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A Self-Starting Hybrid Linear Multistep Method for a Direct Solution of the General Second-Order Initial Value Problem

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***Abstract* :** *A self- starting hybrid linear multistep method for direct solution of the general second-order initial value problem is considered. The continuous method is used to obtain Multiple Finite Difference Methods (MFDMs) (each of order 7) which are combined as simultaneous numerical integrators to provide a direct solution to IVPs over sub-intervals which do not overlap. The convergence of the MFDMs is discussed by conveniently representing the MFDMs as a block method and verifying that the block method is zero-stable and consistent. The superiority of the MFDMs over published work is established numerically.*

Keywords*: Multiple Finite Difference Methods, Second Order, Boundary Value Problem, Block Methods, Multistep Methods*

I. Introduction

The mathematical formulation of physical phenomena in science and engineering often leads to initial value problems of the form:

$$y'' = f(x, y), \; y(a) = y_0 \;, \; y'(a) = \eta_0 \tag{1}$$

However, only a limited number of analytical methods are available for solving (1) directly without reducing to a first order system of initial value problems. Some authors have proposed solution to higher order initial value problems of ordinary differential equations using different approaches [1-5]. In particular Awoyemi and Idowu [2] developed a class of hybrid collocation method for third order ordinary differential equations. Awoyemi[1] derived a p-stable linear multistep method for general third order initial value problems of ordinary differential equations which is to be used in form of predictor-corrector forms and like most linear multistep methods, they require starting values from Runge-Kutta methods or any other one-step methods. The predictors are also developed in the same way as correctors. Moreover, the block methods in Fatunla [3] are discrete and are proposed for non-stiff special second order ordinary differential equations in form of a predictor- corrector integration process. Also like other linear multistep methods they are usually applied to the initial value problems as a single formula but they are not self-starting; and they advance the numerical integration of the ordinary differential equations in one-step at a time, which leads to overlapping of the piecewise polynomials solution Model. There is the need to develop a method which is self-starting, eliminating the use of predictors with better accuracy and efficiency. This study, therefore propose a block hybrid multistep method for the direct solution of third order initial value problems of ordinary differential equations.

Recently, several researches [6-10] proposed LMMs for the direct solution of the general second and third order IVPs, which were showed to be zero stable and were implemented without the need for either predictors or starting values from other methods. Jator [11] used the LMMs developed for IVPs and additional methods obtained from the same continuous k-step LMM to solve third order BVPs with Dirichlet and Neumann boundary conditions and also Yahaya and Mohammed [12] developed a 5-step block method for special second order ordinary differential equations. We extended their methods into hybrid form by adding one off-step point at collocation. The 5-step block hybrid method is zero-stable, consistent and convergent.

II. Development Of Methods.

In this section, our objective is to derive hybrid linear multi-step method (HLMM) of the form

$$\sum_{j=0}^{r-1} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{s-1} \beta_j f_{n+j} + h^2 \beta_v f_{n+v} \tag{2}$$

Where α_j, β_j and β_v are unknown constants and v is not an integer. We note that α_k =1, $\beta_j \neq 0$, α_0 and β_0 do not both vanish. In order to obtain (2), we proceed by seeking to approximate the exact solution Y(x) of the form

$$Y(x) = \sum_{j=0}^{r+s-1} l_j x^j , \tag{3}$$

Where $x \in [a,b]$, l_j are unknown coefficients to be determined and $1 \le r < k$, $s > 0$ are the number of interpolation and collocation points respectively. We then construct our continuous approximation by imposing the following conditions.

$$Y(x_{n+j}) = y_{n+j} , \quad j = 0, 1, 2, ..., r-1 \tag{4}$$

$$Y''(x_{n+\mu}) = f_{n+\mu} \tag{5}$$

Equation (4) and (5) lead to a system of (r+s) equations which is solved by Cramer's rule to obtain l_j. Our continuous approximation is constructed by substituting the values of l_j into equation (3). After some manipulation, the continuous method is expressed as

$$Y(x) = \sum_{j=0}^{r-1} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{s-1} \beta_j(x) f_{n+j} + h^2 \beta_v(x) f_{n+v} \tag{6}$$

