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On R-Closed Maps and R-Homeomorphisms in Topological Spaces

P. Krishna[1], Dr. J. Antony Rex Rodrigo[2]
[1]*Department of Mathematics, Cape Institute of Technology, Levengipuram, Tirunelveli District, Tamil Nadu, India*
[2]*Department of Mathematics, V.O.Chidambaram College, Thoothukudi, Tamil Nadu, India*

***ABSTRACT**: The aim of this paper is to introduce R-closed maps, R-open maps, R-homeomorphisms, R*-homeomorphisms, strongly R-continuous, perfectly R-continuous and study their properties. Using these new types of maps, several characterizations and properties have been obtained.*

***Key words and phrases**- R-closed maps, R-open maps, R-homeomorphism, R*-homeomorphism, strongly R-closed maps, perfectly R-closed maps.*

I. Introduction

In the course of generalizations of the notion of homeomorphism, Sheik John[1] have introduced ω-closed maps and ω-homeomorphism.Devi et al. [2] have studied semi-generalized homeomorphisms and also they have introduced α-homeomorphisms in topological spaces.In this paper, we first introduce R-closed maps in topological spaces and then we introduce R-homeomorphism. We also introduce strongly R-closed maps, perfectly R-closed maps and R*-homeomorphism. We conclude that the set of all R*-homeomorphism forms a group under the operation of composition of maps.

II. Prelimineries

Throughout this paper(X,τ),(Y,σ) and (Z,ζ) will always denote topological spaces on which no separation axioms are assumed, unless otherwise mentioned. When A is a subset of (X,τ),cl(A),Int(A) denote the closure, the interior of A, respectively. We recall the following definitions and some results, which are used in the sequel.

Definition 2.1

Let (X,τ) be a topological space.A subset A of the space X is said to be
(i) Pre open [3] if $A \subseteq \text{Int}(\text{cl}(A))$ and preclosed if $\text{cl}(\text{Int}(A)) \subseteq A$.
(ii) Semi open [4] if $A \subseteq \text{cl}(\text{Int}(A))$ and semiclosed if $\text{Int}(\text{cl}(A)) \subseteq A$.
(iii) α -open [5] if $A \subseteq \text{Int}(\text{cl}(\text{Int}(A)))$ and α-closed if $\text{cl}(\text{Int}(\text{cl}(A)) \subseteq A$.
(iv) Semi preopen [6] if $A \subseteq \text{cl}((\text{Int}(\text{cl}(A)))$ and semi preclosed if $\text{Int}(\text{cl}(\text{Int}(A))) \subseteq A$.

Definition 2.2

Let (X, τ) be a topological space.A subset $A \subseteq X$ is said to be
(i) a generalized closed set [7] (briefly g-closed) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ); the complement of a g- closed set is called a g-open set.
(ii) an α-generalized closed set [8] (briefly αg-closed) if $\alpha\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ); the complement of a αg- closed set is called a αg-open set.
(iii) a generalized semi preclosed set [9] (briefly gsp-closed) if $\text{spcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ); the complement of a gsp- closed set is called a gsp-open set.
(iv) an ω-closed set [1] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in (X,τ); the complement of a ω-closed set is called a ω-open set.
(v) a generalized preclosed set [10] (briefly gp-closed) if $\alpha\text{cl}(A) \subseteq \text{int}U$ whenever $A \subseteq U$ and U is α-open in (X,τ); the complement of a gp- closed set is called a gp-open set.
(vi) a generalized pre regular closed set [11] (briefly gpr-closed) if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is preopen in (X,τ); the complement of a gpr- closed set is called a gpr-open set.
(vii) an R-closed [12] if $\alpha\text{cl}(A) \subseteq \text{int}(U)$ whenever $A \subseteq U$ and U is ω-open in (X,τ); the complement of R-closed set is called an R-open set.

