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Wiener Polynomial Of Splitting Graph Of Sunlet Graph

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Abstract
The sum of distances between all unordered pairs of vertices in a connected graph is called the Wiener index. In this paper, we found the Wiener polynomial and index of splitting graph of sunlet graph .
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I. Introduction

Chemical graph theory, a significant branch of graph theory, has extensive applications in chemistry. One of its central concepts is the topological index, a numerical descriptor derived from the molecular graph of a compound. These indices play a crucial role in modelling chemical and physical properties of molecules, particularly in Quantitative Structure–Property Relationship (QSPR) and Quantitative Structure–Activity Relationship (QSAR) studies.

In this work, we consider a finite, simple, and connected graph G, meaning that it contains no loops or multiple edges. The vertex set and edge set of $G$are denoted by $V(G)$and $E(G)$, respectively. The degree of a vertex a, represented by $\delta(a)$, is defined as the number of edges incident to it.

The distance between two vertices $a$and $b$in $V(G)$, denoted by $d(a,b)$, is the length (i.e., the number of edges) of the shortest path connecting them. The diameter of the graph, denoted by $d(G)$, is the maximum distance between any pair of vertices in G. Furthermore, for each integer $k \in \{0,1,\ldots,d(G)\}$, let $d(G;k)$represent the number of unordered vertex pairs that are at distance exactly k.

The Wiener index is one of the earliest and most fundamental topological indices used in chemistry. It was introduced in 1947 by Harold Wiener, who applied it to study the physicochemical properties of paraffin compounds (alkanes). The Wiener index of a graph G, denoted by $W(G)$, is defined as the sum of distances between all unordered pairs of vertices in the graph:

$$W(G) = \sum_{a,b \in V(G)} d(a,b).$$

In addition, the Wiener polynomial provides a generating function for the distances in the graph. It is defined as:

$$W(G;x) = \sum_{a,b \in V(G)} x^{d(a,b)}.$$

II. Splitting Graph Of A Graph:

The splitting graph is introduced by Prof. E. Sampath Kumar, Prof. H.B. Walikar in [8]. For a graph *G*, Splitting graph is denoted by *S*(*G*). For each vertex *v* of a graph *G*, take a new vertex $v^{/}$ .Join $v^{/}$ to all the vertices of *G* adjacent to *v*. The graph *S*(*G*) thus obtained is called Splitting graph or Duplicate graph of *G*. In [8], they have studied some properties of *S*(*G*) and obtained the characterization of *S*(*G*). Prof. Jan Mycielski, in his work in sets and logic used a graph called *shadow graph S*(*G*). For a graph *G* , the *shadow graph S*(*G*) is obtained from *G* by adding for each vertex *v* of *G*, a new vertex $v^{/}$ called shadow vertex of *v*, and joining $v^{/}$ to the neighbour of *v* in *G*.

***Observations on S*(*G*):**

1. A vertex of *G* and its duplicate vertices are not adjacent in *S*(*G*).
2. All duplicate vertices induces a null graph.

3. $d(v_i)\ in\ G\ = d(v_i^{/})\ in\ S(G)$.

4. $d(v_i)\ in S(G) = 2d(v_i^{/})\ in\ S(G)$

5. By definition of splitting graph, $d(v_i, v_j / G) = d(v_i, v_j / S(G))$

***Note* [9]**: As the splitting graph consists of the vertices of given graph and its duplicate vertices, for convenience sake, partition the vertex set of *S* (*G*) into the sets *A*, *B* as below.

$$A = \{v_i / v_i \in V(G)\}$$
$$B = \{v_i^{/} / v_i^{/} \in V(S(G)) - V(G)\}$$

In order to demonstrate the validity of the theorem, we adopt the following notation:

$d_A(G,i) =$ *number of pairs of vertices in the set A, at a distance i*

$d_B(G,i) =$ *number of pairs of vertices in the set B, at a distance i*

$d_{AB}(G,i) =$ *number of pairs of vertices of which one is in the set A and the other is in the set B, that are at a distance i*

