1$.

We say that a function $f(z) \in T$ is in the class $\mathrm{Q}(\alpha_1, \alpha_2, \beta_1, \lambda, A, B)$ if it satisfies the following subordination condition:

$$(1-\lambda)[L^k(\alpha_1, \alpha_2, \beta_1) f(z)]' + \lambda[L^k(\alpha_1 + 1, \alpha_2, \beta_1) f(z)]' \prec \frac{(1+Az)}{(1+Bz)} \tag{1.8}$$

for $z \in U$, where $\lambda \geq 0, \; -1 \leq A < B \leq 1$.

II. Coefficient Estimates

Theorem 1. Let the function $f(z)$, defined by (1.6), is in the class $Q(\alpha_1,\alpha_2,\beta_{1,}\lambda,A,B)$ if and only if

$$\sum_{m=2}^{\infty} mc_m\phi(k,m)a_m \leq (B-A)\alpha_1, \tag{2.1}$$

where

$$c_m = [\alpha_1+\lambda(m-1)](1+B)$$

and

$$\phi(k,m)=\frac{\Gamma(\beta_1)(\alpha_1)_{m-1}\Gamma(\alpha_2+k(m-1))}{\Gamma(\alpha_2)\Gamma(\beta_1+k(m-1))(1)_{m-1}} \tag{2.2}$$

Proof. Let $f(z)\in\ Q(\alpha_1,\alpha_2,\beta_{1,}\lambda,A,B)$. Then

$$h(z)=(1-\lambda)[L^k(\alpha_1,\alpha_2,\beta_1)f(z)]'+\lambda[L^k(\alpha_1+1,\alpha_2,\beta_1)f(z)]'=\frac{1+A\omega(z)}{1+B\omega(z)} \tag{2.3}$$

$$-1\leq A<B\leq 1, 0<B\leq 1, Z\in U,$$

$\omega\in H=\{\omega\ an\ analytic, \omega(0)=0\ and\ |\omega(z)|<1, Z\in U\}$.

From (2.3), we get

$$\omega(z)=\frac{1-h(z)}{Bh(z)-A}.$$

Therefore

$$h(z)=1-\sum_{m=2}^{\infty}\frac{m[\alpha_1+\lambda(m-1)]}{\alpha_1}\frac{\Gamma(\beta_1)(\alpha_1)_{m-1}\Gamma(\alpha_2+k(m-1))}{\Gamma(\alpha_2)\Gamma(\beta_1+k(m-1))(1)_{m-1}}a_m z^{m-1}$$

and $|\omega(z)|<1$ implies

$$\left|\frac{\sum_{m=2}^{\infty}\frac{m[\alpha_1+\lambda(m-1)]}{\alpha_1}\frac{\Gamma(\beta_1)(\alpha_1)_{m-1}\Gamma(\alpha_2+k(m-1))}{\Gamma(\alpha_2)\Gamma(\beta_1+k(m-1))(1)_{m-1}}a_m z^{m-1}}{(B-A)-B\sum_{m=2}^{\infty}\frac{m[\alpha_1+\lambda(m-1)]}{\alpha_1}\frac{\Gamma(\beta_1)(\alpha_1)_{m-1}\Gamma(\alpha_2+k(m-1))}{\Gamma(\alpha_2)\Gamma(\beta_1+k(m-1))(1)_{m-1}}a_m z^{m-1}}\right|<1 \tag{2.4}$$

Hence

$$\mathrm{Re}\left\{\frac{\sum_{m=2}^{\infty}\frac{m[\alpha_1+\lambda(m-1)]}{\alpha_1}\frac{\Gamma(\beta_1)(\alpha_1)_{m-1}\Gamma(\alpha_2+k(m-1))}{\Gamma(\alpha_2)\Gamma(\beta_1+k(m-1))(1)_{m-1}}a_m z^{m-1}}{(B-A)-B\sum_{m=2}^{\infty}\frac{m[\alpha_1+\lambda(m-1)]}{\alpha_1}\frac{\Gamma(\beta_1)(\alpha_1)_{m-1}\Gamma(\alpha_2+k(m-1))}{\Gamma(\alpha_2)\Gamma(\beta_1+k(m-1))(1)_{m-1}}a_m z^{m-1}}\right\}<1. \tag{2.5}$$