Definition 2.3

A function $f: (X,\tau) \to (Y,\sigma)$ is called

(i) g-continuous [13] if $f^{-1}(V)$ is g-closed in(X,τ) for every closed set V in (Y,σ)
(ii) ω-continuous [1] if $f^{-1}(V)$ is ω-closed in(X,τ) for everyclosed set V in (Y,σ)
(iii) gsp-continuous [9] if $f^{-1}(V)$ is gsp-closed in(X,τ) for every closed set V in (Y,σ)
(iv) gp-continuous [14] if $f^{-1}(V)$ is gp-closed in(X,τ) for every closed set V in (Y,σ)
(v) gpr-continuous [11] if $f^{-1}(V)$ is gpr-closed in(X,τ) for every closed set V in (Y,σ)
(vi) α-continuous [15] if $f^{-1}(V)$ is α-closed in(X,τ) for every closed set V in (Y,σ)
(vii) Contra-continuous [16] if $f^{-1}(V)$ is closed in(X,τ) for every open set V in (Y,σ)
(viii) ω-irresolute [1] if $f^{-1}(V)$ is ω-closed in(X,τ) for everyω- closed set V in (Y,σ)
(ix) closed [17] (resp. g-closed [18], pre-closed [10], gp-closed [10], gpr-closed [19], gsp-closed, α-closed, αg-closed) if f(V) is closed (resp. g-closed, pre-closed, gp-closed, gpr-closed, gsp-closed, α-closed, αg-closed) in (Y,σ) for every closed set V in (X,τ).
(x) contra-open [20]if f(V) is closed in (Y,σ) for every open set V in (X,τ).
(xi) α-irresolute [15] if $f^{-1}(V)$ is an α-open set in (X,τ) for each α-open set V of (Y,σ).
(xii) α-quotient map [15] if f is α-continuous and $f^{-1}(V)$ is open set in (X,τ) implies V is an α-open set in (Y,σ).
(xiii) α*-quotient map[15] if f is α-irresolute and $f^{-1}(V)$ is an α-open set in (X,τ) implies V is an open set in (Y,σ).
(xiv) an R-continuous [12] if $f^{-1}(V)$ is R-closed in (X, τ) for every closed set V of (Y,σ).

Definition 2.4
A space (X,τ) is called
(i) a $T_{1/2}$ space [21] if every g-closed set is closed.
(ii) a T_ω space [22] if every ω-closed set is closed.
(iii) $gsT^{\#}_{1/2}$ space [18] if every #g-semi-closed set is closed.

III. R- Closed Maps And R-Open Maps

Definition 3.1 A map $f:(X,\tau)\rightarrow(Y,\sigma)$ is said to be an R-closed map (R-open map) if the image f (A) is R-closed (R-open) in (Y,σ) for each closed (open) set A in (X,τ).
Example 3.2 Taking X= {a,b,c,d},Y={a,b,c}.Let τ ={X, ϕ ,{a,b}} and σ ={Y, ϕ ,{a}}.Define f(a)=f(b)=a,f(c)=b,f(d)=c.Then f is a R-closed map. Taking X= {a,b,c,d},Y={a,b,c}.Let τ={X,ϕ,{c},{d},{c,d}} and σ={Y,ϕ,{a,b}}.Define f(a)=f(b)=a,f(c)=b,f(d)=c.Then f is not a R-closed map.
Proposition 3.3 For any bijection $f:(X,\tau)\rightarrow(Y,\sigma)$ the following statements are equivalent.
(i) $f^{-1}:(Y,\sigma)\rightarrow(X,\tau)$ is R-continuous.
(ii) f is an R-open map.
(iii) f is an R-closed map.
Proof: (i)⇒(ii) Let U be an open set of (X,τ).By hypothesis,$(f^{-1})^{-1}(U)$ is R-open in (Y,σ) (by theorem 4.10 [12]).Thus f(U) is R-open in (Y,σ).Hence f is R-open.
(ii) ⇒(iii) Let F be a closed set of (X,τ).Then F^c is open in (X,τ).By hypothesis, $f(F^c)$ is R-open. Thus f(F) is R-closed. Thus f is R-closed.
(iii) ⇒ (ii) Let F be a closed set in (X,τ).Then f(F) is R-closed in (Y,σ).That is $(f^{-1})^{-1}(F)$ is R-closed in (Y,σ).Thus f^{-1} is R-continuous.
Proposition 3.4 A mapping f: $(X,\tau)\rightarrow(Y,\sigma)$ is R-closed if and only if R-cl(f(A)) ⊆ f(cl(A)) for every subset A of (X,τ).
Proof: Suppose that f is R-closed and A ⊆ X. Then f(cl(A)) is R-closed in (Y,σ).We have A⊆ cl(A).Thus f(A) ⊆ f(cl(A)).Then R-cl(f(A)) ⊆ R-cl(f(cl(A)))=f(cl(A)). Conversely, let A be any closed set in (X,τ).Then A=cl(A).Thus f(A)=f(cl(A)). But R-cl(f(A)) ⊆ f(cl(A))=f(A).Also f(A) ⊆ R-cl(f(A)).Thus f(A) is R-closed and hence f is R-closed.