$d_A(v_i, v_j) =$ *distance between the vertices* v_i, v_j *among the vertices of the set A*

$d_B(v_i, v_j) =$ *distance between the vertices* v_i, v_j *among the vertices of the set B*

$d_{AB}(v_i, v_j) =$ *distance between the vertices* v_i, v_j *among the vertices of the set A, B*

With the above notation, the Wiener polynomial of Splitting graph is given as

$W(S(G,q)) = a_1 q + a_2 q^2 + a_3 q^3 + ... + a_i q^i$

It can be easily observed that $a_i = d_A(G,i) + d_B(G,i) + d_{AB}(G,i)$

By computing these three terms we found the coeficients of wiener polynomial.

Note: In the following proofs consider*j* as *n* whenever $j \equiv 0(\bmod n)$.

Theorem: The wiener polynomial of splitting graph of sunlet graph $S(C_n \otimes K_2)$ is $\sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}$

$W(S(C_n \otimes K_2)) = 4W(C_n \otimes K_2) + 2n(-q + q^2 + q^3),\ n \geq 5$

Proof: Let $v_1, v_2, ... v_n \in V(C_n), a_1, a_2, ..., a_n$ be the pendent vertices that are adjacent to v_i in $(C_n \otimes K_2)$. $v_1^{/}, v_2^{/}, ... v_n^{/} \in V(C_n),\ a_1^{/}, a_2^{/}, ... a_n^{/}$ be the corresponding duplicate vertices. For convenience sake we partition the vertex set into the following subsets and find the distances between them.

$A =$ *Set of vertices of sunlet graph*

$B =$ *Set of duplicate vertices of sunlet graph*

Further the sets *A*,*B* are partitioned as

$A1 =$ *Set of vertices of cycle of sunlet graph* $B1 =$ *Set of duplicate vertices of cycle.*

$A2 =$ *Set of pendent vertices of sunlet graph* $B2 =$ *Set of duplicate vertices of degree one*

The distances between the vertices from the above sets are given in ten combitions as follows.

Case1: For odd *n*

The distance between the vertices of A1is given as in theorem2

Wiener polynomial in this case is given by

$$\sum_{\{v_i, v_j\} \in V(A1)} q^{(v_i, v_j)} = nq^1 + nq^2 + nq^3 + ... \quad + nq^{\frac{n-1}{2}} = n \sum_{i=1}^{\frac{n-1}{2}} q^i$$

***case* 2**: The distance function between the vertices of A2 is given as $d_{A2}(a_i, a_j) = x,\ 3 \leq x \leq \dfrac{n+3}{2},\ 1 \leq i, j \leq n,\ j = i + x - 2(\bmod n)$.

$d_{A2}(G,i) = n,\ 3 \leq i \leq \dfrac{n+3}{2}$

$$\sum_{\{a_i,a_j\}\in V(A2)} q^{(a_i,a_j)} = nq^3 + nq^4 + \dots \qquad + nq^{\frac{n+3}{2}} \;=\; n\sum_{i=3}^{\frac{n+3}{2}} q^i$$

***case*3**:
The distance function ,between the vertices of A1, A2 is given as

$$d_{A1,A2}(v_i,a_j) = x,\ 1 \le x \le \frac{n+1}{2},\ 1 \le i,j \le n,\ j = i + x - 1\ (\bmod n)$$

$$j = n - x + i + 1 (\bmod n), 2 \le x \le \frac{n+1}{2}.$$

$$d_{A1A2}(G,i) = \begin{cases} n\ for\ i = 1 \\ 2n, 2 \le i \le \frac{n+1}{2} \end{cases}$$

$$\sum_{\{v_i,a_j\}\in V(A1A2)} q^{(v_i,a_j)} = nq^1 + 2\left(nq^2 + nq^3 + \dots \qquad + nq^{\frac{n+1}{2}}\right) = nq^1 + \ 2n\sum_{i=2}^{\frac{n+1}{2}} q^i$$

