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Rational Approximation Of Irrational Numbers

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***Abstract**:*
It has always been interesting to find rational approximate fractions like p/q for any irrational number where p ,q are integers. The methods involve finding approximations by their decimal values or by resolving them into continued fractions and taking successive convergents for each irrational number. In my present paper I have derived some formulas which are able to give approximate fractions directly without individually resolving them into continued fractions.
***Keywords**: Irretional numbers , rational approximations basic formula , approximation formulas.*

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I. Introduction/ Basic formula

Any irrational number of the form $\sqrt{\frac{x+k}{x}}$ can be expressed as an infinite continued fraction as below

$\sqrt{\frac{x+k}{x}} = 1+\cfrac{1}{\frac{2x}{k}+\cfrac{1}{2+\cfrac{1}{\frac{2x}{k}+\cfrac{1}{2+\cdots\ldots\ldots\infty}}}}$ where $\sqrt{\frac{x+k}{x}}$ is real number

Proof:-

Let y = $\cfrac{1}{x+\cfrac{1}{n+\cfrac{1}{x+\cfrac{1}{n+\cdots\ldots\ldots\infty}}}}$

So y = $\frac{1}{x+\frac{1}{n+y}}$

Or y = $\frac{n+y}{xn+xy+1}$

Or $xny + xy^2 + y = n + y$

Or $xy^2 + xny - n = 0$

Or y = $\frac{-xn\pm\sqrt{(xn)^2+4xn}}{2x}$

Or y = $-\frac{n}{2} + \sqrt{\frac{x^2n^2+4xn}{4x^2}}$ (taking only + sign instead of $\pm$ sign)

Or Y = $-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{n}{x}}$

Replacing x by $\frac{4x}{kn}$ we get

Y = $-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{kn^2}{4x}}$

Or y = $-\frac{n}{2} + \frac{n}{2}\sqrt{1+\frac{k}{x}}$

Or y = $-\frac{n}{2} + \frac{n}{2}\sqrt{\frac{x+k}{x}}$

This is true for all values of n so we take n=2 then

y = $-1 + \sqrt{\frac{x+k}{x}}$

Or y +1 = $\sqrt{\frac{x+k}{x}}$

Or $\sqrt{\frac{x+k}{x}}$ =1+y

Or $\sqrt{\frac{x+k}{x}}$ $=1+\cfrac{1}{\frac{2x}{k}+\cfrac{1}{2+\cfrac{1}{\frac{2x}{k}+\cfrac{1}{2+\cdots\ldots\ldots\infty}}}}$

since replacing x by $\frac{4x}{nk}$ and taking n=2 x is finally replaced by $\frac{4x}{2\bullet k}=\frac{2x}{k}$
Hence the above result /formula.

II. Approximation Formulas

Based on the above basic formula I we draw five approximation formulas which gradually provide more approximate values of the irrational number $\sqrt{\frac{x+k}{x}}$

1^{st} Approximation Formula

Taking 1 + one set of $\left(\frac{1}{\frac{2x}{k}+\frac{1}{2}}\right)$

$\sqrt{\frac{x+k}{x}}$ $=1+\frac{1}{\frac{2x}{k}+\frac{1}{2}}=1+\frac{1}{\frac{4x+k}{2k}}=1+\frac{2k}{4k+k}$

Or $\sqrt{\frac{x+k}{x}}=\frac{4x+3k}{4x+k}$ ………. **F1**

2^{nd} Approximation Formula

Taking 1 + two sets of $\left(\frac{1}{\frac{2x}{k}+\frac{1}{2}}\right)$

$\sqrt{\frac{x+k}{x}}$ $=1+\cfrac{1}{\frac{2x}{k}+\cfrac{1}{2+\cfrac{1}{\frac{2x}{k}+\frac{1}{2}}}}=1+\cfrac{1}{\frac{2x}{k}+\cfrac{1}{2+\frac{2k}{4x+k}}}$

Or $=1+\cfrac{1}{\frac{2x}{k}+\cfrac{1}{\frac{8x+4k}{4x+k}}}=1+\cfrac{1}{\frac{2x}{k}+\frac{4x+k}{8x+4k}}$

Or $=1+\cfrac{1}{\frac{16x^2+12xk+k^2}{8xk+4k^2}}=1+\frac{8xk+4k^2}{16x^2+12xk+k^2}$