Theorem 3.5 A map f: $(X,\tau)\rightarrow(Y,\sigma)$ is R-closed if and only if for each subset S of (Y,σ) and for each open set U containing $f^{-1}(S)$ there is an R-open set V of (Y,σ) such that S⊆ V and $f^{-1}(V)$ ⊆ U.
Proof: Suppose that f is R-closed. Let S⊆ Y and U be an open set of (X,τ) such that $f^{-1}(S)$ ⊆ U.Then $V=(f(U^c))^c$ is an R-open set containing S such that $f^{-1}(V)$ ⊆ U. Conversely, let F be a closed set of (X,τ).Then $f^{-1}((f(F))^c)$ ⊆ F^c and F^c is open. By assumption, there exists an R-open set V of (Y,σ) such that $(f(F))^c$ ⊆ V and $f^{-1}(V)$ ⊆ F^c and so F⊆ $(f^{-1}(V))^c$. Hence V^c⊆ f(F) ⊆ $f((f^{-1}(V))^c)$ ⊆ V^c. Thus f(F)=V^c .Since V^c is R-closed, f(F) is R-closed and therefore f is R-closed.
Proposition 3.6 The composition of two R-closed maps need not be R-closed.

Taking X= {a,b,c,d},Y={a,b,c} and Z={a,b,c}.Let τ ={X, ϕ ,{a},{b},{c},{a,b},{a,c},{b,c},{a,b,c}} and σ ={Y, ϕ ,{b},{a,b}} and ζ ={X, ϕ ,{b},{a,b},{b,c}}.Define f(a)=a,f(b)=f(c)=b,f(d)=c and g(a)=c,g(b)=b,g(c)=a.Then g(f({b,c,d}))=g({b,c})={a,b} is not a R-closed set.Thus gof is not a R-closed map.

Theorem 3.7 Let f: (X,τ)→(Y,σ) , g: (Y,σ)→(Z,ζ) be R-closed maps and (Y,σ) be a T_R space. Then their composition gof: (X,τ) → (Z,ζ) is R-closed.

Proof: Let A be a closed set of (X,τ).By assumption f(A) is R-closed in (Y,σ).Since (Y,σ) is a T_R space, f(A) is closed in (Y,σ) and again by assumption g(f(A)) is R-closed in (Z,ζ).Thus gof(A) is R-closed in (Z,ζ).Hence gof is R-closed.

Proposition 3.8 Let f: (X,τ)→(Y,σ) be a closed map and g:(Y,σ)→ (Z,ζ) be an R-closed map then gof: (X,τ) → (Z,ζ) is R-closed.

Proof: Let U be a closed set of (X,τ).Hence f(U) is closed in (Y,σ).Now (gof)(U)=g(f(U)) which is R-closed in (Z,ζ).

Remark 3.9 If f is R-closed map and g is a closed map then gof need not be a R-closed map.

Taking X= {a,b,c,d},Y={a,b,c} and Z={a,b,c}.Let τ ={X, ϕ ,{a},{b},{c},{a,b},{a,c},{b,c},{a,b,c}}, σ ={Y, ϕ ,{b},{a,b}} and ζ ={Z, ϕ ,{b},{a,b},{b,c}}.Define f(a)=a,f(b)=f(c)=b,f(d)=c and g(a)=c,g(b)=b,g(c)=a.Then f is a R-closed map and g is a closed map.Here g(f({b,c,d}))=g({b,c})={a,b} is not a R-closed set.Thus gof is not a R-closed map.

Definition 3.10 A map f: (X,τ) → (Y,σ)is called strongly R-continuous if the inverse image of every R-open set in (Y,σ) is open in (X,τ).

Example 3.11 Taking X= {a,b,c,d},Y={a,b,c}.Let τ ={X, ϕ ,{a},{b},{c},{a,b},{a,c},{b,c},{a,b,c}} and σ={X,ϕ,{c},{a,c}}.Define f(a)=a,f(b)=f(c)=b,f(d)=c.Then f is strongly R-continuous.

Theorem 3.12 Let f: (X,τ)→(Y,σ) and g:(Y,σ)→ (Z,ζ) be two mappings such that their composition gof: (X,τ) → (Z,ζ) be an R-closed mapping. Then the following statements are true.

(i) If f is continuous and surjective then g is R-closed.

(ii) If g is R-irresolute and injective then f is R-closed.

(iii) If f is ω-continuous,surjective and (X,τ) is a Tω-space then g is R-closed.

(iv) If g is strongly R-continuous and injective then f is closed.

Proof: (i) Let A be a closed set of (Y,σ).Since f is continuous $f^{-1}(A)$ is closed in (X,τ).Thus (gof)($f^{-1}(A)$) is R-closed in (Z,ζ),since gof is R-closed. This gives g(A) is R-closed in (Z,ζ),since f is surjective. Hence g is an R-closed map.

(ii)Let B be a closed set of (X,τ).Since gof is R-closed, (gof)(B) is R-closed in (Z,ζ).Thus g^{-1}((gof)(B)) =f(B) is R-closed in (Y,σ),since g is injective and R-irresolute. Thus f is an R-closed map.