Where $\alpha_j(x), \beta_j(x)$ and $\beta_v(x)$ are continuous coefficients. We note that since equation (1) involves first and second derivatives, the first derivative formula

$$Y'(x) = \frac{1}{h} \left(\sum_{j=0}^{r-1} \alpha_j'(x) y_{n+j} + h^3 \sum_{j=0}^{s-1} \beta_j'(x) f_{n+j} + h^3 \beta_v'(x) f_{n+v} \right) \tag{7}$$

Equation (7) is easily obtained from (6) and is then used to provide the first and second derivatives for the methods by imposing the condition

$$Y'(x) = z(x) \tag{8}$$

$$Y'(a) = z_0 \tag{9}$$

III. Specification Of The Methods

Our methods are obtained from section two and expressed in the form of (2) given by

$$\bar{y}(x) = \sum_{j=0}^{r-1} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{s-1} \beta_j(x) f_{n+j} + h^2 \beta_v(x) f_{n+v} \tag{10}$$

with the following specification $r = 2, s = 7, k = 5, \ \gamma_i(x) = x^i , \ i = 0, 1, ..., 12$ we also express the continuous form as follows:

: $Y(x) = \left(4 - \frac{x-x_n}{h}\right) y_{n+3} + \left(-3 + \frac{x-x_n}{h}\right) y_{n+4} + \left\{\frac{493}{7560} h - \frac{27271}{90720} h(x - x_n) + \frac{1}{2}(x - x_n)^2 - \frac{451}{1080} \frac{(x-x_n)^3}{h} +\right.$

257312960x−xn4h2−9160x−xn5h3+616480x−xn6h4−1315120x−xn7h5+130240x−xn8h6fn+31392940h2−5248935280hx−xn+1514x−xn3h−271336x−xn4h2+9473360x−xn5h3−671260x−xn6h4+377056x−xn7h5−14704x−xn8h6fn+1+267140h2−2980hx−xn−32x−xn3h+361240x−xn4h2−149240x−xn5h3+47360x−xn6h4−172x−xn7h5+11680x−xn8h6fn+2+859315h2−60013780hx−xn+53x−xn3h−391216x−xn4h2+199240x−xn5h3−103540x−xn6h4+11504x−xn7h5−11008x−xn8h6fn+3+169840 h2+653310080hx−xn−158x−xn3h+20396x−xn4h2−491480x−xn5h3+181720x−xn6h4−311008x−xn7h5+1672x−xn8h6fn+4+3526615h2−1050419845hx−xn+256189x−xn3h −43842835x−xn4h2+1621x−xn5h3−5442835x−xn6h4+321323x−xn7h5−86615x−xn8h6fn+92+−3140h2+67560hx−xn−310x−xn3h +83240x−xn4h2−83480x−xn5h3+245x−xn6h4−295040x−xn7h5+13360x−xn8h6fn+5

(11)

, The MFDMs are obtained by evaluating (11) at $x = \left\{ x_n, x_{n+1}, x_{n+2}, x_{n+\frac{9}{2}}, x_{n+5} \right\}$ to obtain the following

$$y_{n+5} = -y_{n+3} + 2y_{n+4} - \frac{2}{2835}h^2 f_n + \frac{223}{35280}h^2 f_{n+1} - \frac{37}{1260}h^2 f_{n+2} + \frac{1231}{7560}h^2 f_{n+3} + \frac{821}{1260}h^2 f_{n+4} + \frac{3536}{19845}h^2 f_{n+\frac{9}{2}} + \frac{157}{5040}h^2 f_{n+5} \quad (12)$$

$$y_{n+2} = 2y_{n+3} - y_{n+4} - \frac{31}{45360}h^2 f_n - \frac{151}{17640}h^2 f_{n+1} + \frac{283}{2520}h^2 f_{n+2} + \frac{1489}{1890}h^2 f_{n+3} + \frac{659}{5040}h^2 f_{n+4} - \frac{496}{19845}h^2 f_{n+\frac{9}{2}} + \frac{1}{504}h^2 f_{n+5} \quad (13)$$