Or $\sqrt{\frac{x+k}{x}}=\frac{16x^2+20xk+5k^2}{16x^2+12xk+k^2}$ ……………..**F2 (i)**

Or alternatively $=\frac{2(2x+k)\,(4x+3k)-k^2}{2(2x+k)\,(4x+k)-k^2}$ ………………**F2 (ii)**

Note: F2 (ii) can easily be derived from **F2 (i)**

3^{rd} Approximation Formula

Taking 1+ three sets of $\left(\frac{1}{\frac{2x}{k}+\frac{1}{2}}\right)$

$\sqrt{\frac{x+k}{x}}$ $=1+\cfrac{1}{\frac{2x}{k}+\cfrac{1}{2+\cfrac{1}{\frac{2x}{k}+\cfrac{1}{2+\cfrac{1}{\frac{2x}{k}+\frac{1}{2}}}}}}$ and simplifying as in **F1** and **F2** above we get

$\sqrt{\frac{x+k}{x}}=\frac{64x^3+112kx^2+56k^2x+7k^3}{64x^3+80kx^2+24k^2x+k^3}$ ………..**F3 (i)**

Alternatively = $\frac{(4x+3k)^2(4x+k)-k^2(4x+3k)+k^3}{(4x+k)^2(4x+3k)-k^2(4x+k)-k^3}$ ……………..**F3 (ii)**

4^{th} Approximation Formula

Taking 1+ four sets of $\left(\frac{1}{\frac{2x}{k}+\frac{1}{2}}\right)$ and simplifying as above

$\sqrt{\frac{x+k}{x}}=\frac{256x^4+576x^3k+432x^2k^2+120xk^3+9k^4}{256x^4+448x^3k+240x^2k^2+40xk^3+k^4}$ …………..**F4 (i)**

Alternatively = $\frac{4x(4x+3k)^3+3k^3(4x+3k)}{4(x+k)(4x+k)^3-3k^3(4x+k)}$ ……………..**F4 (ii)**

5^{th} Approximation Formula

Taking 1+ five sets of $\left(\frac{1}{\frac{2x}{k}+\frac{1}{2}}\right)$ and simplifying we get

$\sqrt{\frac{x+k}{x}} = \frac{1024x^5+2816x^4k+2816x^3k^2++1232x^2k^3+220xk^4+11k^5}{1024x^5+2304x^4k+1792x^3k^2+560x^2k^3+60xk^4+k^5}$ …………..**F5 (i)**

Alternatively = $\frac{(4x+3k)^3(4x+k)^2-2k^2(4x+3k)^2(4x+k)+16x(x+k)k^3+2k^5}{(4x+k)^3(4x+3k)^2-2k^2(4x+k)^2(4x+3k)-16x(x+k)k^3-2k^5}$ ……………..**F5 (ii)**

From above it can be seen that alternative forms from F2 to F5 are symmetrical, easier to apply and remember.

Note:-more approximation formulas can be derived taking $\sqrt{\frac{x+k}{x}}$ = 1 + ……n sets of $\left(\frac{1}{\frac{2x}{k}+\frac{1}{2}}\right)$

III. Cited Examples

These examples demonstrate as to how we can apply these formulas to obtain rational approximations of irrational numbers keeping in view the following

a) When k is small in comparison to x that is $\frac{k}{x}$ is small we can directly apply these formulas from F1 to F5 to obtain gradually nearer rational approximations.

b) When k is relatively not small in comparison to x we can break up the original irrational number into a combination of multiple irrational numbers whose $\frac{k}{x}$ are smaller as will be shown in some examples.

c) We can apply any appropriate formula from F1 to F5 to obtain rational approximations according as the value of $\frac{k}{x}$.

Example 1-

Find rational approximations of $\sqrt{\frac{32}{29}}$

$\sqrt{\frac{32}{29}} = \sqrt{\frac{29+3}{29}}$ here x= 29, k = 3, 4x+3k = 125, 4x+k=119, 2x+k=61, x+k =32

By using formula F1 = $\frac{4x+3k}{4x+k} = \frac{125}{119}$ = 1.05042 which is correct up to 3 decimal places.

By using formula F2 (ii) $\sqrt{\frac{x+k}{x}} = \frac{2(2x+k)(4x+3k)-k^2}{2(2x+k)(4x+k)-k^2}$

Or = $\frac{2\bullet 61\bullet 125-3^2}{2\bullet 61\bullet 119-3^2} = \frac{15241}{14509}$ = 1.05045144 which is correct up to 6 decimal places.