***case*4**: Distance function between the vertices of the set B1is given as

$$d_{B1}(v_i^{/}, v_j^{/}) = 3\ for\ i = j + 1$$

$$= x,\ 2 \le x \le \frac{n-1}{2},\ 1 \le i,j \le n,\ j = i + x\ (\bmod n)$$

$$d_{B1}(G,i) = \begin{cases} 2n\ for\ i = 3 \\ n, 2 \le i \le \frac{n-1}{2}, n \ne 3 \end{cases}$$

Thus the wiener polynomial in this case becomes

$$\sum_{\{v_i^{/},v_j^{/}\}\in V(B1)} q^{(v_i^{/},v_j^{/})} = nq^3 + \left(nq^2 + nq^3 + \dots \qquad + nq^{\frac{n-1}{2}}\right) = \ nq^3 + n\sum_{i=2}^{\frac{n-1}{2}} q^i$$

***case*5**: Distance function between the vertices B2is given as

$$d_{B2}(a_i^{/}, a_j^{/}) = x,\ 3 \le x \le \frac{n+3}{2},\ 1 \le i,j \le n,\ j = i + x - 2 (\bmod n)$$

$$d_{B2}(G,i) = n, 3 \le i \le \frac{n+3}{2}$$

Thus the wiener polynomial in this case becomes

$$\sum_{\{a_i^{/},a_j^{/}\}\in V(B2)} q^{(a_i^{/},a_j^{/})} = nq^3 + nq^4 + \dots \qquad + nq^{\frac{n+3}{2}} \;=\; n\sum_{i=3}^{\frac{n+3}{2}} q^i$$

***case*6**: Distance between the duplicate vertices of the sets B1,B2 is given as distance function

$$d_{B1B2}(a_i^{/}, v_j^{/}) = x, 1 \le x \le \frac{n+1}{2},\ 1 \le i,j \le n,\ j = i + x - 1 (\bmod n)$$

$$j = n - x + i + 1 (\bmod n), 2 \le x \le \frac{n+1}{2}$$

$$d_{B1B2}(G,i) = \begin{cases} n\ for\ i = 1 \\ 2n, 2 \le i \le \frac{n+1}{2},, i \ne 3 \end{cases}$$

Thus the wiener polynomial in this case becomes

$$\sum_{\{a_i^{/},v_j^{/}\}\in V(B1B2)} q^{(a_i^{/},v_j^{/})} = nq^1 + 2\left(nq^2 + nq^3 + \ldots \qquad + nq^{\frac{n+1}{2}} \right) = nq^1 + 2n\sum_{i=2}^{\frac{n+1}{2}} q^i$$

***Case*7**: Distance between the vertices of A1,B2 is given as

$$d_{A1B2}(v_i, a_j^{/}) = x,\ 1 \le x \le \frac{n+1}{2},\ 1 \le i, j \le n,\ j = i + x - 1\ (\mathrm{mod}\, n)$$

$$j = n - x + i + 1(\mathrm{mod}\, n), 2 \le x \le \frac{n+1}{2}$$

$$d_{A1B2}(G,i) = \begin{cases} n\ for\ i = 1 \\ 2n, 2 \le i \le \dfrac{n+1}{2} \end{cases}$$

$$\sum_{\{v_i,a_j\}\in V(A1B2)} q^{(v_i,a_j)} = nq^1 + 2\left(nq^2 + nq^3 + \ldots \qquad + nq^{\frac{n+1}{2}} \right) = nq^1 + \ 2n\sum_{i=2}^{\frac{n+1}{2}} q^i$$