We consider real values of z and take $z=r$ with $0\leq r<1$. Then for $r=0$, the denominator (2.5) is positive and so it is positive for all r with $0\leq r<1$, since $\omega(z)$ is analytic for $|z|<1$. Then (2.5) gives

$$\sum_{m=2}^{\infty}\frac{m[\alpha_1+\lambda(m-1)]}{\alpha_1}\frac{\Gamma\beta_1(\alpha_1)_{m-1}\Gamma\alpha_2+k(m-1)}{\Gamma(\alpha_2)\Gamma\beta_1+k(m-1)(1)_{m-1}}(1+B)a_m z^{m-1}\leq(B-A)$$

i.e.

$$mc_m\phi(k,m)a_m z^{m-1}\leq(B-A)\alpha_1, \tag{2.6}$$

where

$$c_m=[\alpha_1+\lambda(m-1)](1+B),$$

and

$$\phi(k,m)=\frac{\Gamma\beta_1(\alpha_1)_{m-1}\Gamma\alpha_2+k(m-1)}{\Gamma(\alpha_2)\Gamma\beta_1+k(m-1)(1)_{m-1}}.$$

Letting $r\to 1$ in (2.6), we get (2.1).

Conversely, let $f(z)\in T$ and satisfies (2.1). For $|z|=r, 0\leq r<1$, we have (2.6) by (2.1), since $r^{m-1}<1$. So that, we have

$$\left|\sum_{m=2}^{\infty}\frac{m[\alpha_1+\lambda(m-1)]}{\alpha_1}\phi(k,m)a_m z^{m-1}\right|\leq(B-A)-B\sum_{m=2}^{\infty}\frac{m[\alpha_1+\lambda(m-1)]}{\alpha_1}\phi(k,m)a_m z^{m-1}$$

$$\leq\left|(B-A)-B\sum_{m=2}^{\infty}\frac{m[\alpha_1+\lambda(m-1)]}{\alpha_1}\phi(k,m)a_m z^{m-1}\right|$$

Which gives (2.4) and hence follows that

$$(1-\lambda)[L^k(\alpha_1,\alpha_2,\beta_1)f(z)]'+\lambda[L^k(\alpha_1+1,\alpha_2,\beta_1)f(z)]'=\frac{1+A\omega(z)}{1+B\omega(z)}$$

$\omega \in H, z \in U, -1 \le A < B \le 1, 0 < B \le 1.$
That is $f(z) \in Q(\alpha_1, \alpha_2, \beta_1, \lambda, A, B)$. Finally the function $f(z)$ given by

$$f(z) = z - \frac{(B-A)\alpha_1}{m c_m \Phi(k,m)} z^m, \quad (m \ge 2) \tag{2.7}$$

is an extremal function for the theorem.
The theorem is completely proved.

III. Some properties of $Q(\alpha_1, \alpha_2, \beta_1, \lambda, A, B)$

Theorem 2. $Q(\alpha_1 + 1, \alpha_2, \beta_1, \lambda, A, B) \subset Q(\alpha_1, \alpha_2, \beta_1, \lambda, A, B)$ for
$-1 \le A < B \le 1, 0 < B \le 1, \lambda \ge 0.$
Proof. Let the function $f(z)$ defined by (1.6) be in the class $Q(\alpha_1 + 1, \alpha_2, \beta_1, \lambda, A, B)$. Then by Theorem (1), we have

$$\sum_{m=2}^{\infty} \frac{m[\alpha_1 + \lambda(m-1)]}{\alpha_1} \phi(k,m)(1+B)a_m$$

$$\le \sum_{m=2}^{\infty} \frac{m[(\alpha_1+1) + \lambda(m-1)]}{(\alpha_1+1)} \frac{\Gamma\beta_1(\alpha_1+1)_{m-1}\Gamma\alpha_2 + k(m-1)}{\Gamma(\alpha_2)\Gamma\beta_1 + k(m-1)(1)_{m-1}} (1+B)a_m$$

$$= \sum_{m=2}^{\infty} \frac{m[(\alpha_1+1) + \lambda(m-1)](\alpha_1+m)}{(\alpha_1+2)(\alpha_1+1)} \phi(k,m)(1+B)a_m \le (\mathrm{B} - \mathrm{A})$$

i.e. $$\sum_{m=2}^{\infty} \frac{m[\alpha_1 + \lambda(m-1)]}{\alpha_1+1} \phi(k,m)(1+B)a_m \le (\mathrm{B} - \mathrm{A})$$