(iii)Let C be a closed set of (Y,σ).Since f is ω–continuous,$f^{-1}(C)$ is ω–closed in (X,τ).Since (X,τ) is a Tω -space,$f^{-1}(C)$ is closed in (X,τ) and so as in (i) g is an R-closed map.

(iv)Let D be a closed set of (X,τ).Since gof is R-closed,(gof)(D) is R-closed in (Z,ζ).Since g is strongly R-continuous,g^{-1}((gof)(D)) is closed in (Y,σ).Thus f is a closed map.

Regarding the restriction f_A of a map f: (X,τ)→(Y,σ) to a subset A of (X,τ),we have the following theorem.

Theorem 3.13 If f: (X,τ)→(Y,σ) is R-closed and A is a closed subset of (X,τ) ,then f_A: (A,τ_A) →(Y,σ) is R-closed.

Proof: Let B be a closed set of A. Then B=A⋂F for some closed set F of (X,τ) and so B is closed in (X,τ).By hypothesis, f(B) is R-closed in (Y,σ).But f(B)=f_A(B) and hence f_A is an R-closed map.

Theorem 3.14 If f: (X,τ)→(Y,σ) is a continuous R-closed map from a normal space (X,τ) onto a space (Y,σ) then (Y,σ) is normal.

Proof: Let A and B be two disjoint closed sets of (Y,σ).Let A and B are disjoint closed sets of (Y,σ).Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets of (X,τ),since f is continuous. Therefore there exists open sets U and V such that $f^{-1}(A)$⊆U and $f^{-1}(B)$⊆V,since X is normal.Using theorem 3.5, there exists R-open sets C,D in (Y,σ) such that A⊆C,B⊆D,$f^{-1}(C)$⊆U and $f^{-1}(D)$⊆V.Since A and B are closed, A and B are α-closed and ω–closed.By the result, C is R-open if and only if cl(A)⊆ α-int(C) whenever A⊆C and A is ω-closed, we get cl(A)⊆ α-int(C)⊆Int(C).Thus A⊆int(C) and B⊆int(D). Hence Y is normal.

Proposition 3.15 If f: (X,τ)→(Y,σ) is an α-irresolute where (X,τ) is a discrete space and (Y,σ) is a $T_{1/2}$space then f is an R-irresolute.

Proof: Let U be R-closed in (Y,σ).Then U is αg-closed.Since (Y,σ) is a $T_{1/2}$space, U is α-closed.Since f is α-irresolute and (X,τ) is discrete,$f^{-1}(U)$ is α-closed and open. Hence $f^{-1}(U)$ is R-closed. Thus f is R-irresolute.

Proposition 3.16 If f: (X,τ)→(Y,σ) is a α*-quotient map and (Y,σ) is a $T_{1/2}$space then (Y,σ) is a T_R-space.

Proof: Let U be R-closed in (Y,σ).Then U is αg-closed in (Y,σ).Since (Y,σ) is a $T_{1/2}$ space,$f^{-1}(U)$ is α-closed in (X,τ).Since f is α-irresolute, f($f^{-1}(U)$) is closed. Thus (Y,σ) is a T_R-space.