***case*8**: Distance between the vertices of A1,B1 is given as

$$d_{A1B1}(v_i, v_j^{/}) = 2\ for\ i = j$$

$$= x,\ 1 \le x \le \frac{n-1}{2},\ 1 \le i, j \le n,\ j = i + x\ (\mathrm{mod}\, n)$$

$$j = n - x + i(\mathrm{mod}\, n),\ ,2 \le x \le \frac{n-1}{2}$$

$$d_{A1B1}(G,i) = \begin{cases} 3n\ for\ i = 2 \\ 2n, 1 \le i \le \dfrac{n-1}{2}, i \ne 2 \end{cases}$$

$$\sum_{\{v_i,v_j^{/}\}\in V(A1B1)} q^{(v_i,v_j^{/})} = nq^2 + 2\left(nq + nq^2 + nq^3 + \ldots \qquad + nq^{\frac{n-1}{2}} \right) = nq^2 + \ 2n\sum_{i=1}^{\frac{n-1}{2}} q^i$$

***Case* 9**: Distance between the sets A2, B1 is

$$d_{A2B1}(a_i, v_j^{/}) = \begin{cases} 3\ for\ i = j \\ x,\ 2 \le x \le \dfrac{n+1}{2},\ 1 \le i, j \le n,\ j = i + x - 1\ (\mathrm{mod}\, n) \\ \qquad\qquad\qquad\qquad = n - x + i + 1(\mathrm{mod}\, n) \end{cases}$$

$$d_{A2B1}(G,i) = \begin{cases} n\ for\ i = 3 \\ 2n, 2 \le i \le \dfrac{n+1}{2} \end{cases}$$

$$\sum_{\{v_i,a_j\}\in V(A2B1)} q^{(v_i,a_j)} = nq^3 + 2\left(nq^2 + nq^3 + \ldots \qquad + nq^{\frac{n+1}{2}} \right) = nq3 + \ 2n\sum_{i=2}^{\frac{n+1}{2}} q^i$$

***case*10**: Distance between the sets A2,B2 is

$$d_{A2B2}(a_i,a_j^{/})=\begin{cases} x, 2\le x\le \frac{n+3}{2},\ 1\le i,j\le n,\ j=i+x-2\ (\bmod n) \\ x, j=n-x+i+2(\bmod n) \end{cases}$$

$$d_{A2B2}(G,i)=\begin{cases} n\ for\ i=2 \\ 2n, 3\le i\le \frac{n+3}{2} \end{cases}$$

$$\sum_{\{a_i,a_j^{/}\}\in V(A2B2)} q^{(a_i,a_j^{/})}=nq^2+2\left(nq^3+nq^4+\dots \quad +nq^{\frac{n+3}{2}}\right)=nq^2+\ 2n\sum_{i=3}^{\frac{n+3}{2}}q^i$$

By adding all the polynomials ,we get the wiener polynomial of splitting graph of sun let graph for odd n is
$W(S(C_n\otimes K_2))=4W(C_n\otimes K_2)+2n\left(-q+q^2+q^3\right),\ n\ge 5$

For even n:

***case*1**: The distance between the vertices of A1 is given as in theorem 2

$$\sum_{\{v_i,v_j\}\in V(A1)} q^{(v_i,v_j)}=nq^1+nq^2+nq^3+\dots \quad +nq^{\frac{n}{2}-1}+\frac{n}{2}q^{\frac{n}{2}}=\ n\sum_{i=1}^{\frac{n}{2}}q^i+\frac{n}{2}q^{\frac{n}{2}}$$

***case* 2**: Distance between the vertices of the set A2

$$d_{A2}(a_i,a_j)=x,\ 3\le x\le \frac{n}{2}+2,\ 1\le i,j\le n,\ j=i+x-2(\bmod n)\ .$$

$$d_{A2}(G,i)=n\ for\ 3\le i\le \frac{n+4}{2}$$

$$\sum_{\{a_i,a_j\}\in V(A2)} q^{(a_i,a_j)}=nq^3+nq^4+\dots \quad +nq^{\frac{n+4}{2}}\ =\ n\sum_{i=3}^{\frac{n+2}{2}}q^i+\frac{n}{2}q^{\frac{n+4}{2}}$$