Theorem 3. $Q(\alpha_1, \alpha_2, \beta_1, \lambda_1, A, B) \subseteq Q(\alpha_1, \alpha_2, \beta_1, \lambda_2, A, B)$ for
$-1 \le A < B \le 1, 0 < B \le 1, \lambda_1, \lambda_2 \ge 0.$
Proof. By Theorem 1 we have

$$\sum_{m=2}^{\infty} \frac{m[\alpha_1 + \lambda_1(m-1)]}{\alpha_1} \phi(k,m)(1+B)a_m$$

$$\le \sum_{m=2}^{\infty} \frac{m[\alpha_1 + \lambda_2(m-1)]}{\alpha_1} \phi(k,m)(1+B)a_m$$

$$\le (\mathrm{B} - \mathrm{A}).$$

This completes the proof of Theorem 3.

IV. A set of Distortion inequalities

Theorem 4. Let the function $f(z)$ defined by (1.6) be in the class $Q(\alpha_1, \alpha_2, \beta_1, \lambda, A, B)$. Then we have for $|z| = r < 1$,

$$r - \frac{(B-A)}{2c_2}\left[\frac{\Gamma(\alpha_2)\Gamma(\beta_1+k)}{\Gamma(\alpha_2+k)\Gamma(\beta_1)}\right] r^2 \le |f(z)| \le r + \frac{(B-A)}{2c_2}\left[\frac{\Gamma(\alpha_2)\Gamma(\beta_1+k)}{\Gamma(\alpha_2+k)\Gamma(\beta_1)}\right] r^2 \tag{4.1}$$

and

$$1 - \frac{(B-A)}{c_2}\left[\frac{\Gamma(\alpha_2)\Gamma(\beta_1+k)}{\Gamma(\alpha_2+k)\Gamma(\beta_1)}\right] r \le |f'(z)| \le 1 - \frac{(B-A)}{c_2}\left[\frac{\Gamma(\alpha_2)\Gamma(\beta_1+k)}{\Gamma(\alpha_2+k)\Gamma(\beta_1)}\right] r \tag{4.2}$$

The result is sharp for the function

$$f(z) = z - \frac{(B-A)\alpha_1}{2c_2}\left[\frac{\Gamma(\alpha_2)\Gamma(\beta_1+k)}{\Gamma(\alpha_2+k)\Gamma(\beta_1)}\right] z^2, \quad z = \pm r \tag{4.3}$$

Proof. Since $mc_m\phi(k)$ is an increasing function of m $(m \ge 2)$, and
$f(z) \in Q(\alpha_1, \alpha_2, \beta_1, \lambda_1, A, B)$, by Theorem 1, we have

$$2c_2\phi(k,2)\sum_{m=2}^{\infty} a_m \le \sum_{m=2}^{\infty} mc_m\phi(k,m)a_m \le (B-A)\alpha_1$$

$$\sum_{m=2}^{\infty} a_m \le \frac{(B-A)}{2c_2}\left[\frac{\Gamma(\alpha_2)\Gamma(\beta_1+k)}{\Gamma(\alpha_2+k)\Gamma(\beta_1)}\right] \tag{4.4}$$

$$|f(z)| \le r + \sum_{m=2}^{\infty} a_m r^m \le r + r^2 \sum_{m=2}^{\infty} a_m \le r + \frac{(B-A)}{2c_2}\left[\frac{\Gamma(\alpha_2)\Gamma(\beta_1+k)}{\Gamma(\alpha_2+k)\Gamma(\beta_1)}\right] r^2$$

and

$$|f(z)| \ge r - \sum_{m=2}^{\infty} a_m r^m \ge r - r^2 \sum_{m=2}^{\infty} a_m \ge r - \frac{(B-A)}{2c_2}\left[\frac{\Gamma(\alpha_2)\Gamma(\beta_1+k)}{\Gamma(\alpha_2+k)\Gamma(\beta_1)}\right] r^2.$$

