Neutrosophic Set and Neutrosophic Topological Spaces

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Abstract: *Neutrosophy has been introduced by Smarandache [7, 8] as a new branch of philosophy. The purpose of this paper is to construct a new set theory called the neutrosophic set. After given the fundamental definitions of neutrosophic set operations, we obtain several properties, and discussed the relationship between neutrosophic sets and others. Finally, we extend the concepts of fuzzy topological space [4], and intuitionistic fuzzy topological space [5, 6] to the case of neutrosophic sets. Possible application to superstrings and ζ^{∞} space–time are touched upon.*

Keywords: *Fuzzy topology; fuzzy set; neutrosophic set; neutrosophic topology*

I.   Introduction

The fuzzy set was introduced by Zadeh [9] in 1965, where each element had a degree of membership. The intuitionstic fuzzy set (Ifs for short) on a universe X was introduced by K. Atanassov [1, 2, 3] in 1983 as a generalization of fuzzy set, where besides the degree of membership and the degree of non- membership of each element. After the introduction of the neutrosophic set concept [7, 8]. In recent years neutrosophic algebraic structures have been investigated. Neutrosophy has laid the foundation for a whole family of new mathematical theories generalizing both their classical and fuzzy counterparts, such as a neutrosophic set theory.

II.   Terminologies

We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [7, 8], and Atanassov in [1, 2, 3]. Smarandache introduced the neutrosophic components T, I, F which represent the membership, indeterminacy, and non-membership values respectively, where $\left]0^-,1^+\right[$ is nonstandard unit interval.

2.1   Definition. [3,4]

Let T, I,F be real standard or nonstandard subsets of $\left]0^-,1^+\right[$, with

Sup_T=t_sup, inf_T=t_inf
Sup_I=i_sup, inf_I=i_inf
Sup_F=f_sup, inf_F=f_inf
n-sup=t_sup+i_sup+f_sup
n-inf=t_inf+i_inf+f_inf,
T, I, F are called neutrosophic components

III.   Neutrosophic Sets and Its Operations

We shall now consider some possible definitions for basic concepts of the neutrosophic set and its operations.

3.1 Definition

Let X be a non-empty fixed set. A neutrosophic set (NS for short) A is an object having the form $A = \left\{\left\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\right\rangle : x \in X\right\}$ Where $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x)$ which represent the degree of member ship function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$), and the degree of non-member ship (namely $\gamma_A(x)$) respectively of each element $x \in X$ to the set A.

3.1   Remark

A neutrosophic $A = \left\{< x, \mu_A(x), \sigma_A(x), \gamma_A(x) > : x \in X\right\}$ can be identified to an ordered triple $< \mu_A, \sigma_A, \gamma_A >$ in $\left]^-0,1^+\right[$ on. X.

3.2 Remark

For the sake of simplicity, we shall use the symbol $A = <x, \mu_A, \sigma_A, \gamma_A>$ for the

NS $A = \{<x, \mu_A(x), \sigma_A(x), \gamma_A(x)>: x \in X\}$

3.1 Example

Every IFS A a non-empty set X is obviously on NS having the form

$A = \{<x, \mu_A(x), 1 - (\mu_A(x) + \gamma_A(x)), \gamma_A(x)>: x \in X\}$

Since our main purpose is to construct the tools for developing neutrosophic set and neutrosophic topology, we must introduce the NSS 0_N and 1_N in X as follows:

0_N may be defined as:

(0_1) $0_N = \{\langle x, 0, 0, 1 \rangle : x \in X\}$

(0_2) $0_N = \{\langle x, 0, 1, 1 \rangle : x \in X\}$

(0_3) $0_N = \{\langle x, 0, 1, 0 \rangle : x \in X\}$

(0_4) $0_N = \{\langle x, 0, 0, 0 \rangle : x \in X\}$

1_N may be defined as:

(1_1) $1_N = \{\langle x, 1, 0, 0 \rangle : x \in X\}$

(1_2) $1_N = \{\langle x, 1, 0, 1 \rangle : x \in X\}$

(1_3) $1_N = \{\langle x, 1, 1, 0 \rangle : x \in X\}$

(1_4) $1_N = \{\langle x, 1, 1, 1 \rangle : x \in X\}$

3.2 Definition

Let $A = \langle \mu_A, \sigma_A, \gamma_A \rangle$ a NS on X, then the complement of the set A $(C(A)$, for short $)$ maybe defined as three kinds of complements

(C_1) $C(A) = \{\langle x, 1 - \mu_A(x), 1 - \gamma_A(x) \rangle : x \in X\}$,

(C_2) $C(A) = \{\langle x, \gamma_A, \sigma_A(x), \mu_A(x) \rangle : x \in X\}$

(C_3) $C(A) = \{\langle x, \gamma_A, 1 - \sigma_A(x), \mu_A(x) \rangle : x \in X\}$

One can define several relations and operations between NSS follows:

3.3 Definition

Let x be a non-empty set, and NSS A and B in the form $A = \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle$, $B = \langle x, \mu_B(x), \sigma_B(x), \gamma_B(x) \rangle$, then we may consider two possible definitions for subsets $(A \subseteq B)$

$(A \subseteq B)$ may be defined as

(1) $A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x), \gamma_A(x) \ge \gamma$ and $\sigma_A(x) \le \sigma_B(x)$ $\forall x \in X$

(2) $A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x), \gamma_A(x) \ge \gamma_B(x)$ and $\sigma_A(x) \ge \sigma_B(x)$

3.1 Proposition

For any neutrosophic set A the following are holds

(1) $0_N \subseteq A$, $0_N \subseteq 0_N$

(2) $A \subseteq 1_N$, $1_N \subseteq 1_N$

3.4. Definition

Let X be a non-empty set, and $A = <x, \mu_A(x), \gamma_A(x), \sigma_A(x)>$, $B = <x, \mu_B(x), \sigma_B(x), \gamma_B(x)>$ are NSS. Then

(1) $A \bigcap B$ maybe defined as:

(I_1) $A \bigcap B = <x, \mu_A(x).\mu_B(x), \sigma_A(x).\sigma_B(x),$

$\gamma_A(x).\gamma_B(x)>$

(I_2) $A \bigcap B = <x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x),$

$\gamma_A(x) \vee \gamma_B(x)>$

(I_3) $A \bigcap B = <x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \gamma_A(x) \vee \gamma_B(x)>$

(2) $A \cup B$ may be defined as:

(U_1) $A \cup B = <x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x),$

$\gamma_A(x) \wedge \gamma_B(x)>$

(U_2) $A \cup B = <x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \wedge \sigma_B(x),$

$\gamma_A(x) \wedge \gamma_B(x)>$

(3) $[\ ]A = <x, \mu_A(x), \sigma_A(x), 1 - \mu_A(x)>$

(4) $<>A = <x, 1 - \gamma_A(x), \sigma_A(x), \gamma_A(x)>$

We can easily generalize the operations of intersection and union in definition 3.4 to arbitrary family of *NSS* as follow:

3.5 Definition

Let $\{Aj : j \in J\}$ be a arbitrary family of *NSS* in X, then

(1) $\bigcap Aj$ maybe defined as:

(i) $\bigcap Aj = \left\langle x, \underset{j \in J}{\wedge} \mu_{Aj}(x), \underset{j \in J}{\wedge} \sigma_{Aj}(x), \vee \gamma_{Aj}(x) \right\rangle$

(ii) $\bigcap Aj = \left\langle x, \wedge \mu_{Aj}(x), \vee \sigma_{Aj}(x), \vee \gamma_{Aj}(x) \right\rangle$

(2) $\bigcup Aj$ maybe defined as:

(i) $\bigcup Aj = \left\langle x, \vee, \wedge, \wedge \right\rangle$

(ii) $\bigcup Aj = \left\langle x, \vee, \wedge, \wedge \right\rangle$

3.6. Definition

Let A and B are neutrosophic sets then

$A | B$ may be defined as

$A | B = \left\langle x, \mu_A \wedge \gamma_B, \sigma_A(x) \sigma_B(x), \gamma_A \vee \mu_B(x) \right\rangle$