$$y_{n+1} = 3y_{n+3} - 2y_{n+4} - \frac{2}{945}h^2 f_n + \frac{923}{11760}h^2 f_{n+1} + \frac{439}{420}h^2 f_{n+2} + \frac{4171}{2520}h^2 f_{n+3} + \frac{121}{420}h^2 f_{n+4} - \frac{496}{6615}h^2 f_{n+\frac{9}{2}} + \frac{17}{1680}h^2 f_{n+5} \quad (14)$$

$$y_n = 4y_{n+3} - 3y_{n+4} - \frac{493}{7560}h^2 f_n + \frac{3139}{2940}h^2 f_{n+1} + \frac{267}{140}h^2 f_{n+2} + \frac{859}{315}h^2 f_{n+3} + \frac{169}{840}h^2 f_{n+4} + \frac{352}{6615}h^2 f_{n+\frac{9}{2}} + \frac{3}{140}h^2 f_{n+5} \quad (15)$$

$$y_{n+\frac{9}{2}} = \frac{1}{2}y_{n+3} - \frac{3}{2}y_{n+4} - \frac{2531}{7741440}h^2 f_n + \frac{17807}{6021120}h^2 f_{n+1} + \frac{2001}{143360}h^2 f_{n+2} + \frac{103039}{1290240}h^2 f_{n+3} + \frac{273841}{860160}h^2 f_{n+4} - \frac{3769}{211680}h^2 f_{n+\frac{9}{2}} + \frac{339}{57344}h^2 f_{n+5} \quad (16)$$

In particular, to start the initial value problem for n = 0, we obtain the following equations from (9):

$$z_n = -\frac{1}{635040}\frac{1}{h}\left\{635040y_{n+3} - 635040y_{n+4} + 190897h^2 f_n + 944802h^2 f_{n+1} + 230202h^2 f_{n+2} + 1008168h2fn+3 - 411579h2fn+4 + 336128h2fn+92 - 75978h2fn+5 \right. \quad (17)$$

It is worth noting that the derivatives are provided as follows:

$$z_{n+1} = -\frac{1}{90720}\frac{1}{h}\left\{-90720y_{n+3} + 90720y_{n+4} + 928h^2 f_n - 33975h^2 f_{n+1} - 114228h^2 f_{n+2} - 61878h^2 f_{n+3} - 29772h^2 f_{n+4} + 15104 f_{n+\frac{9}{2}} - 2979h^2 f_{n+5}\right\}$$

$$z_{n+2} = -\frac{1}{211680}\frac{1}{h}\left\{211680y_{n+3} - 211680y_{n+4} + 539h^2 f_n - 6234h^2 f_{n+1} + 93534h^2 f_{n+2} + 220248 f_{n+3} + 1239h^2 f_{n+4} + 11008h^2 f_{n+\frac{9}{2}} - 2814h^2 f_{n+5}\right\}$$

$$z_{n+3} = \frac{1}{635040}\frac{1}{h}\left\{-635040y_{n+3} + 635040y_{n+4} + 560h^2 f_n - 5121h^2 f_{n+1} + 24948h^2 f_{n+2} - 226842 f_{n+3} - 16560h^2 f_{n+4} + 64768h^2 f_{n+\frac{9}{2}} - 10773h^2 f_{n+5}\right\}$$

$$z_{n+4} = \frac{1}{635040}\frac{1}{h}\left\{635040y_{n+3} + 635040y_{n+4} + 497h^2 f_n - 4446h^2 f_{n+1} + 20538h^2 f_{n+2} - 110040 f_{n+3} - 286587 f_{n+4} + 73984h^2 f_{n+\frac{9}{2}} - 11466h^2 f_{n+5}\right\}$$

$$z_{n+\frac{9}{2}} = \frac{1}{2903040}\frac{1}{h}\left\{2903040y_{n+3} + 2903040y_{n+4} + 1517h^2 f_n - 13977h^2 f_{n+1} + 68202h^2 f_{n+2} - 425706 f_{n+3} - 2137383h^2 f_{n+4} - 382976h^2 f_{n+\frac{9}{2}} - 12717h^2 f_{n+5}\right\}$$

$$z_{n+5} = \frac{1}{211680}\frac{1}{h}\left\{211680y_{n+3} - 211680y_{n+4} + 224h^2 f_n - 1941h^2 f_{n+1} + 8484h^2 f_{n+2} - 40194 f_{n+3} - 124068h^2 f_{n+4} - 122624h^2 f_{n+\frac{9}{2}} - 37401h^2 f_{n+5}\right\}$$

