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On Generalized Half Canonical Cosine Transform

A. S. Gudadhe[1] and A.V. Joshi [2]
[1] *Govt. Vidarbha Institute of Science and Humanities, Amravati. (M. S.)*
[2] *Shankarlal Khandelwal College, Akola - 444002 (M. S.)*

Abstract: *As generalization of the fractional Cosine transform (FRCT), the Canonical Cosine Transform (CCT) has been used in several areas, including optical analysis and signal processing. For practical purpose half canonical cosine transform is more useful. Hence in this paper we have proved some important results Differentiation property, Modulation property, Scaling property, Derivative property, Parseval's Identity for half canonical cosine transform (HCCT).*

Keywords: *Linear canonical transform, Fractional Fourier Transform.*

I. Introduction:

The idea of the fractional powers of Fourier operator appeared in mathematical literature as early in 1930. It has been rediscovered in quantum mechanics by Namias [5]. He had given a systematic method for the development of fractional integral transforms by means of Eigenvalues. Later on numbers of integral transforms are extended in its fractional domain. For examples Almeida [2] had studied fractional Fourier transform, Akay [1] developed fractional Mellin transform, Sontakke, Gudadhe [8] studied number of property of fractional Hartley transform, Gudadhe, Joshi[3] studied number of property of generalized canonical cosine transform etc. These fractional transforms found number of applications in signal processing, image processing, quantum mechanics etc.

Recently further generalization of fractional Fourier transform known as linear canonical transform was introduced by Moshinsky [4] in 1971. Pei, Ding [5] had studied its eigen value aspect.
Linear canonical transform is a three parameter linear integral transform which has several special cases as fractional Fourier transform, Fresnel transform, Chirp transform etc. Linear canonical transform is defined as,

$$[LCTf(t)](s) = \sqrt{\frac{1}{2\pi ib}} \cdot \int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cdot e^{\frac{i}{2}(a/b)t^2} \cdot e^{-i(s/b)t} f(t)\, dt, \quad \text{for } b \neq 0$$

$$= \sqrt{d}\ e^{\frac{i}{2}(cd)s^2} \cdot f(d \cdot s), \text{ for } b = 0, \text{ with } ad - bc = 1,$$

Where a, b, c, and d are real parameters independent on s and t.
Pei and Ding [6] had defined canonical cosine transform (CCT) as

$$[CCTf(t)](s) = \sqrt{\frac{1}{2\pi ib}} \cdot \int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cdot \cos\left(\frac{s}{b}t\right) f(t)\, dt,$$

1. Testing Function Space ξ :

An infinitely differentiable complex valued function ϕ on R^n belongs to $\xi(R^n)$, if for each compact set, $I \subset S_\alpha$ where $S_\alpha = \{t :\in R^n, |t| \leq \alpha, \alpha > 0\}$ and for $k \in R^n$,

$$\gamma_{\xi,k}\,\phi(t) = \sup_{t \in I} \left|D^k \phi(t)\right| < \infty.$$

II. Half Canonical Cosine Transform:

2.1 Definition:

The Generalized Half Canonical Cosine Transform $f \in \xi^1(R^n)$ can be defined by,
{HCCT *f (t)*} (s) = < *f(t)*, K_{HC} (t, s) > where,

$$K_{HC}(t,s) = \sqrt{\frac{2}{\pi . ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2 \cdot} \cos\left(\frac{s}{b}t\right)$$

Hence the generalized half canonical cosine transform of $f \in \xi^1(R^n)$ can be defined by,

$$\{\ HCCTf(t)\ \}(s) = \sqrt{\frac{2}{\pi i b}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_0^\infty \cos\left(\frac{s}{b}t\right) \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)\, dt,$$

Since the range of integration for the half canonical cosine transform is just [0, ∞] and not for (-∞, ∞) using half canonical cosine transform is more convenient than using the canonical transform to deal with the even function.