3.2. Proposition

For all A, B two neutrosophic sets then the following are true

(1) $C(A \cap B) = C(A) \cup C(B)$

(2) $C(A \cup B) = C(A) \cap C(B)$

IV.   Neutrosophic Topological Spaces

Here we extend the concepts of fuzzy topological space [4], and intuitionistic fuzzy topological space [5, 7] to the case of neutrosophic sets.

4.1 Definition

A neutrosophic topology (*NT* for short) an a non empty set X is a family τ of neutrosophic subsets in X satisfying the following axioms

(NT_1) $O_N, 1_N \in \tau$,

(NT_2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,

(NT_3) $\bigcup G_i \in \tau$ $\forall \{G_i : i \in J\} \subseteq \tau$

In this case the pair (X, τ) is called a neutrosophic topological space (*NTS* for short) and any neutrosophic set in τ is known as neuterosophic open set (*NOS* for short) in X. The elements of τ are called open neutrosophic sets, A neutrosophic set F is closed if and only if it C (F) is neutrosophic open.

4.1 Example

Any fuzzy topological space (X, τ_0) in the sense of Chang is obviously a *NTS* in the form $\tau = \{A : \mu_A \in \tau_0\}$ wherever we identify a fuzzy set in X whose members ship function is μ_A with its counterpart.

4.1. Remark Neutrosophic topological spaces are very natural generalizations of fuzzy topological spaces allow more general functions to be members of fuzzy topology.

4.3 Example

Let $X = \{x\}$ and

$A = \{\langle x, 0.5, 0.5, 0.4 \rangle : x \in X\}$

$B = \{\langle x, 0.4, 0.6, 0.8 \rangle : x \in X\}$

$D = \{\langle x, 0.5, 0.6, 0.4 \rangle : x \in X\}$

$C = \left\{ \langle x, 0.4, 0.5, 0.8 \rangle : x \in X \right\}$

Then the family $\tau = \left\{ O_n, 1_n, A, B, C, D \right\}$ of NSs in X is neutrosophic topology on X

4.4 Example

Let (X, τ_0) be a fuzzy topological space in changes sense such that τ_0 is not indiscrete suppose now that $\tau_0 = \{0_N, 1_N\} \cup \{V_j : j \in J\}$ then we can construct two *NTSS* on X as follows

a) $\tau_0 = \{0_N, 1_N\} \cup \{< x, V_j, \sigma(x), 0 >: j \in J\}.$

b) $\tau_0 = \{0_N, 1_N\} \cup \{< x, V_j, 0, \sigma(x), 1 - V_j >: j \in J\}.$

4.1 Proposition

Let (X, τ) be a *NTS* on X, then we can also construct several *NTSS* on X in the following way:

a) $\tau_{o,1} = \{[\ ]G : G \in \tau\},$

b) $\tau_{o,2} = \{<>G : G \in \tau\},$

Proof

a) (NT_1) and (NT_2) are easy.

(NT_3) Let $\left\{ [\ ]G_j : j \in J, G_j \in \tau \right\} \subseteq \tau_{0,1}$. Since

$\cup G_j = \left\langle\!\!\left\langle x, \vee \mu_{G_j}, \vee \sigma_{G_j}, \wedge \gamma_{G_j} \right\rangle\!\!\right\rangle or \left\langle\!\!\left\langle x, \vee \mu_{G_j}, \wedge \sigma_{G_j}, \wedge \gamma_{G_j} \right\rangle\!\!\right\rangle or \left\langle\!\!\left\langle x, \vee \mu_{G_j}, \wedge \sigma_{G_j}, \vee \gamma_{G_j} \right\rangle\!\!\right\rangle \in \tau$, we have