Theorem 3.17 If f: $(X,\tau)\rightarrow(Y,\sigma)$ is R-closed and g: $(X,\tau) \rightarrow (Z,\zeta)$ is a continuous map that is constant on each set $f^{-1}(y)$ for $y\in Y$,then g induces a R-continuous map h: $(Y,\sigma)\rightarrow (Z,\zeta)$ such that hof=g.
Proof: Since 'g' is constant on $f^{-1}(y)$ for each $y\in Y$,the set $g(f^{-1}(y))$ is a one point set in (Z,ζ).If h(y) denote this point ,it is clear that 'h' is well defined and for each $x\in X$, h(f(x))=g(x).
We claim that 'h' is R-continuous. Let U be closed in (Z,ζ) ,then $g^{-1}(U)$ is closed in (X,τ).
But $g^{-1}(U)=f^{-1}(h^{-1}(U))$ is closed in (X,τ).Since 'f' is R-closed,$h^{-1}(V)$ is R-closed. Hence h is R-continuous.
Theorem 3.18 If f: $(X,\tau)\rightarrow(Y,\sigma)$ is a contra closed and α-closed map then 'f' is a R-closed map.
Proof: Let V be a closed set in (X,τ).Then f(V) is α-closed and open. Hence V is R-closed. Thus V is a R-closed map.
The converse need not be true as seen from the following example.
Consider X=Y={a,b,c}. Let τ={X,ϕ,{b},{a,b}} and σ={Y,ϕ,{a},{a,b}}.Define f(a)=c,f(b)=a,f(c)=b.Then f is R-closed map but not a contra closed map.
Proposition 3.19 Every R-closed map is αg-closed,gsp-closed,gp-closed and gpr-closed.
Proof: Since every R-closed set is αg-closed,gsp-closed,gp-closed and gpr-closed,we get the proof.
Remark 3.20 The converse need not be true as seen from the following examples.
Consider X=Y={a,b,c}. Let τ={X,ϕ,{a,b}} and σ={Y,ϕ,{a,c}}.Define f(a)=b,f(b)=c,f(c)=a.Then f is gsp-closed but not R-closed.
Consider X=Y={a,b,c}. Let τ={X,ϕ,{a,c}} and σ={Y,ϕ,{a,b}}.Define f(a)=b,f(b)=a,f(c)=c.Then f is gp-closed but not R-closed.
Consider X=Y={a,b,c}. Let τ={X,ϕ,{a}} and σ={Y,ϕ,{a},{b,c}}.Define f(a)=a,f(b)=b,f(c)=b.Then f is αg-closed but not R-closed.
Consider X={x,y,z} and Y={a,b,c}. Let τ={X,ϕ,{y,z}} and σ={Y,ϕ,{a,b}}.Define f(x)=a,f(y)=b,f(z)=c.Then f is gpr-closed but not R-closed.
Remark 3.21 The concept of R-closed map and g-closed maps are independent.
Consider X={x,y,z} and Y={a,b,c}. Let τ ={X, ϕ ,{x,z}} and σ ={Y, ϕ ,{a},{a,b}}.Define f(x)=c,f(y)=b,f(z)=a.Then f is R-closed but not g-closed. Consider X={x,y,z} and Y={a,b,c}. Let τ={X,ϕ,{y,z}} and σ={Y,ϕ,{a},{a,c }}.Define f(x)=b,f(y)=a,f(z)=c.Then f is g-closed but not R-closed.
Remark 3.22 The concept of R-closed map and pre-closed maps are independent.
Consider X={x,y,z} and Y={a,b,c}. Let τ={X,ϕ,{z}} and σ={Y,ϕ,{a}}.Define f(x)=a,f(y)=b,f(z)=c.Then f is R-closed but not pre-closed. Consider X={x,y,z} and Y={a,b,c}. Let τ={X,ϕ,{x,z}} and σ={Y,ϕ,{a,b}}.Define f(x)=b,f(y)=a,f(z)=c.Then f is pre-closed but not R-closed.
Theorem 3.23 If f: $(X,\tau)\rightarrow(Y,\sigma)$ is R-closed , g:$(Y,\sigma)\rightarrow (Z,\zeta)$ is R-closed and (Y,σ) is a T_R-space then their composition gof: $(X,\tau) \rightarrow (Z,\zeta)$ is R-closed.
Proof: Let V be a closed set in (X,τ).Then f(V) is R-closed in (Y,σ).Since (Y,σ) is a T_R-space, f(V) is closed in (Y,σ).Hence g(f(V))=gof(V) is R-closed in (Z,ζ).Thus gof is a R-closed map.
Theorem 3.24 If f: $(X,\tau)\rightarrow(Y,\sigma)$ is a ω-closed map (resp. g-closed, #gs-closed map) ,g:$(Y,\sigma)\rightarrow (Z,\zeta)$ is a R-closed map and Y is a Tω-space ($T_{1/2}$ space, $gsT^{\#}_{1/2}$ space),then their composition gof: $(X,\tau) \rightarrow (Z,\zeta)$ is a R-closed map.
Proof: Let V be a closed set in (X,τ).Then f(V) is ω-closed (g-closed, #gs-closed) set in (Y,σ).Since (Y,σ) is a Tω-space ($T_{1/2}$ space,$gsT^{\#}_{1/2}$ space), f(V) is a closed set in (Y,σ).Since g is R-closed, g(f(V))=(gof)(V) is R-closed in (Z,ζ).Thus gof is a R-closed map.
Theorem 3.25 If f: $(X,\tau)\rightarrow(Y,\sigma)$ is a closed map and g:$(Y,\sigma)\rightarrow (Z,\zeta)$ be a R-closed map then their composition gof: $(X,\tau) \rightarrow (Z,\zeta)$ is R-closed.
Proof: Let V be a closed set in (X,τ).Then f(V) is a closed set in (Y,σ).Hence g(f(V))= (gof)(V) is R-closed in (Z,ζ). Thus gof is a R-closed map.
Remark 3.26 If f: $(X,\tau)\rightarrow(Y,\sigma)$ is R-closed and g:$(Y,\sigma)\rightarrow (Z,\zeta)$ is closed, then their composition gof: $(X,\tau) \rightarrow (Z,\zeta)$ need not be a R-closed map.
For example, consider X= {a,b,c},Y={a,b,c} and Z={a,b,c}.Let τ={X, ϕ ,{b},{a,b}}, σ={Y, ϕ ,{a},{a,b}} and ζ={Z, ϕ,{c},{b,c}}.Define f(a)=c, f(b)=a, f(c)=b and g(a)=c,g(b)=b,g(c)=a.Then f is R-closed and g is closed.Here $(gof)^{-1}\{a\}=\{a\}$, which is not R-closed.Thus gof is not a R-closed map.
Theorem 3.27 Let f: $(X,\tau)\rightarrow(Y,\sigma)$ and g:$(Y,\sigma)\rightarrow (Z,\zeta)$ be two mappings such that their composition gof: $(X,\tau) \rightarrow (Z,\zeta)$ be a R-closed mapping. Then the following statements are true if:
(i) f is continuous and surjective, then g is R-closed.
(ii) g is R-irresolute and injective, then f is R-closed.
(iii) f is ω-continuous and (X,τ) is a Tω space, then g is R-closed.
(iv) f is g-continuous, surjective and (X,τ) is a $T_{1/2}$ space, then g is R-closed.
(v) f is R-continuous, surjective and (X,τ) is a T_R space, then g is R-closed.

