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Generalized Fixed Point Theorems For Four Weakly Compatible Self-Maps With The CLR Property In S_b-Metric Spaces

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Abstract:
In this article, we establish certain fixed point theorems for four pairwise self-maps that are weakly compatible in an S_b-metric space. We combine a contractive condition together with weak compatibility of pairs of mappings. Our findings expand upon and generalise some of the previous findings in the literature.

Keywords: *Coincidence Point; Common Fixed Point; S_b-metric Space; Weak Compatibility*

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I. Introduction And Preliminaries

Brouwer was the first to develop the fixed point theorem in an n-dimensional Euclidean space. The existence theorems in the theory of differential equations were then established in 1922 by Birkhoff and Kellogg using Brouwer's fixed point theorem. Schaiider extended Brouwer's fixed point theorem to the case where E is a compact convex subset of a normed space. The famous fixed point theorem for contraction maps was later established by Banach. The proof of this theorem is straightforward and doesn't require much of topological knowledge. After that, several researchers extended fixed point results to b - , S – and S_b – metric spaces.

The idea of S_b-metric space is due to Nizar Souayah et al **[1]** and S. Radenovic et al **[2].** It is introduced by combining S-metric space **[5]** and b-metric space **[3,4].** After that several authors proved various fixed point theorems in S_b-metric spaces **[10-14].**
Now we give the definition and some topological properties of S_b-metric Space.

Definition 1.1. **[2]** Let X be a non-empty set and $b \geq 1$ be a real number. An S_b-metric on X is a function S: $X^3 \to [0,\infty)$ that satisfy the following conditions, for each $u, v, w, a \in X$,
(S_b1) $S(u,v,w) = 0$ if and only if $u = v = w$,
(S_b2) $S(u,v,w) \leq b[S(u,u,a) + S(v,v,a) + S(w,w,a)].$
In this case, the pair *(X, S)* is called an S_b-metric space.
Since every S-metric is an S_b-metric with $b = 1$, it is clear that S_b-metric spaces are the generalisations of S-metric spaces.

Example 1.2. **[2]** Let $X = R$ and let the function $S_b : X^3 \to [0,\infty)$ be defined as $S_b(u,v,w) = \{|u-w| + |v-w|\}^2$ is an S_b-metric on X with $b = 4$.

Definition 1.3. **[2]** Let *(X, S)* is an S_b-metric space. A sequence $\{u_n\}$ in X

a. is said to converge to some $u \in X$, if to each $\varepsilon > 0$ there exists a number $n_0 \in N$ such that $S(u_n,u_n,u) < \in$ whenever $n > n_0$, that is, $S(u_n,u_n,u) \to 0$ as $n \to \infty.$
Here, we write $u_n \to u$ as $n \to \infty$ or $\lim_{n\to\infty} u_n = u.$

b. is said to be a Cauchy sequence, if to each $\in > 0$ there exists a number $n_0 \in N$ such that $S(u_n,u_n,u_m) < \in$ whenever $m, n > n_0$, that is, $S(u_n,u_n,u_m) \to 0$ as $m, n \to \infty.$

The S_b-metric space X is said to be complete if every Cauchy sequence is convergent in X.

Lemma 1.4. **[2]** Let *(X,S)* is an S_b-metric space. Then for every $u, v, w \in X$,
(1) $S(u,u,v) \leq bS(v,v,u)$
(2) $S(u,u,v) \leq 2bS(u,u,w) + bS(v,v,w) \leq 2bS(u,u,w) + b^2 S(w,w,v).$

Gerald Jungck **[6]** first proposed the idea of compatibility in 1986 as a generalisation of commutative property. Later, the concept of weak compatibility of mappings was put forward by Jungck and Rhoades **[7].** Additionally, they demonstrated that two mappings that are compatible are always weakly compatible, but not the other way around. Subsequently, in 2002, Aamri and Moutawakil [**8**] established the (E.A) property, which has been widely used by the authors to establish fixed points. Later on, in 2011, the common limit range (CLR) property was introduced by Sintunavarat et al. [**9**]. By using this property, it was proved that the closedness of the range of any of the underlying mappings is not necessary for having fixed points.

Definition 1.5. Let *(X, S)* be an S_b-metric space and *F, G* be two self-maps of *X*. Then the pair *(F, G)* is

(i) said to be compatible **[6]** if $\lim_{n\to\infty} S(FGu_n, FGu_n, GFu_n) = 0$ for every sequence $\{u_n\}$ in *X* such that $\lim_{n\to\infty} Fu_n = \lim_{n\to\infty} Gu_n = t \in X$

(ii) said to be weakly compatible **[7]** if $FGu = GFu$ for every $u \in X$ such that $Fu = Gu.$

(iii) said to satisfy (E.A) property [**8**] if there exists a sequence $\{u_n\}$ in X such that $\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Qu_n = t, t \in X.$

Definiton 1.6. (Sintunavarat and Kumam, [**9**]) The pair *(F,G)* of self mappings of *X*, where *X* is an S_b-metric space is said to satisfy the common limit in the range of *G* (CLR_G)- property if there exists a sequence $\{u_n\}$ in *X* such that $\lim_{n\to\infty} Fu_n = \lim_{n\to\infty} Gu_n = Gu$, $u \in X.$

Definition 1.7. Let *X* be a non-empty set and *F, G* be two self-maps of *X*. Then a point $u \in X$ is said to be a coincidence point of *F* and *G,* if $Fu = Gu.$ We denote $C(F,G) = \{u \in X : Fu = Gu\}$

Some fixed point theorems for four pairwise weakly compatible self-maps in a complete Sb-metric space are established in this study.