2.1.1 Inversion theorem for Half Canonical Cosine Transform:

If $\{HCCT\ f(t)\}(s)$ half canonical cosine transform of *f(t)* is given by,

$$\{HCCT\ f(t)\}(s) = \sqrt{\frac{2}{\pi i b}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cdot \int_0^\infty \cos\left(\frac{s}{b}t\right) \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)\, dt \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.1.1)$$

then $$f(t) = \sqrt{\frac{\pi i}{2b}} \cdot e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2} \int_0^\infty e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cos\left(\frac{s}{b}t\right) \{HCCT\ f(t)\}(s)\, ds \quad \ldots\ldots\ldots..(2.1.2)$$

Proof: The half canonical cosine transform of *f(t)* is given by

$$\{HCCT\ f(t)\}(s) = \sqrt{\frac{2}{\pi i b}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_0^\infty \cos\left(\frac{s}{b}t\right) \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)\, dt$$

$$F_{HC}(s) = \sqrt{\frac{2}{\pi i b}}\ e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cos\left(\frac{s}{b}\right)t\ f(t)\, dt$$

$$\therefore F_{HC}(s)\sqrt{\frac{\pi i b}{2}} \cdot e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2} = \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)\ \cos\left(\frac{s}{b}t\right) dt$$

$$\therefore C_1(s) = \int_0^\infty g(t) \cdot \cos\left(\frac{s}{b}t\right) dt$$

where, $C_1(s) = F_{HC}(s)\sqrt{\frac{\pi i b}{2}} \cdot e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}$ and $g(t) = e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cdot f(t).$

$$C_1(s) = \int_0^\infty g(t) \cdot \cos\left(\frac{s}{b}t\right)\ dt = \int_0^\infty g(t) \cdot \cos(\eta t)\ d\eta.$$

where, $\left(\frac{s}{b}\right) = \eta, \therefore d\eta = \frac{1}{b} ds$

By using inverse formula, $g(t) = \int_0^\infty C_1(s) \cdot \cos(\eta.t)\ d\eta$

$$e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cdot f(t) = \int_0^\infty F(s)\ \sqrt{\frac{\pi i b}{2}}\ e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cdot \cos(\eta.t)\ d\eta$$

$$f(t) = e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2} \int_0^\infty F_{HC}(s)\ \sqrt{\frac{\pi i b}{2}}\ e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cos(\eta.t)\ d\eta.$$

$$f(t) = e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2} \int_0^\infty e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2} \sqrt{\frac{\pi i b}{2}}\ F_{HC}(s) \cos\left(\frac{s}{b}t\right) \frac{1}{b} ds.$$

$$f(t) = e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2} \sqrt{\frac{\pi . i}{2b}} \int_0^\infty e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cos\left(\frac{s}{b}t\right) F_{HC}(s) ds$$

$$f(t) = \sqrt{\frac{\pi . i}{2b}} \cdot e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2} \int_0^\infty e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cos\left(\frac{s}{b}t\right) \{HCCTf(t)\}(s).ds$$

III. Some Operational Results

3.1.1 Differentiation property of half canonical cosine transformations:

If {HCCT *f(t)*}(s) denotes generalized half canonical cosine transform of *f(t),* then

$$\left\{HCCT\ (f^1(t))\right\}(s) = i.\left(\frac{s}{b}\right)\{HCST\ f(t)\}(s) - i\left(\frac{a}{b}\right)\left\{HCCT\ t.f(t)\right\}(s)$$

Proof: We have,

$$\{HCCT\ f^1(t)\}(s) = \sqrt{\frac{2}{\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cdot \cos\left(\frac{s}{b}t\right) \cdot f^1(t)dt$$

By integration by parts, we get,

$$\{HCCT\ f^1(t)\}(s) = \sqrt{\frac{2}{\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \left\{\left[e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cos\left(\frac{s}{b}t\right) f(t)\right]_0^\infty - \left[\int_0^\infty \frac{d}{dt}\left(e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cos\left(\frac{s}{b}t\right)\right) \cdot \int_0^\infty f^1(t)dt\right]\right\}$$

$$= \sqrt{\frac{2}{\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \left\{[0 - \int_0^\infty \left\{\left[-\sin\left(\frac{s}{b}t\right)\right]\left(\frac{s}{b}\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} + \cos\left(\frac{s}{b}t\right)\frac{d}{dt}\left(e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\right)\right\} f(t)dt\right\}$$