***case*3**: Distance between the vertices of A1,A2 is given as

$$d_{A1A2}(v_i,a_j)=x, 1\le x\le \frac{n+2}{2},\ 1\le i,j\le n,\ j=i+x-1\ (\bmod n)$$

$$=n-x+i+1(\bmod n), 2\le x\le \frac{n+2}{2}\ .$$

$$d_{A1A2}(G,i)=\begin{cases} n\ for\ i=1 \\ 2n, 2\le i\le \frac{n+2}{2} \end{cases}$$

$$\sum_{\{v_i,a_j\}\in V(A1A2)} q^{(v_i,a_j)}=nq^1+2\left(nq^2+nq^3+\dots \quad +nq^{\frac{n+2}{2}}\right)=nq^1+\ 2n\sum_{i=2}^{\frac{n+2}{2}}q^i$$

***case*4**: Distance between the vertices of B1 is given as

$$d_{B1}(v_i^{/},v_j^{/})=\begin{cases} 3\ for\ i=j \\ x,\ 2\le x\le \frac{n}{2},\ 1\le i,j\le n,\ j=i+x(\bmod n) \\ \frac{n}{2}, 1\le i\le \frac{n}{2}, \frac{n}{2}+1\le j\le n, \end{cases}$$

$$d_{B1}(G,i)=\begin{cases} 2n \; for \; i=3 \\ n, 2\leq i\leq \frac{n-2}{2}, i\neq 3 \\ \frac{n}{2} \; for \; i=\frac{n}{2} \end{cases}$$

Thus the wiener polynomial in this case becomes

$$\sum_{\{v_i^{/},v_j^{/}\}\in V(B1)} q^{(v_i^{/},v_j^{/})} = nq^3 + \left(nq^2 + nq^3 + \ldots \qquad + nq^{\frac{n-2}{2}} \right) + \frac{n}{2}q^{\frac{n}{2}} = \quad nq^3 + n\sum_{i=2}^{\frac{n-2}{2}} q^i + \frac{n}{2}q^{\frac{n}{2}}$$

***case*5**: Distance function between the vertices of B2 is given as

$$d_{B2}(a_i^{/},a_j^{/}) = x,\; 3\leq x\leq \frac{n+4}{2},\; 1\leq i,j\leq n,\; j=i+x-2(\mathrm{mod}\, n).$$

$$d_{B2}(G,i) = n \; for \; 3\leq i\leq \frac{n+4}{2}$$

$$\sum_{\{a_i^{/},a_j^{/}\}\in V(B2)} q^{(a_i^{/},a_j^{/})} = nq^3 + nq^4 + \ldots \qquad + nq^{\frac{n+4}{2}} \;=\; n\sum_{i=3}^{\frac{n+2}{2}} q^i + \frac{n}{2}q^{\frac{n+4}{2}}.$$

***case* 6**: Distance function between the vertices of B1,B2 is given as

$$d_{B1B2}(a_i^{/},v_j^{/}) = \begin{cases} 3 \; for \; i=j \\ x,\; 2\leq x\leq \frac{n+2}{2},\; 1\leq i,j\leq n,\; j=i+x-1(\mathrm{mod}\, n) \end{cases}$$

$$d_{B1B2}(G,i)=\begin{cases} 3n \; for \; i=3 \\ 2n, 2\leq i\leq \frac{n+2}{2} \end{cases}$$

Thus the wiener polynomial in this case becomes

$$\sum_{\{a_i^{/},v_j^{/}\}\in V(B1B2)} q^{(a_i^{/},v_j^{/})} = nq^3 + 2\left(nq^2 + nq^3 + \ldots \qquad + nq^{\frac{n+1}{2}} \right) = \; nq^3 + 2n\sum_{i=2}^{\frac{n+2}{2}} q^i$$