II. Main Results

Theorem 2.1*: Let A,B,P and Q be self-maps of an* S_b*-metric space (X,S) with* $b > 1$ *such that* $b^k S(Au, Au, Bv) \le M_b(u,v)$ *for all* $u, v \in X$ *, where k > 1 is a constant and*

$$M_b(u,v) = \max\{S(Pu,Pu,Qv), S(Au,Au,Pu), S(Bv,Bv,Qv), \frac{S(Au,Au,Qv) + S(Pu,Pu,Bv)}{2b}\}. \tag{2.1}$$

If either

(i) The pair (A,P) satisfy CLR_P *-property and* $A(X) \subseteq Q(X)$ *; or*

(ii) The pair (B, Q) satisfy CLR_Q *- property and* $B(X) \subseteq P(X),$

Then $C(A,P) \neq \phi$ *and* $C(B,Q) \neq \phi.$

Furthermore, if the pairs (A, P) and (B,Q) are weakly compatible then the maps A, B, P and Q have a unique common fixed point.

Proof : Firstly, we consider the assumption (i).

From the (CLR_P) property of the pair *(A,P),* we can see that there must be a sequence $\{u_n\}$ in *X* such that

$$\lim_{n\to\infty} Au_n = \lim_{n\to\infty} Pu_n = Pz = q, \text{ for some } z, q \in X. \tag{2.2}$$

Since $A(X) \subseteq Q(X),$ there must be a sequence $\{v_n\}$ in *X* such that $Au_n = Qv_n$.

$$\text{Hence } \lim_{n\to\infty} Qv_n = q. \tag{2.3}$$

Now, we claim that $\lim_{n\to\infty} Bv_n = q.$

By replacing *u, v* with u_n, v_n respectively in (3.1) and using $Qv_n = Au_n$, we get

$$b^k S(Au_n, Au_n, Bv_n) \le M_b(u_n, v_n) \tag{2.4}$$

where,

$$M_b(u_n,v_n) = \max\left\{S(Pu_n,Pu_n,Qv_n), S(Au_n,Au_n,Pu_n), S(Bv_n,Bv_n,Qv_n), \frac{S(Au_n,Au_n,Qv_n) + S(Pu_n,Pu_n,Bv_n)}{2b}\right\}$$

$$= \max\left\{S(Pu_n,Pu_n,Au_n), S(Au_n,Au_n,Pu_n), S(Bv_n,Bv_n,Au_n), \frac{S(Pu_n,Pu_n,Bv_n)}{2b}\right\}$$

$$\le \max\left\{\begin{array}{l} S(Pu_n,Pu_n,Au_n), bS(Pu_n,Pu_n,Au_n), S(Bv_n,Bv_n,Au_n), \\ \frac{b[2S(Pu_n,Pu_n,Au_n) + S(Bv_n,Bv_n,Au_n)]}{2b} \end{array}\right\}$$

$$= \max\left\{ bS(Pu_n,Pu_n,Au_n), S(Bv_n,Bv_n,Au_n), S(Pu_n,Pu_n,Au_n)+\frac{1}{2}S(Bv_n,Bv_n,Au_n)\right\}.$$

On taking limit superior in (2.4), we get

$$\limsup_{n\to\infty} b^k S(Au_n,Au_n,Bv_n) \le \limsup_{n\to\infty} M_b(u_n,v_n)$$

$$\le \limsup_{n\to\infty} \max\left\{ bS(Pu_n,Pu_n,Au_n), S(Bv_n,Bv_n,Au_n), S(Pu_n,Pu_n,Au_n)+\frac{1}{2}S(Bv_n,Bv_n,Au_n)\right\}$$

$$= \lim_{n\to\infty}\sup S(Bv_n,Bv_n,Au_n) \le \lim_{n\to\infty}\sup bS(Au_n,Au_n,Bv_n), \text{since } S(Au_n,Au_n,Pu_n)\to 0 \text{ as } n\to\infty,$$

Since $b^k > b > 1,$ we get $\limsup_{n\to\infty} S(Au_n,Au_n,Bv_n)=0$ and hence $\lim_{n\to\infty} S(Au_n,Au_n,Bv_n)=0.$

Also, $S(q,q,Bv_n) \le b\left[2S(q,q,Au_n)+d(Bv_n,Au_n,Au_n)\right] \to 0$ as $n\to\infty.$