$$= \sqrt{\frac{2}{\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \left\{-\int_0^\infty \left[-e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \sin\left(\frac{s}{b}t\right)\left(\frac{s}{b}\right) + \cos\left(\frac{s}{b}t\right) \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cdot \frac{i}{2}\left(\frac{a}{b}\right)2t\right] f(t)dt\right\}$$

$$= i\left\{\sqrt{\frac{2}{\pi ib}}\left(\frac{s}{b}\right)(-i)\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \sin\left(\frac{s}{b}t\right) f(t)dt - \sqrt{\frac{2}{\pi ib}}\left(\frac{a}{b}\right)\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cos\left(\frac{s}{b}t\right)(t\ f(t))dt\right\}$$

$$\left\{HCCT\ (f^1(t))\right\}(s) = i.\left(\frac{s}{b}\right)\{HCST\ f(t)\}(s) - i\left(\frac{a}{b}\right)\left\{HCCT\ t.f(t)\right\}(s)$$

3.1.2 Modulation property of half canonical cosine transform:

If *{HCCT f(t)} (s)* denotes generalized half canonical cosine transform of *f(t) t*hen,

$$\left\{HCCT \cos zt.f(t)\right\}(s) = \frac{e^{-\frac{i}{2}(db)z^2}}{2}\left\{\left[HCCT\ f(t)\cdot e^{idsz}\right]\left(\frac{s+bz}{b}\right) + \left[HCCT\ f(t)\cdot e^{-idsz}\right]\left(\frac{s-bz}{b}\right)\right\}$$

Proof: We have,

$$\{HCCT\ \cos zt.f(t)\}(s) = \sqrt{\frac{2}{\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} .\cos\left(\frac{s}{b}t\right) \cdot \cos zt \cdot f(t)\,dt$$

$$\{HCCT\ \cos zt.f(t)\}(s) = \sqrt{\frac{2}{\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \frac{2}{2}\cos\left(\frac{s}{b}t\right) \cdot \cos zt \cdot f(t)\,dt$$

$$\{HCCT\ \cos zt.f(t)\}(s) = \frac{1}{2}\sqrt{\frac{2}{\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} .\left(\cos\left(\frac{s}{b}+z\right)t + \cos\left(\frac{s}{b}-z\right)t\right) f(t)\,dt$$

$$\{HCCT\ \cos zt.f(t)\}(s) = \frac{1}{2}\left\{\begin{array}{l}\sqrt{\frac{2}{\pi ib}}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)(s+bz)^2} e^{-idsz} e^{-\frac{i}{2}(db)z^2} \cos\left(\frac{s}{b}+z\right)t.\ f(t)\,dt + \\ \sqrt{\frac{2}{\pi ib}}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)(s-bz)^2} e^{idsz} e^{-\frac{i}{2}(db)z^2} \cos\left(\frac{s}{b}-z\right)t..f(t)\,dt\end{array}\right.$$

$$\{HCCT\ \cos zt.f(t)\}(s) = \frac{e^{-\frac{i}{2}(db)z^2}}{2}\left\{\begin{array}{l}\sqrt{\frac{2}{\pi ib}}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)(s+bz)^2} \cos\left(\frac{s+bz}{b}\right)t.e^{-idsz}\ f(t)\,dt + \\ \sqrt{\frac{2}{\pi ib}}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)(s-bz)^2} \cos\left(\frac{s-bz}{b}\right)t.e^{idsz}.f(t)\,dt\end{array}\right.$$

$$\left\{HCCT \cos zt.f(t)\right\}(s) = \frac{e^{-\frac{i}{2}(db)z^2}}{2}\left\{\left[HCCT\ f(t)\cdot e^{idsz}\right]\left(\frac{s+bz}{b}\right) + \left[HCCT\ f(t)\cdot e^{-idsz}\right]\left(\frac{s-bz}{b}\right)\right\}$$