Also by Theorem 1, we have

$$c_2\phi(k,2)\sum_{m=2}^{\infty} m a_m \le (B-A)\alpha_1$$

i.e. $$\sum_{m=2}^{\infty} m a_m \le \frac{(B-A)\alpha_1}{c_2}\left[\frac{\Gamma(\alpha_2)\Gamma(\beta_1+k)}{\Gamma(\alpha_2+k)\Gamma(\beta_1)}\right] \tag{4.5}$$

Thus

$$|f'(z)| \le 1 + \sum_{m=2}^{\infty} m a_m r^{m-1} \le 1 + r\sum_{m=2}^{\infty} m a_m$$
$$\le 1 + \frac{(B-A)}{c_2}\left[\frac{\Gamma(\alpha_2)\Gamma(\beta_1+k)}{\Gamma(\alpha_2+k)\Gamma(\beta_1)}\right] r$$

and

$$|f'(z)| \ge 1 - \sum_{m=2}^{\infty} m a_m r^{m-1} \ge 1 - r\sum_{m=2}^{\infty} m a_m$$
$$\ge 1 - \frac{(B-A)}{c_2}\left[\frac{\Gamma(\alpha_2)\Gamma(\beta_1+k)}{\Gamma(\alpha_2+k)\Gamma(\beta_1)}\right] r.$$

The theorem is completely proved.

V. Closure Theorems

Let the function $f_i(z)$ be defined, for $i = 1,2, \dots \nu$ by

$$f_i(z) = z - \sum_{m=2}^{\infty} a_{m,i} z^m, \qquad \sum_{m=2}^{\infty} a_{m,i} \ge 0. \tag{5.1}$$

Theorem 5. Let the functions $f_i(z)$ defined by (5.1) be in the class $Q(\alpha_1, \alpha_2, \beta_1, \lambda, A_i, B_i)$, for $i = 1,2, \dots \nu$. Then the function $h(z)$ defined by

$$h(z) = z - \frac{1}{\nu}\sum_{m=2}^{\infty}\left(\sum_{i=1}^{\nu} a_{m,i} z^m\right) \tag{5.2}$$

is in the class $Q(\alpha_1, \alpha_2, \beta_1, \lambda, A_i, B_i)$, where

$A = \underset{1 \le i \le \nu}{min}\{A_i\}$ and $B = \underset{1 \le i \le \nu}{max}\{B_i\}$ (5.3)

Proof. Since $f_i(z) \in Q(\alpha_1, \alpha_2, \beta_1, \lambda, A_i, B_i)$ for $i = 1,2, \dots \nu$, we have

$$\sum_{m=2}^{\infty} m[\alpha_1 + \lambda_1(m-1)]\phi(k,m)(1+B_i)a_{m,i} \le (B_i - A_i)\alpha_1 \tag{5.4}$$

hence

$$\sum_{m=2}^{\infty} m[\alpha_1 + \lambda_1(m-1)]\phi(k,m)\left[\frac{1}{\nu}\sum_{i=1}^{\infty} a_{m,i}\right]$$
$$= \frac{1}{\nu}\sum_{i=1}^{\nu}\left[\sum_{m=2}^{\infty} m[\alpha_1 + \lambda_1(m-1)]\phi(k,m)a_{m,i}\right]$$
$$\le \frac{1}{\nu}\sum_{i=1}^{\nu} \frac{(B_i - A_i)(\alpha_1)}{(1+B_i)} \le \frac{(B-A)(\alpha_1)}{(1+B)} \tag{5.5}$$

$$\sum_{m=2}^{\infty} m[\alpha_1 + \lambda(m-1)]\phi(k,m)(1+B)\left[\frac{1}{\nu}\sum_{i=1}^{\infty} a_{m,i}\right] \le (B-A)(\alpha_1) \tag{5.6}$$

A and B is given by (5.3).Hence $f(z) \in Q(\alpha_1, \alpha_2, \beta_1, \lambda, A, B)$.
The theorem is completely proved.