IV. Analysis And Implementation Of The Method

Following Fatunla [13] and Lambert [4] we define the local truncation error associated with the conventional form of (2) to be the linear difference operator

$$L[y(x);h] = \sum_{j=0}^{k} \{\alpha_j y(x+jh) - h^2 \beta_j y'''(x+jh)\} + h^2 \beta_v f_{n+v}$$

(18)

Assuming that y(x) is sufficiently differentiable, we can expand the terms in (18) as a Taylor series about the point x to obtain the expression

$$L[y(x);h] = C_0 y(x) + C_1 hy' + \ldots, + C_q h^q y^q(x) + \ldots,$$

(19)

Where the constant coefficients $C_q, \quad q = 0,1,\ldots$ are given as follows: $C_q, \quad q = 0, 1,\ldots$

$$C_0 = \sum_{j=0}^{k} \alpha_j ,$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j ,$$

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$$C_q = \left[\frac{1}{q!} \sum_{j=1}^{k} j^q \alpha_j - q(q-1) \sum_{j=1}^{k} j^{q-2} \beta_j \right].$$

According to Henrici [14], we say that the method (5) has order p if

$$C_0 = C_1 = \ldots = C_P = C_{P+1} = 0, \quad C_{P+2} \neq 0$$

Our calculations reveal that the methods (12) to (16) have order p = 7 and error constants given by the vector

$$\left(C_9 - \frac{229}{846720}, \frac{31}{282240}, \frac{31}{169344}, \frac{797}{423360} \right)^T$$

In order to analyze the methods for zero-stability, we normalize (12) to (17) and write them as a block method given by the matrix difference equation

$$A^0 Y_{\mu+1} = A^1 Y_\mu + h^2 \left[B^0 F_{\mu+1} + B^1 F_\mu \right] \tag{20}$$

Where $Y_{\mu+1} = (y_{n+1}, \ldots, y_{n+3})^T$, $Y_\mu = (y_{n-3} \ldots, y_n)^T$,.

$F_{\mu+1} = (f_{n+1}, \ldots, f_{n+3})^T$, $F_\mu = (f_{n-3} \ldots, f_n)^T$ *and* $n = 0,5,..$

and matrices A^0 and A^1 are defined as follows:

A^0 is an identity matrix of dimension 6

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

It is worth noting that zero-stability is concerned with the stability of the difference system in the limit as h tends to zero. Thus, as $h \to 0$, the method (20) tends to the difference system

$A^0 Y_{\mu+1} - A^1 Y_\mu = 0$ Whose first characteristic polynomial $\rho(R)$ is given by

$$\rho(R) = \det(RA^0 - A^1) = R^5(R-1)$$

(21)

Following Fatunla [13], the block method (20) is zero-stable, since from (21),

$\rho(R) = 0$ *Satisfy* $|R_j| \leq 1 \; j = 1,\ldots,k$ and for those roots with $|R_j|$ =1, the multiplicity does not exceed 2.

The block method (20) is consistent as it has order $P > 1$. According to Henrici [14], we can safely assert the convergence of the block method (20).