3**.1.3** If *{HCCT f(t)} (s)* denotes generalized half canonical cosine transform of *f(t)* then,

$$\{HCCT \sin zt. f(t)\}(s) = (-i)\frac{e^{-\frac{i}{2}(db)z^2}}{2}\left\{\left[HCST\ f(t)\cdot e^{-idsz}\right](\frac{s+bz}{b}) - \left[HCST\ f(t)\cdot e^{idsz}\right](\frac{s-bz}{b})\right\}$$

Proof: We have,

$$\{HCCT \sin zt. f(t)\}(s) = \sqrt{\frac{2}{\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} .\cos\left(\frac{s}{b}t\right)\cdot \sin zt\cdot f(t)\,dt$$

$$\{HCCT \sin zt. f(t)\}(s) = \sqrt{\frac{2}{\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\ \frac{2}{2}\cos\left(\frac{s}{b}t\right)\cdot \sin zt\cdot f(t)\,dt$$

$$\{HCCT \sin zt. f(t)\}(s) = \frac{1}{2}\sqrt{\frac{2}{\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} .\left(\sin\left(\frac{s}{b}+z\right)t - \sin\left(\frac{s}{b}-z\right)t\right) f(t)\,dt$$

$$\{HCCT \sin zt. f(t)\}(s) = \frac{1}{2}\left\{\begin{array}{l} \sqrt{\frac{2}{\pi ib}} \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)(s+bz)^2} e^{-idsz} e^{-\frac{i}{2}(db)z^2} \sin\left(\frac{s}{b}+z\right)t.\ f(t)\,dt \\ -\sqrt{\frac{2}{\pi ib}} \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)(s-bz)^2} e^{idsz} e^{-\frac{i}{2}(db)z^2} \sin\left(\frac{s}{b}-z\right)t..f(t)\,dt \end{array}\right\}$$

$$\{HCCT \sin zt. f(t)\}(s) = \frac{(-i)e^{-\frac{i}{2}(db)z^2}}{2}\left\{\begin{array}{l} (-i)\sqrt{\frac{2}{\pi ib}} \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)(s+bz)^2} \sin\left(\frac{s+bz}{b}\right)t.e^{-idsz}\ f(t)\,dt \\ -(-i)\sqrt{\frac{2}{\pi ib}} \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)(s-bz)^2} \cos\left(\frac{s-bz}{b}\right)t.e^{idsz}\ .f(t)\,dt \end{array}\right\}$$

$$\{HCCT \sin zt. f(t)\}(s) = \frac{(-i)e^{-\frac{i}{2}(db)z^2}}{2}\left\{\left[HCST\ f(t)\cdot e^{-idsz}\right](\frac{s+bz}{b}) - \left[HCST\ f(t)\cdot e^{idsz}\right](\frac{s-bz}{b})\right\}$$

3.1.4 Scaling property of half canonical cosine transform:

If *{HCCT f(t)}(s)* denotes generalized half canonical cosine transform, then,

$$\{HCCT\,[f(kt)]\}(s) = \frac{1}{k} e^{\left(1-\frac{1}{k}\right)\frac{i}{2}\left(\frac{d}{b}\right)s^2}\left[HCCT\left\{f(t)\cdot e^{\left(\frac{1}{k}-1\right)\frac{i}{2}\left(\frac{a}{bk}\right)t^2}\right\}\right](\frac{s}{bk})$$

Proof: We have,

$$\{HCCT\ f(kt)\}(s) = \sqrt{\frac{2}{\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cos\left(\frac{s}{b}t\right) f(kt)dt$$

Putting, $kt = T \Rightarrow dt = \frac{1}{k}dT$

$$\{HCCTf(kt)\}(s) = \sqrt{\frac{2}{\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)\frac{T^2}{k^2}} \cdot\cos\left(\frac{s}{b}\frac{T}{k}\right) f(T)\,\frac{dT}{k}$$

$$= \sqrt{\frac{2}{\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{-\frac{i}{2}\left(\frac{d}{bk}\right)s^2} e^{\frac{i}{2}\left(\frac{d}{bk}\right)s^2}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{bk^2}\right)T^2} e^{-\frac{i}{2}\left(\frac{a}{bk}\right)T^2} e^{\frac{i}{2}\left(\frac{a}{bk}\right)T^2}\cdot\cos\left(\frac{s}{bk}T\right)\ f(T)\,\frac{dT}{k}$$