Theorem 6. Let the function $f_i(z)$ $(i = 1,2, \dots \nu)$ defined by (5.1) be in the class $Q(\alpha_1, \alpha_2, \beta_1, \lambda, A, B)$. Then the function $h(z)$ defined by

$$h(z) = \sum_{i=1}^{\nu} d_i f_i(z) \qquad (5.7)$$

is also in the same class $Q(\alpha_1, \alpha_2, \beta_1, \lambda, A, B)$, where

$$\sum_{i=1}^{\nu} d_i = 1 \qquad (5.8)$$

Proof. By (5.7), we have

$$h(z) = z - \sum_{m=2}^{\infty} \left(\sum_{i=1}^{\nu} d_i a_{m,i} \right) z^m \qquad (5.9)$$

Since $f_i(z) \in Q(\alpha_1, \alpha_2, \beta_1, \lambda, A, B)$ for every $i = 1,2, \dots \nu$,

$$\sum_{m=2}^{\infty} m c_m \phi(k, m) a_{m,i} \le (B - A)\alpha_1 \qquad (5.10)$$

$$\sum_{m=2}^{\infty} m c_m \phi(k, m) \left(\sum_{i=1}^{\nu} d_i a_{m,i} \right)$$

$$= \sum_{i=1}^{\nu} d_i \left(\sum_{m=2}^{\infty} m c_m \phi(k, m) a_{m,i} \right)$$

$$\le \sum_{i=1}^{\nu} d_i (B - A)\alpha_1 = (B - A)\alpha_1 \qquad (5.11)$$

Hence $h(z) \in Q(\alpha_1, \alpha_2, \beta_1, \lambda, A, B)$.
The theorem is completely proved.

Theorem 7. Let $f_1(z) = z$ and $f_m(z) = z - \frac{(B-A)\alpha_1}{m c_m \phi(k,m)} z^m \qquad (m \ge 2) \qquad (5.12)$
Then $f(z) \in Q(\alpha_1, \alpha_2, \beta_1, \lambda, A, B)$ if and only if it can be expressed as

$$f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z), \qquad (5.13)$$

where $\mu_m \ge 0 \quad (m \ge 1)$ and $\sum_{m=1}^{\infty} \mu_m = 1$.

Proof. Let

$$f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z) = z - \sum_{m=2}^{\infty} \frac{(B-A)\alpha_1 \mu_m}{m c_m \phi(k,m)} z^m. \qquad (5.14)$$

Then by Theorem 1, we have

$$\sum_{m=2}^{\infty} \frac{(B-A)\alpha_1 \mu_m}{m c_m \phi(k,m)} . \frac{m c_m \phi(k,m)}{(B-A)\alpha_1}$$

$$= \sum_{m=2}^{\infty} \mu_m = 1 - \mu_l \le 1. \qquad (5.15)$$

Hence by Theorem 1, $f(z) \in Q(\alpha_1, \alpha_2, \beta_1, \lambda, A, B)$.
Conversely, let $f(z)$ defined by (1.6) belongs to the class $Q(\alpha_1, \alpha_2, \beta_1, \lambda, A, B)$. Then

$$a_m \le \frac{(B-A)\alpha_1}{m c_m \phi(k,m)} \qquad (m \ge 2) \qquad (5.16)$$

$$\mu_m = \frac{m c_m \phi(k,m)}{(B-A)\alpha_1} a_m \qquad (m \ge 2), \qquad (5.17)$$

and

$$\mu_1 = 1 - \sum_{m=2}^{\infty} \mu_m.$$

The theorem is completely proved.

VI. Radii of close-to-convexity, Starlikeness and Convexity

Theorem 8. Let the function $f(z)$ defined by (1.6) be in the class $Q(\alpha_1, \alpha_2, \beta_1, \lambda, A, B, \rho)$. Then $f(z)$ is close-to-convex of order $\rho(0 \le \rho < 1)$ in $|z| < r_1(\alpha_1, \alpha_2, \beta_1, \lambda, A, B, \rho)$, where

$$r_1(\alpha_1, \alpha_2, \beta_1, \lambda, A, B, \rho) = \underset{m}{inf} \left[\frac{(1-\rho)c_m \phi(k,m)}{(B-A)\alpha_1}\right]^{1/m-1}, \qquad (m \ge 2). \qquad (6.1)$$

The result is sharp for the function $f(z)$ defined by (2.7).