By using formula F3 (ii) $\sqrt{\frac{x+k}{x}} = \frac{(4x+3k)^2(4x+k)-k^2(4x+3k)+k^3}{(4x+k)^2(4x+3k)-k^2(4x+k)-k^3}$

Or = $\frac{125^2\bullet 119-9\bullet 125+27}{119^2\bullet 125-9\bullet 119-27} = \frac{1858277}{1769027}$ = 1.0504514628663 which is correct up to 10 decimal places.

By using formula F4 (ii) $\sqrt{\frac{x+k}{x}} = \frac{4x(4x+3k)^3+3k^3(4x+3k)}{4(x+k)(4x+k)^3-3k^3(4x+k)}$

Or = $\frac{4\bullet 29\bullet 125^3+3\bullet 3^3\bullet 125}{4\bullet 32\bullet 119^3-3\bullet 3^3\bullet 119} = \frac{226572625}{215690713}$ = 1.0504514628777735 which is correct up to 13 decimal places.

By using formula F5 (ii) $\sqrt{\frac{x+k}{x}} = \frac{(4x+3k)^3(4x+k)^2-2k^2(4x+3k)^2(4x+k)+16x(x+k)k^3+2k^5}{(4x+k)^3(4x+3k)^2-2k^2(4x+k)^2(4x+3k)-16x(x+k)k^3-2k^5}$

Or = $\frac{125^3\bullet 119^2-2\bullet 3^2\bullet 125^2\bullet 119+16\bullet 29\bullet 32\bullet 3^3+2\bullet 3^5}{119^3\bullet 125^2-2\bullet 3^2\bullet 119^2\bullet 125-16\bullet 29\bullet 32\bullet 3^3-2\bullet 3^5} = \frac{27625135757}{26298345743}$ = 1.050451462877780448 which is correct up to 17 decimal places.

Example 2-

Find approximations of $\sqrt{\frac{44}{39}}$.

Solution: Here k =5 and $\frac{k}{x} = \frac{5}{39}$ which is slightly higher so we break up $\sqrt{\frac{44}{39}}$ as

$\sqrt{\frac{44}{39}} = \sqrt{\frac{44}{40}} \bullet \sqrt{\frac{40}{39}} = \sqrt{\frac{11}{10}} \bullet \sqrt{\frac{40}{39}}$

For $\sqrt{\frac{11}{10}}$ x=10, k=1, 2x+k =21, 4x+3k= 43, 4x+k=41

So by F2(ii) $\sqrt{\frac{11}{10}} = \frac{2(2x+k)(4x+3k)-k^2}{2(2x+k)(4x+k)-k^2}$

Or = $\frac{2 \bullet 21 \bullet 43 - 1^2}{2 \bullet 21 \bullet 41 - 1^2} = \frac{1805}{1721}$

For $\sqrt{\frac{40}{39}}$ x= 39 , k=1, 2x+k=79 , 4x+3k=159 ,4x+k=157

So by F2(ii) $\sqrt{\frac{40}{39}} = \frac{2 \bullet 79 \bullet 159 - 1^2}{2 \bullet 79 \bullet 157 - 1^2} = \frac{25121}{24805}$

So $\sqrt{\frac{44}{39}} = \frac{1805 \bullet 25121}{1721 \bullet 24805}$ = 1.062169993 which is correct up to 7 decimal places.

By applying F3, F4, F5 we can obtain more accurate approximations.

Example 3 – Find approximations of $\sqrt{\frac{480}{53}}$

Solution: - here $\sqrt{\frac{480}{53}} = \sqrt{\frac{53+427}{53}}$ so here $\frac{k}{x} = \frac{427}{53}$ which is very high

So we transform $\sqrt{\frac{480}{53}}$ as = $\sqrt{\frac{480 \bullet 9}{53 \bullet 9}} = 3\sqrt{\frac{480}{53 \bullet 9}} = 3\sqrt{\frac{160}{159}}$

Now for $\sqrt{\frac{160}{159}}$ x =159, k=1 so $\frac{k}{x} = \frac{1}{159}$ which is quite small

So by applying F2(ii) for $\sqrt{\frac{160}{159}} = \frac{2(2x+k)\,(4x+3k) - k^2}{2(2x+k)\,(4x+k) - k^2}$

Or $\sqrt{\frac{160}{159}} = \frac{2 \bullet 319 \bullet 639 - 1^2}{2 \bullet 319 \bullet 637 - 1^2} = \frac{407681}{406405}$

So $\sqrt{\frac{480}{53}} = 3 \bullet \frac{407681}{406405}$ = 3.0094191754530579 which is correct up to 13 decimal places.