$$= \frac{1}{k} e^{\left(1-\frac{1}{k}\right)\frac{i}{2}\left(\frac{d}{b}\right)s^2}\sqrt{\frac{2}{\pi ib}}\ e^{\frac{i}{2}\left(\frac{d}{bk}\right)s^2}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{bk}\right)t^2}\ \cos\left(\frac{s}{bk}\right)t\left[f(t)\cdot e^{\left(\frac{1}{k}-1\right)\frac{i}{2}\left(\frac{a}{bk}\right)t^2}\right]dt$$

$$\{HCCT\,[f(kt)]\}(s) = \frac{1}{k} e^{\left(1-\frac{1}{k}\right)\frac{i}{2}\left(\frac{d}{b}\right)s^2}\left[HCCT\left\{f(t)\cdot e^{\left(\frac{1}{k}-1\right)\frac{i}{2}\frac{a}{bk}t^2}\right\}\right](\frac{s}{bk})$$

3.1.5 Derivative property of half canonical cosine transform:

If *{HCCT f(t)}(s)* denotes generalized half canonical cosine transform, then,

$$\frac{d}{ds}\left[\{HCSTf(t)\}(s)\right]=i\left\{s.\left(\frac{d}{b}\right)\{HCCTf(t)\}(s)-\frac{1}{b}\{HCST[t.f(t)]\}(s)\right\}$$

Proof: We have,

$$\frac{d}{ds}\{HCCT\ \ f(t)\}(s)=\frac{d}{ds}\left\{\sqrt{\frac{2}{\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cos\left(\frac{s}{b}t\right)f(t)dt\right\}$$

$$\frac{d}{ds}\{HCCT\ \ f(t)\}(s)=\sqrt{\frac{2}{\pi ib}}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\frac{\partial}{\partial s}\left(e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}.\cos\left(\frac{s}{b}t\right)\right)f(t)dt$$

$$=\sqrt{\frac{2}{\pi ib}}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\left(-\left(\frac{t}{b}\right)e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}.\sin\left(\frac{s}{b}t\right)+i\left(\frac{d}{b}\right).s.e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cos\left(\frac{s}{b}t\right)\right)f(t)dt$$

$$=\sqrt{\frac{2}{\pi ib}}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}i.\left(\frac{-1}{i}\left(\frac{t}{b}\right)e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}.\sin\left(\frac{s}{b}t\right)+\left(\frac{d}{b}\right).s.e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cos\left(\frac{s}{b}t\right)\right)f(t)dt$$

$$=i\left\{-(-i)\sqrt{\frac{2}{\pi ib}}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\left(\frac{1}{b}\right)e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}.\sin\left(\frac{s}{b}t\right)[t.f(t)]dt+s.\left(\frac{d}{b}\right)\sqrt{\frac{2}{\pi ib}}\cdot\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cos\left(\frac{s}{b}t\right)f(t)dt\right\}$$

$$=i\{-\frac{1}{b}[HCST[t.f(t)]\}(s)+s.\left(\frac{d}{b}\right)\{HCCTf(t)\}(s)$$

$$\frac{d}{ds}\left[\{HCSTf(t)\}(s)\right]=i\left\{s.\left(\frac{d}{b}\right)\{HCCTf(t)\}(s)-\frac{1}{b}\{HCST[t.f(t)]\}(s)\right\}$$

IV. Parseval's Identity for half canonical cosine transform:

If *f(t)* and *g(t)* are the inversion canonical half cosine transform of $F_{HC}(s)$ and $G_{HC}(s)$ respectively, then (1)

$$\int_0^\infty f(t).\overline{g(t)}\ dt=(-i)\frac{\pi}{2}\int_0^\infty F_{HC}(s).\overline{G_{HC}(s)}\ ds$$ and (2) $$\int_0^\infty .\left|f(t)\right|^2 dt=(-i)\frac{\pi}{2}\int_0^\infty .\left|F_{HC}(s)\right|^2 ds$$