Proof. We must show that

$$\left|f'(z) - 1\right| \le 1 - \rho \text{ for } |z| < r_1(\alpha_1, \alpha_2, \beta_1, \lambda, A, B, \rho)$$

$$\left|f'(z) - 1\right| \le \sum_{m=2}^{\infty} m a_m |z|^{m-1}$$

Hence

$\left|f'(z) - 1\right| \le 1 - \rho$ if

$$\sum_{m=2}^{\infty} \frac{m}{(1-\rho)} a_m |z|^{m-1} \le 1. \qquad (6.2)$$

By Theorem 1, we have

$$\sum_{m=2}^{\infty} \frac{m c_m \phi(k,m)}{(B-A)\alpha_1} a_m \le 1. \qquad (6.3)$$

By (6.2) and (6.3), we have

$$\frac{m}{(1-\rho)} |z|^{m-1} \le \frac{m c_m \phi(k,m)}{(B-A)\alpha_1}$$

or

$$|z| \le \left[\frac{(1-\rho)c_m \phi(k,m)}{(B-A)\alpha_1}\right]^{1/m-1} \qquad (m \ge 2). \qquad (6.4)$$

The theorem is completely proved.

Theorem 9. Let the function $f(z)$ defined by (1.6) be in the class $Q(\alpha_1, \alpha_2, \beta_1, \lambda, A, B)$, then $f(z)$ is starlike of order $\rho(0 \le \rho < 1)$ in $|z| < r_2(\alpha_1, \alpha_2, \beta_1, \lambda, A, B, \rho)$, where

$$r_2(\alpha_1, \alpha_2, \beta_1, \lambda, A, B, \rho) = \underset{m}{inf} \left[\frac{(1-\rho)c_m \phi(k,m)}{(B-A)\alpha_1}\right]^{1/m-1}, \qquad (m \ge 2). \qquad (6.5)$$

The result is sharp for the function $f(z)$ defined by (2.7).

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le 1 - \rho \quad \text{for } |z| < r_2(\alpha_1, \alpha_2, \beta_1, \lambda, A, B, \rho)$$

we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{\sum_{m=2}^{\infty} m-1)a_m |z|^{m-1}}{1-\sum_{m=2}^{\infty} a_m |z|^{m-1}}.$$

Thus

$\left|\frac{zf'(z)}{f(z)} - 1\right| \le 1 - \rho$ if

$$\sum_{m=2}^{\infty} \frac{(m-\rho)a_m |z|^{m-1}}{(1-\varphi)} \le 1. \qquad (6.6)$$

By using (6.3) and (6.6), we have

$$\frac{(m-\rho)|z|^{m-1}}{(1-\varphi)} \le \frac{m c_m \phi(k,m)}{(B-A)\alpha_1}$$

or

$$|z| \le \left[\frac{(1-\rho)mc_m\phi(k,m)}{(B-A)(m-\rho)\alpha_1}\right]^{1/m-1} \qquad (m \ge 2). \qquad (6.7)$$

The theorem is completely proved.

Corollary 1. Let the function $f(z)$ defined by (1.6) be in the class $Q(\alpha_1, \alpha_2, \beta_1, \lambda, A, B)$, then $f(z)$ is convex of order $\rho(0 \le \rho < 1)$ in $|z| < r_3(\alpha_1, \alpha_2, \beta_1, \lambda, A, B, \rho)$, where

$$r_3(\alpha_1, \alpha_2, \beta_1, \lambda, A, B, \rho) = \inf_m \left[\frac{(1-\rho)c_m\phi(k,m)}{(m-\rho)(B-A)\alpha_1}\right]^{1/m-1}, \qquad (m \ge 2). \qquad (6.8)$$

The result is sharp for the function $f(z)$ defined by (2.7).