Example 4- where k is a negative number

Find approximations of $\sqrt{24}$

Solution:- $\sqrt{24} = \sqrt{\frac{24 \bullet 25}{25}} = 5\sqrt{\frac{24}{25}}$ = here k=-1 , x =25 , 2x+k=49 , 4x+k=99 , 4x+3k=97

So by **F2 (ii)** $\sqrt{\frac{24}{25}} = \frac{2(2x+k)\,(4x+3k) - k^2}{2(2x+k)\,(4x+k) - k^2} = \frac{2 \bullet 49 \bullet 97 - (-1^2)}{2 \bullet 49 \bullet 99 - (-1^2)} = \frac{98 \bullet 97 - 1}{98 \bullet 99 - 1} = \frac{9505}{9701}$

So $\sqrt{24} = 5 \bullet \frac{9505}{9701}$ = 4.898979487 which is correct up to 8 decimal places.

Also by **F4 (ii)** $\sqrt{\frac{24}{25}} = \frac{4x(4x+3k)^3 + 3k^3(4x+3k)}{4(x+k)(4x+k)^3 - 3k^3(4x+k)}$

Or $\sqrt{\frac{24}{25}} = \frac{4 \bullet 25 \bullet 97^3 + 3(-1^3) \bullet 97}{4 \bullet 24 \bullet 99^3 - 3(-1^3) \bullet 99} = \frac{91267009}{93149001}$

So $\sqrt{24} = 5 \bullet \frac{91267009}{93149001}$ = 4.898979485566356208 which is correct up to 16 decimal places.

Example 5 :-

If π is approximately equal to $\frac{355}{113}$ find approximations for $\sqrt{\pi}$

Solution: - $\sqrt{\pi} = \sqrt{\frac{355}{113}} = \sqrt{\frac{355 \bullet 3}{113 \bullet 3}} = \sqrt{3}\sqrt{\frac{355}{339}} = \sqrt{\frac{339+16}{339}}$

So for $\sqrt{\frac{339+16}{339}}$ x =339 , k=16 , 4x+k=1372 , 4x+3k = 1404 , 2x+k= 694

So by **F2 (ii)** $\sqrt{\frac{355}{339}} = \frac{2(2x+k)\,(4x+3k) - k^2}{2(2x+k)\,(4x+k) - k^2} = \frac{2 \bullet 694 \bullet 1404 - 16^2}{2 \bullet 694 \bullet 1372 - 16^2} = \frac{347 \bullet 351 - 16}{347 \bullet 343 - 16}$ (dividing by 16)

Or $\sqrt{\frac{355}{339}} = \frac{121781}{119005}$ so $\sqrt{\pi} = \sqrt{3} \bullet \frac{121781}{119005}$

Now taking $\sqrt{3}$ approximately equal to $\frac{1351}{780}$

So $\sqrt{\pi} = \frac{1351}{780} \bullet \frac{121781}{119005} = \frac{164526131}{92823900}$ = 1.772454411 which is correct up to 6 decimal places.

Note:- Resolving $\frac{164526131}{92823900}$ into continued fraction $\mathbf{\frac{296}{167}}$ =1.77245509 is also a good approximation for $\sqrt{\pi}$ up to 4 decimal places.

Example 6 :-

The accuracy of these formulas depend on the value of the ratio $\frac{k}{x}$. Smaller the value of $\frac{k}{x}$, more accurate the approximations of $\sqrt{\frac{x+k}{x}}$ .Below examples show how an irrational number with higher $\frac{k}{x}$ can be broken up into combination of multiple irrational numbers having smaller values of $\frac{k}{x}$.