Proof:
(i) Let f be continuous and surjective. Let A be a closed set in (Y, σ). Since f is continuous, $f^{-1}(A)$ is closed in (X,τ).Since gof is R-closed,(gof)(f^{-1}(A))=g(A) (since f is a surjective) is R-closed in (Z,ζ).Thus g is a R-closed map.
(ii) Let A be closed in (X,τ). Since gof is R-closed, (gof)(A) is R-closed in (Z,ζ).Since g is R-irresolute, (g^{-1}(gof)(A)) is R-closed in (Y, σ).since g is injective, f is a R-closed map.
(iii) Let A be closed in (Y, σ). Since f is ω-continuous, $f^{-1}(A)$ is ω-closed in (X,τ) and (X,τ) is a Tω space. Thus $f^{-1}(A)$ is closed in (X,τ). Since gof is R-closed and f is a surjective, (gof)(f^{-1}(A))=g(A) is R-closed in (Z,ζ), Thus g is a R-closed map.
(iv) Let A be closed in (Y, σ). Since f is g-continuous, $f^{-1}(A)$ is g-closed in (X,τ).Since (X,τ) is a $T_{1/2}$ space,$f^{-1}(A)$ is closed in (X,τ).Since gof is R-closed and f is a surjective, (gof)(f^{-1}(A))=g(A) is R-closed in (Z,ζ). Thus g is a R-closed map.
(v) Let A be a closed set in (Y, σ). Since f is R-continuous, $f^{-1}(A)$ is R-closed in (X,τ). Since (X,τ) is a T_R space, $f^{-1}(A)$ is closed in (X,τ). Since gof is R-closed and f is a surjective, (gof)(f^{-1}(A))=g(A) is R-closed in (Z,ζ). Thus g is a R-closed map.
Theorem 3.28 Let f: (X,τ)→(Y,σ) be a R-open map, then for each x∈X and for each neighbourhood U of x in (X,τ), there exist a R-neighbourhood W of f(x) in (Y, σ) such that W⊆ f(U).
Proof: Let x∈X and U be an arbitrary neighbourhood of x.Then there exist an open set V in (X,τ) such that x∈ V ⊆ U. By assumption f(V) is a R-open set in (Y, σ).Further f(x)∈ f(V) ⊆ f(U).Clearly f(U) is a R-neighbourhood of f(x) in (Y, σ) and so the theorem holds by taking W=f(V).