It is vital to note that the main method given by (10) can be used as a numerical integrator directly and singly in the conventional way on overlapping sub-intervals. However, our method is implemented more efficiently by combining methods (12) to (16), each of order seven with relatively small error constants, as simultaneous integrators for IVPs without looking for any other methods to provide the starting values. We proceed by explicitly obtaining initial conditions at $x_{n+5},\ n=0,5,\ldots,N-5$ using the computed values $y(x_{n+5})=y_{n+5}, z(x_{n+5})=z_{n+5}$ over sub-intervals $[x_0, x_5],\ldots[x_{n-5},\ x_N]$ which do not overlap (see [10]). For instance, $n=0,(y_1, y_2, y_3, y_4, y_5)^T$ are simultaneously obtained over the sub-interval $[x_0, x_5]$ as y_0 is known from the IVP, for $n=5,(y_6, y_7, y_8, y_9, y_{10})^T$ are simultaneously obtained over the sub-interval $[x_5, x_{10}]$, as y_5 is known from the previous block, and so on. Hence, the sub-intervals do not over-lap and the solutions obtained in this manner are more accurate that those obtained in the conventional way.

V. Numerical Examples

In this section, we have tested the performance of the method on three problems by considering nonlinear IVPs (Examples 4.1), linear non-homogeneous ODE (Example 4.2), mildly stiff problem (Example 4.3). For each example, we find absolute errors of the approximate solution.

Example 4.1 We consider the equation

$$y'' = x(y')^2 \quad y(0)=1,\ y'(0)=\frac{1}{2}$$

$$Exact\ \ Solution:\ y(x)=1+\frac{1}{2}\ln\left(\frac{(2-x)}{(2+x)}\right)$$

It is obvious that, our method performs better than those given in Awoyemi [3,4] despite the fact that we used a larger step size *h* = 0.05. Hence, for this example, our method is clearly superior. The details of the numerical results at some selected points are given in Table 4.1

Table 4.1

x	Awoyemi [1] Order p = 6 h = 0.003125	Awoyemi and Kayode[2] Order p = 8 h = 0.003125	Jator[6] Order p = 6 h = 0.05	Our Methods Order p=7 *h* = 0.05
0.1	0.26075×10^{-9}	0.66391×10^{-13}	0.71629×10^{-11}	0
0.2	1.98167×10^{-9}	0.20012×10^{-9}	$0.15091. \times 10^{-10}$	0
0.3	6.50741×10^{-9}	1.72007×10^{-9}	0.45286×10^{-10}	0
0.4	15.5924×10^{-9}	5.89464×10^{-9}	1.08084×10^{-10}	0
0.5	31.5045×10^{-9}	14.4347×10^{-9}	1.78186×10^{-10}	0
0.6	56.3746×10^{-9}	41.8664×10^{-9}	4.44344×10^{-10}	0
0.7	96.1640×10^{-9}	53.1096×10^{-9}	7.44460×10^{-10}	0
0.8	156.868×10^{-9}	91.1317×10^{-9}	15.0098×10^{-10}	0
0.9	248.698×10^{-9}	149.242×10^{-9}	37.5797×10^{-10}	0
1.0	387.984×10–9	237.189×10^{-9}	47.4108×10^{-10}	0

Example 4.2 We consider the non-homogeneous ODE given by

$$y'' - 4y' + 8y = x^3 \quad y(0)=2,\ y'(0)=4$$

$$Exact\ \ Solution:\ y(x)=e^{2x}\left(2\cos 2x-\frac{3}{64}\sin 2x\right)+\frac{3}{32}x\ +\frac{3}{16}x^2+\frac{1}{8}x^3$$

Although the numerical results of this problem were not compared with another method, the results were compared with the theoretical solution as shown in Table 4.2.

Table 4.2

Exact solution Y(x)	Numerical solution Y(x)	Present Error	Error by Jator [7]
2	2	0	0.00000×10^{-6}
2.394112577	2.39411253	4.69999999E-08	5.10704×10^{-6}
2.748141333	2.748141222	1.11000000E-07	14.9586×10^{-6}
3.00786694	3.007866777	1.63000000E-07	27.8532×10^{-6}
3.101762405	3.101762202	2.03000000E-07	42.8908×10^{-6}
2.939543102	2.939542877	2.25000000E-07	67.0307×10^{-6}

Example 4.3. We consider the mildly stiff IVP

$$y'' + 1001y' + 1000y = 0 \quad y(0) = 1, \ \ y'(0) = -1$$

$$Exact \ \ Solution : \ y(x) = e^{-x}$$

Although the numerical results for this problem were not compared with another method, the results were compared with the theoretical solution as shown in Table 4.3.

