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On The Existence of Solution of A Tuberculosis Epidemic Model

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Abstract: *We show that there exists a unique solution of the tuberculosis model proposed in Blower et.al. [2] Under certain conditions. We formulate and prove a theorem on existence and uniqueness of solution.*
Keywords: *tuberculosis, mathematical model, epidemic, existence and uniqueness of solution*

I. Introduction

The development of tuberculosis (TB) due to Mycobacterium tuberculosis is a complex, multi-stage process. Three possible routes of infection are primary progression after a recent infection, re-activation of a latent infection or exogenous re-infection of a previously infected individual [5]. Tuberculosis is easily spread by body fluids, which means that it can become airborne simply by someone coughing, sneezing or spitting [6].A newly infected person may take 3 to 4 weeks before transmitting the disease to others [9].Many people may not realize they are infected as the infection is usually latent but may develop later into active disease [10].

The global burden of tuberculosis has increased over the years despite widespread implementation of control measures including BCG vaccination and the WHO's DOTS (directly observed therapy, short course) which focuses on case finding and short course therapy [4].The incidence of the disease is rising in Africa, Eastern Europe and Asia. In these regions, the emergence of multi-drug resistant TB strains and the convergence of HIV and TB epidemics have contributed largely to the spread of the disease [15].Studies have shown that optimal TB control strategies may vary depending on the predominant route to disease [8, 12, 13, 14, 15]. It is therefore important for public health policy makers to understand the relative frequency of each type within specific epidemiological scenario.

The dynamics of tuberculosis epidemic has been a subject of rigorous research among many researchers. Many mathematical models have been proposed to investigate the complex epidemic and endemic behavior of TB [1,2,3,4,5,6,7,9,10,11]. In a paper by Blower et. al. [2], the authors studied the transmission dynamics of TB by the following set of first order differential equations.

$$S^1 = \pi + sI - \beta IS - \mu S \qquad (1)$$
$$E^1 = (1-\rho)\beta IS - (v+\mu)E \qquad (2)$$
$$I^1 = \rho\beta IS + vE - (\mu + \mu_T)I \qquad (3)$$

Where

$\pi =$ Recruitment rate of susceptible individuals
$\mu =$ Natural death rate
$\mu_T =$ Death rate due to TB infection
$v =$ Rate of slow progression
$\rho =$ Rate of fast progression

Their objective was to find a critical threshold parameter called the basic reproduction number denoted by R_0 which determines the virulence of infection. It was shown that if $R_0 < 1$, the infection is temporal and will eventually die out but if $R_0 > 1$, the infection persists and an epidemic result.

In this paper, we formulate a theorem on existence of solution and show that there exists a unique solution of the model system (1)-(3).

II. Methodology

2.1 Statement of Theorem

Consider the system of equations below

$$\left.\begin{aligned} x_1^1 &= f_1(t, x_1, x_2, \ldots, x_n),\ x_1(t_0) = x_{10} \\ x_2^1 &= f_2(t, x_1, x_2, \ldots, x_n),\ x_2(t_0) = x_{20} \\ &\vdots \\ x_n^1 &= f_n(t, x_1, x_2, \ldots, x_n),\ x_n(t_0) = x_{n0} \end{aligned}\right\} \qquad (4)$$

We may write (4) in compact form as

$$x^1 = f(t, x), \quad x(t_0) = x_0 \qquad (5)$$

Theorem 1 [8] Let D^1 denotes the region

$$|t - t_0| \leq a, \|x - x_0\| \leq b, x = (x_1, x_2, \dots, x_n), x_0 = (x_{10}, x_{20}, \dots, x_{n0}) \quad (6)$$

And suppose that f(t,x) satisfies the Lipschitz condition

$$\|f(t, x_1) - f(t, x_2)\| \leq k\|x_1 - x_2\| \quad (7)$$

Whenever the pairs (t, x_1) and (t, x_2) belong to D^1, where k is a positive constant. Then, there is a constant $\delta > 0$ such that there exists a unique continuous vector solution $\underline{x}$ (t) of the system (5) in the interval $|t - t_0| \leq \delta$. It is important to note that the condition (7) is satisfied by requirement that $\frac{\partial f_i}{\partial x_j}$, $i, j = 1, 2, \dots, n$ be continuous and bounded in D^1.

2.2 Existence and Uniqueness of Solution

We are interested in the region

$$1 \leq \in \leq R \quad (8)$$

We look for a bounded solution of the form

$$0 < R < \infty \quad (9)$$

We shall prove the following existence theorem

Theorem 2.2

Let D denote the region defined in (8) such that (8)-(9) hold. Then there exists a solution of model system (1) (3) which is bounded in the region D

Then there exists a unique solution of the model system (1)-(3) which is bounded in D.

Proof

Let

$$f_1 = \pi - \beta IS - \mu S$$
$$f_2 = (1 - \rho)\beta IS - (v + \mu)E$$
$$f_3 = \rho\beta IS + vE - (\mu + \mu_T)I$$

It suffices to show that $\frac{\partial f_i}{\partial x_j}$,$i, j = 1,2,3$ are continuous

Consider the partial derivatives below

$$\left|\frac{\partial f_1}{\partial S}\right| = |-\mu| < \infty$$
$$\left|\frac{\partial f_1}{\partial E}\right| = 0 < \infty$$
$$\left|\frac{\partial f_1}{\partial I}\right| = 0 < \infty$$
$$\left|\frac{\partial f_2}{\partial S}\right| = 0 < \infty\ .$$

$$\left|\frac{\partial f_2}{\partial E}\right| = |-(v + \mu)| < \infty$$
$$\left|\frac{\partial f_2}{\partial I}\right| = 0 < \infty$$

$$\left|\frac{\partial f_3}{\partial S}\right| = 0 < \infty\ .$$
$$\left|\frac{\partial f_3}{\partial E}\right| = |v| < \infty$$
$$\left|\frac{\partial f_2}{\partial I}\right| = |-(\mu + \mu_{T)})| < \infty$$

Clearly all these partial derivatives are continuous and bounded. Hence by Theorem 2.2, there exists a unique solution of equation (1)-(3) in the region D.

III. Conclusion

This paper provides another method of showing that the first order system of equations for tuberculosis epidemics discussed in [2] has a unique solution.

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