IV. R-Homeomorphisms

Definition 4.1 A bijection f: (X,τ)→(Y,σ) is called R-homeomorphism if f is both R-continuous and R-open.
Example 4.2 Taking X= {a,b,c},Y={a,b,c}.Let τ ={X, ϕ ,{a},{a,b}} and σ={Y, ϕ ,{b},{a,b},{b,c}}.Define f(a)=b,f(b)=a,f(c)=c.Then f is R-homeomorphism.
Proposition 4.3 Let f: (X,τ)→(Y,σ) be a bijective R-continuous map.Then the following are equivalent.
(i) f is an R-open map.
(ii) f is an R-homeomorphism.
(iii) f is an R-closed map.
Proof: The proof follows from proposition 3.3
Remark 4.4 The composition of two R-homeomorphisms need not be an R-homeomorphism.
Since the composition of two R-continuous functions need not be a R-continuous function (Remark 6.3 [12]) we get the conclusion.
Definition 4.5 A bijection f: (X,τ)→(Y,σ) is said to be R*-homeomorphism if both f and f^{-1} are R-irresolute.
Example 4.6 Taking X= {a,b,c}, Y={a,b,c}. Let τ={X,ϕ,{c},{a,c},{b,c}} and σ={Y,ϕ,{a},{a,b},{a,c}}.Define f(a)=b,f(b)=c,f(c)=a.Then f is R*-homeomorphism.
Proposition 4.7 Every R*-homeomorphism is R-irresolute.
Proof: It is the consequence of the definition.
Remark 4.8 Every R-irresolute map need not be a R*-homeomorphism.
For example, consider X=Y={a,b,c}. Let τ ={X, ϕ ,{a},{a,b}} and σ ={Y, ϕ ,{b},{a,b},{b,c}}.Define f(a)=b,f(b)=a,f(c)=c.Then f is R-irresolute but not a R*-homeomorphism.
We denote the family of all R-homeomorphisms (resp. R*-homeomorphisms and homeomorphism) of a topological space (X,τ) onto itself by R-h(X,τ) (resp. R*-h(X,τ) and h(X,τ)).
Theorem 4.9 If f: (X,τ)→(Y,σ) is a R*-homeomorphism then R-cl(f^{-1}(B))=f^{-1}(R-cl(B)) for all B⊆Y.is R-closed
Proof: Since f is R*-homeomorphism, f is R-irresolute. Since R-cl(f(B)) in (Y, σ) ,f^{-1} (R-cl(f(B))) is R-closed in (X,τ).Thus R-cl(f^{-1}(B))⊆f^{-1}(R-cl(B)).Again f^{-1} is R-irresolute and R-cl(f^{-1}(B)) is R-closed in (X,τ),$(f^{-1})^{-1}$(R-cl(f^{-1}(B)))=f(R-cl(f^{-1}(B))) is R-closed in (X,τ). Thus B⊆$(f^{-1})^{-1}$(f^{-1}(B))⊆$(f^{-1})^{-1}$(R-cl(f^{-1}(B)))=f(R-cl(f^{-1}(B))).Hence f^{-1}(R-cl(B))⊆R-cl(f^{-1}(B)).
Proposition 4.10 If f: (X,τ)→(Y,σ) and g:(Y, σ)→ (Z,ζ) are R*-homeomorphisms then their composition gof: (X,τ) → (Z,ζ) is also R*-homeomorphism.
Proof: Let U be an R-open set in (Z,ζ).Then g^{-1}(U) is R-open in (Y, σ).Now $(gof)^{-1}$(U)=f^{-1}(g^{-1}(U)) is R-open in (X,τ).Thus gof is R-irresolute.Also for an R-open set G in (X,τ), (gof)(G)=g(f(G))=g(W) where W=f(G).By hypothesis, f(G) is R-open in (Y, σ).Thus g(f(G)) is R-open in (Z,ζ).Hence $(gof)^{-1}$ is R-irresolute and hence gof is R*-homeomorphism.
Theorem 4.11 The set R*-h(X,τ) is a group under the composition of maps.
Proof: Define a binary operation ∗ as follows.∗:R*-h(X,τ) X R*-h(X,τ)→R*-h(X,τ) by f∗g=gof for all f,g∈R*-h(X,τ) and ‘o’ is the usual operation of composition of maps.

By the above result gof∈R*-h(X,τ).We know that the composition of maps is associative and the identity map I: (X,τ)→ (X,τ)∈R*-h(X,τ) serves as the identity element. If f∈R*-h(X,τ) then f^{-1}∈R*-h(X,τ) such that fof^{-1}=$f^{-1}$of=I and so inverse exists for each element of R*-h(X,τ).Thus R*-h(X,τ) is a group under composition of maps.

Theorem 4.12 Let f: (X,τ)→(Y,σ) be an R*-homeomorphism. Then f induces an isomorphism from the group R*-h(X,τ) onto the group R*-h(X,τ).

Proof: Using the map f, we define I_f : R*-h(X,τ)→ R*-h(Y,σ) by I_f(h)=fohof^{-1} for every h∈ R*-h(X,τ).Then I_f is a bijection.

Further for every h_1,h_2∈ R*-h(X,τ), I_f(h_1oh_2)=fo(h_1oh_2)of^{-1}= (foh_1of^{-1})o (foh_2of^{-1})=I_f(h_1)∗I_f(h_2).Thus I_f is a homeomorphism and so it is an isomorphism induced by 'f'.

Theorem 4.13 ∗ is an equivalence relation in R*-h(X,τ).

Proof: By proposition7 transitivity follows. Reflexive and symmetric are immediate.

Corollary 4.14 If f: (X,τ)→(Y,σ) is an R*-homeomorphism then R-cl(f(B))=f(R-cl(B)) for all B⊆ X.

Proof: Since f: (X,τ)→(Y,σ) is an R*-homeomorphism, f^{-1}: (Y,σ) → (X, τ) is also an R*-homeomorphism.