Table 4.3

x	Y(x)-Exact	Y-Numerical	Error
0	1.000000000	1.000000000	0
0.1	0.9048374180	0.9048374180	0
0.2	0.8187307531	0.8187307531	0
0.3	0.7408182207	0.7408182207	0
0.4	0.6703200460	0.6703200460	0
0.5	0.6065306597	0.6065306597	0
0.6	0.5488116361	0.5488116361	0
0.7	0.4965853038	0.4965853038	0
0.8	0.4493289641	0.4493289641	0
0.9	0.4065696597	0.4065696597	0
1.0	0.3678794412	0.3678794412	0

VI. Conclusion

We have derived a five-step continuous HLMM from which MFDMs are obtained and applied to solve $y'' = f(x, y)$ without first adapting the ODE to an equivalent first order system or reducing it to an initial-value problem. The MFDMs are applied as simultaneous numerical integrators over sub-intervals which do not overlap and hence they are more accurate than SFDMs which are generally applied as single formulas over overlapping intervals. We have shown that the methods are zero stable, convergent and which make them suitable candidates for computing solutions on wider intervals. In addition to providing additional methods and derivatives, the continuous HLMM can be used to obtain global error estimates. Our future research will be focused on adapting the MFDMs to solve third order partial differential equations.

References

[1] Awoyemi, D.O., 2003. A P-stable linear multistep method for solving general third order ordinary differential equations. *Int. J. Comput Math., 8: 985-991. DOI: 10.1080/0020716031000079572*

[2] Awoyemi, D. and Idowu,O. 2005. A class hybrid collocation methods for third order of ordinary differential equations, *Int. J. Comput. Math., 82: 1287-1293. DOI: 10.1080/00207160500112902.*

[3] Fatunla, S.O., (1994). A class of block methods for second order IVPs. Int. *J. Comput. Math., 55: 119-133. DOI: 10.1080/00207169508804368*

[4] Lambert, J.D., (1973). *Computational Methods in Ordinary Differential Equations* (John Willey and Sons, New York, USA., ISBN: 10: 0471511943, p: 294.)

[5] Adee, S.O., Onumanyi, P., Sirisena,U.W., and Yahaya,Y.A (2005). Note on starting numerov method more accurately by a hybrid formula of order four for an initial value problem,. *J.Computat. Applied Math., 175: 369-373. DOI: 10.1016/j.cam.2004.06.016.*

[6] Jator, S.N(2007). A sixth order linear multistep method for the direct solution of y00 = f(x, y, y0), *International Journal of Pure and Applied Mathematics, 40, No. 4, 457-472.*

[7] Jator, S.N and Li, J (2007) A self-starting linear multistep method for a direct solution of the general second order initial value problem, *International Journal of Computer Mathematics Vol. 86, No. 5, May 2009, 827–836*

[8] Jator, S.N (2008) Multiple finite difference methods for solving third order ordinary differential equations, *International Journal of Pure and Applied Mathematics, 43, No. 2, 253 - 265.*

[9] Mohammmed, U.,Jiya, M and Mohammed, A.A(2010). A class of six step block method for solution of general second order ordinary differential equations, *Pacific Journal of Science and Technology*. *11(2):pp273-277*.
[10] Mohammmed, U (2011). A class of implicit five step block method for general second order ordinary differential equations. *Journal of Nigerian Mathematical Society (JNMS*)**.** vol 30 p 25-39
[11] Jator, S.N (2008). On the numerical integration of third order boundary value problems by a linear multistep method, *International Journal of Pure and Applied Mathematics, 46, No. 3,375 -388.*
[12] Yahaya Y A and Mohammed U (2010). A 5-step block method for special second order ordinary differential equations, *Journal of Nigerian Mathematical Society (JNMS*)**.** vol 30 p 113-126
[13] Fatunla S.O (1991): Block Method for Second Order Initial Value Problem (IVP), *International Journal of Computer Mathematics, Vol. 41:55-63., England.*
[14] Henrici, P. (1962); *Discrete variable methods for ODE`s*, New York USA, John Wiley and sons.