$\cup ([\ ]G_j) = \left\{ x, \vee \mu_{G_j}, \vee \sigma_{G_j}, \wedge (1 - \mu_{G_j}) \right\} or \left\{ x, \vee \mu_{G_j}, \vee \sigma_{G_j}, (1 - \vee \mu_{G_j}) \right\} \in \tau_{0,1}$

b) This similar to (a)

4.2 Definition

Let $(X, \tau_1), (X, \tau_2)$ be two neutrosophic topological spaces on X. Then τ_1 is said be contained in τ_2 (in symbols $\tau_1 \subseteq \tau_2$) if $G \in \tau_2$ for each $G \in \tau_1$. In this case, we also say that τ_1 is coarser than τ_2.

4.2 Proposition

Let $\{\tau_j : j \in J\}$ be a family of *NTSS* on X. Then $\cap \tau_j$ is A neutrosophic topology on X. Furthermore, $\cap \tau_j$ is the coarsest *NT* on X containing all. τ_j, s

Proof. Obvious

4.3 Definition

The complement of A (C (A) for short) of $NOS.$ A is called a neutrosophic closed set (*NCS* for short) in X.
 Now, we define neutrosophic closure and interior operations in neutrosophic topological spaces:

4.4 Definition

Let (X, τ) be *NTS* and $A = < x, \mu_A(x), \gamma_A(x), \sigma_A(x) >$ be a *NS* in X.

Then the neutrosophic closer and neutrosophic interior of Aare defined by

$NCl(A) = \cap \{K : K \text{ is an NCS in X and } A \subseteq K\}$

$NInt(A) = \cup \{G : G \text{ is an NOS in X and } G \subseteq A\}$. It can be also shown that

It can be also shown that $NCl(A)$ is *NCS* and $NInt(A)$ is a *NOS* in X

a) A is in X if and only if $NCl(A)$.

b) A is NCS in X if and only if $NInt(A) = A.$

4.2 Proposition

For any neutrosophic set A in (x, τ) we have

(a) $NCl(C(A) = C(NInt(A)),$

(b) $NInt(C(A)) = C(NCl(A)).$

Proof.

a) Let $A = \{< x, \mu_A, \sigma_A, \upsilon_A >: x \in X\}$ and suppose that the family of neutrosophic subsets contained in

A are indexed by the family if *NSS* contained in A are indexed by the

family $A = \{< x, \mu_{G_i}, \sigma_{G_i}, \upsilon_{G_i} >: i \in J\}$. Then we see that $NInt(A) = \{< x, \vee \mu_{G_i}, \vee \sigma_{G_i}, \wedge \upsilon_{G_i} >\}$

and hence $C(NInt(A)) = \{< x, \wedge \mu_{G_i}, \vee \sigma_{G_i}, \vee \upsilon_{G_i} >\}$. Since $C(A)$ and $\mu_{G_i} \le \mu_A$ and $\nu_{G_i} \ge \nu_A$ for

each $i \in J$, we obtaining $C(A)$. i.e $NCl(C(A)) = \{< x, \wedge \upsilon_{G_i}, \vee \sigma_{G_i}, \vee \mu_{G_i} >\}$. Hence

$NCl(C(A) = C(NInt(A)$, follows immediately

b) This is analogous to (a).

4.3 Proposition

Let (x, τ) be a *NTS* and A, B be two neutrosophic sets in X. Then the following properties hold:

(a) $\quad NInt(A) \subseteq A,$

(b) $\quad A \subseteq NCl(A),$

(c) $\quad A \subseteq B \Rightarrow NInt(A) \subseteq NInt(B),$

(d) $\quad A \subseteq B \Rightarrow NCl(A) \subseteq NCl(B),$

(e) $\quad NInt(NInt(A)) = NInt(A) \wedge NInt(B),$

(f) $\quad NCl(A \cup B) = NCl(A) \vee NCl(B),$

(g) $\quad NInt(1_N) = 1_N,$

(h) $\quad NCl(O_N) = O_N,$

Proof (a), (b) and (e) are obvious (c) follows from (a) and Definitions.

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