***Case*7**: Distance between the vertices of A1,B2 is given as

$$d_{A1B2}(v_i,a_j^{/}) = x, 1\leq x\leq \frac{n+2}{2},\; 1\leq i,j\leq n,\; j=i+x-1\;(\mathrm{mod}\, n).$$

$$= n-x+i+1(\mathrm{mod}\, n)$$

$$d_{A1B2}(G,i)=\begin{cases} n \; for \; i=1 \\ 2n, 2\leq i\leq \frac{n+2}{2} \end{cases}$$

$$\sum_{\{v_i,a_j^{'}\}\in V(A1B2)} q^{(v_i,a_j^{'})} = nq^1 + 2\left(nq^2 + nq^3 + \ldots \qquad + nq^{\frac{n+2}{2}} \right) = nq^1 + \; 2n\sum_{i=2}^{\frac{n+2}{2}} q^i$$

***Case*8**: Distancebetween the vertices of A1,B1

$$d(v_i, v_j^{/}) = \begin{cases} 2 \ for \ i = j \\ x, 1 \le x \le \frac{n}{2}, \ 1 \le i \le n, \ j = i + x \ (\mathrm{mod}\, n) \\ \qquad\qquad = n - x + i (\mathrm{mod}\, n) \end{cases}$$

$$d_{A1B1}(G,i) = \begin{cases} 3n \ for \ i = 2 \\ 2n, 1 \le i \le \frac{n}{2}, i \ne 2 \\ n, i = \frac{n}{2} \end{cases}$$

The corresponding wiener polynomial is given as

$$\sum_{\{v_i, v_j^{/}\} \in V(A1B1)} q^{(v_i, v_j^{/})} = nq^2 + 2\left(nq + nq^2 + nq^3 + \ldots \qquad + nq^{\frac{n}{2}}\right) = nq^2 + \ 2n\sum_{i=1}^{\frac{n-2}{2}} q^i + nq^{\frac{n}{2}}$$

case9: Distance function between thevertices of A2,B1 is same as that of A1,B2. Thus the polynomial is

$$\sum_{\{v_i, a_j^{'}\} \in V(A2B1)} q^{(v_i, a_j^{'})} = nq^1 + 2\left(nq^2 + nq^3 + \ldots \qquad + nq^{\frac{n+2}{2}}\right) = nq^1 + \ 2n\sum_{i=2}^{\frac{n+2}{2}} q^i$$

Sub 10: Distance between the vertices of A2,B2 is given as

$$d(a_i, a_j^{/}) = \begin{cases} x, \ 2 \le x \le \frac{n+4}{2}, \ 1 \le i, j \le n, \ j = i + x - 2 \ (\mathrm{mod}\, n) \\ x, 3 \le x \le \frac{n+1}{2} \ , \ 1 \le i, j \le n, j = n - x + i + 2 (\mathrm{mod}\, n) \end{cases}$$

$$d_{A2B2}(G,i) = \begin{cases} n \ for \ i = 2 \\ 2n, 3 \le i \le \frac{n+2}{2} \\ \frac{n}{2}, i = \frac{n}{2} \end{cases}$$

$$\sum_{\{a_i, a_j^{/}\} \in V(A2B2)} q^{(a_i, a_j^{/})} = nq^2 + 2\left(nq^3 + nq4\ldots \qquad + nq^{\frac{n+4}{2}}\right) = nq^2 + \ 2n\sum_{i=3}^{\frac{n+2}{2}} q^i + nq^{\frac{n+4}{2}}$$

By adding all the polynomials ,we get the wiener polynomial of splitting graph of sun let graph for odd n is

$$W(S(C_n \otimes K_2)) = 4W(C_n \otimes K_2) + 2n\left(-q + q^2 + q^3\right), \ n \ge 5$$

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