Thus R-cl(((f^{-1})$^{-1}$)(B))=(f^{-1})$^{-1}$(R-cl(B)).Hence R-cl(f(B))=f(R-cl(B)).

Corollary 4.15 If f: (X,τ)→(Y,σ) is an R*-homeomorphism, then f(R-int(B))=R-int(f(B)) for every B⊆X.

Proof: We have(R-int(A))^c =R-cl(A^c).Thus R-int(B) =(R-cl(B^c))^c .Then f(R-int(B)) = f((R-cl(B^c))^c) = (f(R-cl(B^c)))^c = (R-cl(f(B^c)))^c =R-int(f(B)).

Corollary 4.16 If f: (X,τ)→(Y,σ) is an R*-homeomorphism, then f^{-1}(R-int(B)) =R-int(f^{-1}(B)) for every subset B of Y.

Proof: If f: (X,τ)→(Y,σ) is an R*-homeomorphism then f^{-1}: (Y,σ) → (X,τ) is also an R*-homeomorphism,the proof follows from the above corollary.

Definition 4.17 A map f: (X,τ)→(Y,σ) is called perfectly R-continuous if the inverse image of every R-open set in (Y,σ) is both open and closed in (X,τ).

Example 4.18 Taking X= {a,b,c,d},Y={a,b,c}.Let τ={X,ϕ,{a},{b,c,d}} and σ={X,ϕ,{a,c}}.

Define f(a)=b,f(b)=a,f(c)=a,f(d)=c.Then f is perfectly R-continuous.

Proposition 4.19 If f: (X,τ)→(Y,σ) is perfectly R-continuous then it is strongly R-continuous but not conversely.

Proof: Let U be an R-open set in (Y,σ).Since f: (X,τ)→(Y,σ) is perfectly R-continuous,f^{-1}(U) is both open and closed in (X,τ).Therefore f is strongly R-continuous.

Remark 4.20 Every strongly R-continuous mappings need not be perfectly R-continuous.

Taking X= {a,b,c,d},Y={a,b,c}.Let τ ={X, ϕ ,{a},{b},{a,b}} and σ ={X, ϕ ,{a,c}}.Define f(a)=b,f(b)=a,f(c)=c,f(d)=c.Then f^{-1}({b})={a} is open but not closed.Hence f is strongly R-continuous but not perfectly R-continuous.

Remark 4.21 Strongly R-continuous is independent of R-continuous.

Every strongly R-continuous need not be R-continuous.

For example, consider X= {a,b,c},Y={a,b,c}.Let τ ={X, ϕ ,{a},{a,b}} and σ ={X, ϕ ,{a,c}}.Define f(a)=b,f(b)=a,f(c)=c.Then f is strongly R-continuous but not R-continuous.

Conversely, consider X= {a,b,c},Y={a,b,c}.Let τ ={X, ϕ ,{a},{a,b}} and σ ={X, ϕ ,{a,c}}.Define f(a)=c,f(b)=b,f(c)=a.Then f is R-continuous but not strongly R-continuous.

Definition 4.22 A map f: (X,τ)→(Y,σ) is called strongly R-open if f(U) is R-open in (Y,σ) for each R-open set U in (X,τ).

Example 4.23 Consider X=Y={a,b,c}. Let τ ={X, ϕ ,{b},{a,b},{b,c}} and σ ={Y, ϕ ,{a}}.Define f(a)=b,f(b)=a,f(c)=c.Then f is strongly R-open.

Proposition 4.24 Every R*-homeomorphism is strongly R-open.

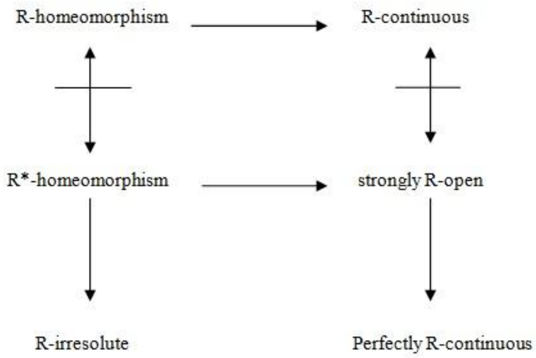
Proof: It is the consequence of the definition.

Remark 4.25 Every strongly R-open map need not be R*-homeomorphism.

Consider X=Y= {a, b, c }. Let τ={X,ϕ,{b},{a,b},{b,c}} and σ={Y,ϕ,{a}}.Define f(a)=b,f(b)=a,f(c)=c.Then f is strongly R-open but not a R*-homeomorphism.

V. Conclusion

We conclude that the set of all R*-homeomorphisms form a group under the composition of mappings. Also from the above discussions we have the following implications. A→ B (A↮B) represents A implies B but not conversly (A and B are independent of each other).



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