VII. Properties involving Hadamard product

Let the function $f_i(z)$ $(i = 1,2, \dots \nu)$ defined by (5.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = z - \sum_{m=2}^{\infty} a_{m,1}a_{m,2}z^m \qquad (7.1)$$

Theorem 10. Let $f_1(z) \in Q(\alpha_1, \alpha_2, \beta_1, \lambda, A_1, B_1)$ and $f_2(z) \in Q(\alpha_1, \alpha_2, \beta_1, \lambda, A_2, B_2)$. Then the modified Hadamard product $f_1 * f_2(z)$ is an element of $Q(\alpha_1, \alpha_2, \beta_1, \lambda, \zeta(\alpha_1, \alpha_2, \beta_1, \lambda, A_1, B_1, A_2, B_2), 1)$. Where $\zeta(\alpha_1, \alpha_2, \beta_1, \lambda, A_1, B_1, A_2, B_2) = 1 - \frac{2\alpha_1(B_1-A_1)(B_2-A_2)}{m\{\alpha_1+\lambda(m-1)\}(1+B_1)(1+B_2)\phi(k,m)}$.

Proof. Employing the technique used earlier by Schit and Silverman[8], we need to find the largest $\zeta(\alpha_1, \alpha_2, \beta_1, \lambda, A_1, B_1, A_2, B_2)$ such that

$$\sum_{m=2}^{\infty} \frac{2m\{\alpha_1+\lambda(m-1)\}\phi(k,m)}{(1-\zeta)\alpha_1} a_{m,1}a_{m,2} \le 1. \qquad (7.2)$$

By Theorem 1, we have

$$\sum_{m=2}^{\infty} \frac{m(1+B_1)\{\alpha_1+\lambda(m-1)\}\phi(k,m)}{(B_1-A_1)\alpha_1} a_{m,1} \le 1, \qquad (7.3)$$

and

$$\sum_{m=2}^{\infty} \frac{m(1+B_2)\{\alpha_1+\lambda(m-1)\}\phi(k,m)}{(B_2-A_2)\alpha_1} a_{m,2} \le 1. \qquad (7.4)$$

From (7.3) and (7.4), by virtue of Cauchy-Schwarz inequality we obtain

$$\sum_{m=2}^{\infty} \frac{m\{\alpha_1+\lambda(m-1)\}\phi(k,m)\sqrt{(1+B_1)(1+B_2)}}{\sqrt{(B_2-A_2)(B_1-A_1)}\alpha_1}\sqrt{a_{m,1}a_{m,2}} \le 1. \qquad (7.5)$$

Hence

$$\sqrt{a_{m,1}a_{m,2}} \le \frac{(1-\zeta)\sqrt{(1+B_1)(1+B_2)}}{2\sqrt{(B_2-A_2)(B_1-A_1)}} \qquad (7.6)$$

From (7.5) we have

$$\sqrt{a_{m,1}a_{m,2}} \le \frac{\sqrt{(B_2-A_2)(B_1-A_1)}\alpha_1}{m\{\alpha_1+\lambda(m-1)\}\phi(k,m)\sqrt{(1+B_1)(1+B_2)}} \qquad (7.7)$$

(7.2) will be satisfied if

$$\frac{\sqrt{(B_2-A_2)(B_1-A_1)}\alpha_1}{m\{\alpha_1+\lambda(m-1)\}\phi(k,m)\sqrt{(1+B_1)(1+B_2)}} \le \frac{(1-\zeta)\sqrt{(1+B_1)(1+B_2)}}{2\sqrt{(B_2-A_2)(B_1-A_1)}}$$

i.e. $\quad \zeta \le 1 - \frac{2\alpha_1(B_1-A_1)(B_2-A_2)}{m\{\alpha_1+\lambda(m-1)\}(1+B_1)(1+B_2)\phi(k,m)}$

ζ is an increasing function for $m \ge 2$. Therefore. Setting $m = 2$ in (7.2)

$$\zeta \le 1 - \frac{\alpha_1(B_1-A_1)(B_2-A_2)}{\{\alpha_1+\lambda\}(1+B_1)(1+B_2)\phi(k,2)},$$

where

$$\phi(k,2) = \frac{\Gamma(\beta_1)(\alpha_1)\Gamma(\alpha_2+k)}{\Gamma(\alpha_2)\Gamma(\beta_1+k)}.$$

The result is sharp for the functions

$$f_1(z) = z - \frac{(B_1-A_1)}{2\{\alpha_1+\lambda\}(1+B_1)}z^2,$$

and

$$f_2(z) = z - \frac{(B_2-A_2)}{2\{\alpha_1+\lambda\}(1+B_2)}z^2.$$

The theorem is completely proved.

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