Proof: By definition of HCCT,

$$\{HCCT\ \ g(t)\}(s)=\sqrt{\frac{2}{\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cos\left(\frac{s}{b}t\right)\cdot g(t)\,dt \quad \text{---------------- (4.1.1)}$$

Using the inversion formula of HCCT

$$g(t)=\sqrt{\frac{\pi i}{2b}}\cdot e^{-\frac{i}{2}\left(\frac{d}{b}\right)t^2}\int_0^\infty e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cos\left(\frac{s}{b}t\right)G_{HC}(s)\,ds$$

Taking complex conjugate we get,

$$\overline{g(t)}=\sqrt{\frac{-\pi i}{2b}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)t^2}\int_0^\infty e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cos\left(\frac{s}{b}t\right)\overline{G_{HC}(s)}\,ds$$

$$\int_0^\infty f(t).\overline{g(t)}\ dt=\int_0^\infty f(t)dt\left(\sqrt{-\frac{\pi i}{2b}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)t^2}\int_0^\infty e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cos\left(\frac{s}{b}t\right)\overline{G_{HC}(s)}\,ds\right)$$

Changing the order of integration, we get,

$$\int_0^\infty f(t).\overline{g(t)}\ dt=\sqrt{-\frac{\pi i}{2b}}\int_0^\infty\overline{G_{HC}(s)}\,ds\frac{1}{\sqrt{\frac{2}{\pi ib}}}\left(\sqrt{\frac{2}{\pi ib}}\ e^{\frac{i}{2}\left(\frac{d}{b}\right)t^2}\int_0^\infty e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cos\left(\frac{s}{b}t\right)f(t)dt\right)$$

$$\int_0^\infty f(t).\overline{g(t)}\ dt=(-i)\frac{\pi}{2}\int_0^\infty\overline{G_{HC}(s)}\,.F_{HC}(s)ds$$

$$\int_0^\infty f(t).\overline{g(t)}\ dt=(-i)\frac{\pi}{2}\int_0^\infty F_{HC}(s).\overline{G_{HC}(s)}\ ds \quad \text{---------------------(4.1.2)}$$

Hence proved

(ii) Putting $f(t) = g(t)$ in equation (4.1.2), we get

$$\int_0^\infty |f(t)|^2 \, dt = (-i)\frac{\pi}{2}\int_0^\infty |F_{HC}(s)|^2 \, ds$$

Table for half canonical cosine transform

S.N.	$f(t)$	$F_{HC}(s)$
1	$f^1(t)$	$i.\left(\frac{s}{b}\right)\{HCST\ f(t)\}(s) - i\left(\frac{a}{b}\right)\{HCCT\ t.f(t)\}(s)$
2	$\cos zt.f(t)$	$\frac{e^{-\frac{i}{2}(db)z^2}}{2}\left\{\left[HCCT\ f(t)\cdot e^{idsz}\right](\frac{s+bz}{b}) + \left[HCCT\ f(t)\cdot e^{-idsz}\right](\frac{s-bz}{b})\right\}$
3	$\sin zt.f(t)$	$(-i)\frac{e^{-\frac{i}{2}(db)z^2}}{2}\left\{\left[HCST\ f(t)\cdot e^{-idsz}\right](\frac{s+bz}{b}) - \left[HCST\ f(t)\cdot e^{idsz}\right](\frac{s-bz}{b})\right\}$
4	$f(kt)$	$\frac{1}{k}e^{\left(1-\frac{1}{k}\right)\frac{i}{2}\left(\frac{d}{b}\right)s^2}\left[HCCT\left\{f(t)\cdot e^{\left(\frac{1}{k}-1\right)\frac{i}{2}\left(\frac{a}{bk}\right)t^2}\right\}\right](\frac{s}{bk})$

V. Conclusion:

In this paper, brief introduction of the generalized half canonical cosine transform are given and its Inversion theorem for half canonical cosine, Differentiation property, Modulation property, Parseval's Identity, Scaling property, Derivative property for half canonical cosine transform obtained which will be useful in solving differential equations occurring in signal processing and many other branches of engineering.

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