(i) $\sqrt{\frac{35}{31}} = \sqrt{\frac{35}{34}} \bullet \sqrt{\frac{34}{32}} \bullet \sqrt{\frac{32}{31}}$

Or $\sqrt{\frac{35}{31}} = \sqrt{\frac{35}{34}} \bullet \sqrt{\frac{17}{16}} \bullet \sqrt{\frac{32}{31}}$

$\frac{k}{x} = \left(\frac{4}{31}\right) \quad \left(\frac{1}{34}\right) \quad \left(\frac{1}{16}\right) \quad \left(\frac{1}{31}\right)$

(ii) $\sqrt{\frac{89}{71}} = \sqrt{\frac{89}{88}} \bullet \sqrt{\frac{88}{80}} \bullet \sqrt{\frac{80}{72}} \bullet \sqrt{\frac{72}{71}}$

$\sqrt{\frac{89}{71}} = \sqrt{\frac{89}{88}} \bullet \sqrt{\frac{11}{10}} \bullet \sqrt{\frac{10}{9}} \bullet \sqrt{\frac{72}{71}}$

$\frac{k}{x}\left(\frac{18}{71}\right) \quad \left(\frac{1}{88}\right) \quad \left(\frac{1}{10}\right) \quad \left(\frac{1}{9}\right) \quad \left(\frac{1}{71}\right)$

(iii) $\sqrt{17} = \sqrt{\frac{17}{16}} \bullet \sqrt{16}$

Or $\sqrt{17} = \sqrt{\frac{17}{16}} \bullet 4$

$\frac{k}{x} \quad \left(\frac{16}{01}\right) \quad \left(\frac{1}{16}\right)$

In all above 3 examples $\frac{k}{x}$ on the right hand side are much smaller than those on the left hand side.

IV. Extension Of The First Approximation Formula F1

For **F1 =** $\sqrt{\frac{x+k}{x}} = \frac{4x+3k}{4x+k} = \frac{(4x+k)+2k}{(4x+k)}$ this is of the form $\frac{X+K}{X}$ where X = 4x+k and K=2k

So by applying formula **F1** $\left(\frac{x+k}{x}\right)^{\frac{1}{4}} = \left\{\frac{(4x+k)+2k}{(4x+k)}\right\}^{\frac{1}{2}}$

Or $= \frac{4(4x+k)+3\bullet 2k}{4(4x+k)+2k}$

Or $= \frac{16x+4k+6k}{16x+4k+2k}$

Or $= \frac{16x+10k}{16x+6k} = \frac{8x+5k}{8x+3k}$

Or $= \frac{4(2x+k)+k}{4(2x+k)-k}$ similarly it can be proved that

$\left(\frac{x+k}{x}\right)^{\frac{1}{8}} = \frac{8(2x+k)+k}{8(2x+k)-k}$

Or in general $\left(\frac{x+k}{x}\right)^{\frac{1}{2^n}} = \frac{2^n(2x+k)+k}{2^n(2x+k)-k}$ …..**F1g (i)**

And in more general form $\left(\frac{x+k}{x}\right)^{\frac{p}{q}} = \frac{\frac{q}{p}(2x+k)+k}{\frac{q}{p}(2x+k)-k}$ …..**F1g (ii)**

Note: These formulas give good approximations when $\frac{k}{x}$ is quite small.

Example 7 :- Find approximation for $\left(\frac{89}{87}\right)^{\frac{1}{8}}$

For $\left(\frac{89}{87}\right)^{\frac{1}{8}}$ x=87, k =2, 2x+k=176

So $\left(\frac{89}{87}\right)^{\frac{1}{8}} = \frac{8(2x+k)+k}{8(2x+k)-k} = \frac{8\bullet 176+2}{8\bullet 176-2} = \frac{1410}{1406} = \frac{705}{703}$ = 1.00284495 which is correct up to 6 decimal places.

Example 8 :- Find approximation for $\left(\frac{97}{93}\right)^{\frac{3}{7}}$.

For $\left(\frac{97}{93}\right)^{\frac{3}{7}}$ x =93, k=4, 2x+k = 190

So $\left(\frac{97}{93}\right)^{\frac{3}{7}} = \frac{\frac{7}{3}\bullet 190+4}{\frac{7}{3}\bullet 190-4} = \frac{1330+12}{1330-12} = \frac{1342}{1318} = \frac{671}{659} = 1.01820948$ which is correct up to 5 decimal places.

V. Conclusion

This paper presents a direct method for obtaining rational approximations of irrational numbers in the form p/q. Unlike traditional approaches based on decimal expansions or continued fractions, the proposed formulas generate approximations directly and more efficiently. The method simplifies the approximation process and provides a useful alternative for studying irrational numbers.

References

[1]. Hall, H. S. And Knight, S. R., *Higher Algebra*.