Thus, $\lim_{n\to\infty} Bv_n = q.$ (2.5)

Now, we prove that $Az = q.$

By taking $u = z, v = v_n$ in (2.1) and using $Pz = q,$ we get

$b^k S(Az,Az,Bv_n) \le M_b(z,v_n),$ (2.6) Here,

$$M_b(z,v_n) = \max\left\{ S(Pz,Pz,Qv_n), S(Az,Az,Pz), S(Bv_n,Bv_n,Qv_n), \frac{S(Az,Az,Qv_n)+S(Pz,Pz,Bv_n)}{2b}\right\}$$

$$= \max\left\{\begin{array}{l} S(q,q,Qv_n), S(Az,Az,q), b\left[2S(Bv_n,Bv_n,q)+S(Qv_n,Qv_n,q)\right], \\ \dfrac{2bS(Az,Az,q)+bS(Qv_n,Qv_n,q)+S(q,q,Bv_n)}{2b} \end{array}\right\}$$

On taking limit as $n\to\infty$ and using (2.3) and (2.5),

we have $\lim_{n\to\infty} M_b(z,v_n) = S(Az,Az,q).$

Therefore (2.6) implies, $\lim_{n\to\infty} b^k S(Az,Az,Bv_n) \le \lim_{n\to\infty} M_b(z,v_n) = S(Az,Az,q).$

That is, $b^k S(Az,Az,q) \le S(Az,Az,q),$ by (2.5).

Hence $S(Az,Az,q)=0,$ because $b^k > b > 1,$ Then $Az = q.$ (2.7)

From (2.2) and (2.7), $q = Az = Pz$ and hence $C(A,P)\ne\phi.$

As $A(X)\subseteq Q(X)$, we have $q = Az = Qw$ for some point $w\in X.$ (2.8)

We claim that $q = Bw.$

By taking $u = z, v = w$ in (2.1), we have $b^k S(Az,Az,Bw) \le M_b(z,w)$ and

$$M_s(z,w) = \max\left\{ S(Pz,Pz,Qw), S(Az,Az,Pz), S(Bw,Bw,Qw), \frac{S(Az,Az,Qw)+S(Pz,Pz,Bw)}{2b}\right\}$$

$$= \max\left\{ S(Bw,Bw,q), \frac{S(q,q,Bw)}{2b}\right\}$$, by using (2.2), (2.7) and (2.8)

$$= \max\left\{ S(Bw,Bw,q), \frac{bS(Bw,Bw,q)}{2b}\right\}$$

$$= S(Bz,Bw,q).$$

Then $b^k S(q,q,Bw) \le bS(q,q,Bw).$

It follows that $S(q,q,Bw)=0,$ because $b^k > b > 1.$ Hence $q = Bw.$ (2.9)

From (2.8) and (2.9), we have $q = Bw = Qw$ and therefore $C(B,Q)\ne\phi.$

Therefore, $Az = Pz = Bw = Qw = q.$ (2.10)

From the weak compatible property of the pairs *(A,P)* and *(B,Q),* it follows that

$Aq = Pq$ and $Bq = Qq$

Now, we will show that $Aq = q.$ (2.11)

From (2.1), we have $b^k S(Aq,Aq,q) = b^k S(Aq,Aq,Bw) \le M_b(q,w),$

where $M_b(q,w) = \max\left\{ S(Pq,Pq,Qw), S(Aq,Aq,Pq), S(Bw,Bw,Qw), \dfrac{S(Aq,Aq,Qw)+S(Pq,Pq,Bw)}{2b}\right\}$

$$= \max\left\{S(Aq,Aq,q), S(Aq,Aq,Aq), S(q,q,q), \frac{S(Aq,Aq,q)+S(Aq,Aq,q)}{2b}\right\}, \text{ by (2.10), (2.11)}$$

$$= S(Aq,Aq,q).$$

Hence $b^k S(Aq,Aq,q) \le S(Aq,Aq,q)$, which follows that $q = Aq = Pq$ because $b^k > b > 1$.

Similarly we can prove that $Bq = q$.

Similarly, from (2.1), we have $b^k S(q,q,Bq) = b^k S(Aq,Aq,Bq) \le M_b(q,q)$,

where $M_b(q,q) = \max\left\{S(Pq,Pq,Qq), S(Aq,Aq,Pq), S(Bq,Bq,Qq), \frac{S(Aq,Aq,Qq)+S(Pq,Pq,Bq)}{2b}\right\}$

$$= \max\left\{S(q,q,Bq), S(q,q,q), S(Bq,Bq,Bq), \frac{S(q,q,Bq)+S(q,q,Bq)}{2b}\right\},$$

$$= S(q,q,Bq),$$

Hence $b^k S(q,q,Bq) \le S(q,q,Bq)$, which follows that $q = Bq = Qq$ because $b^k > b > 1$.

Therefore $Aq = Pq = q = Bq = Qq$.

To prove uniqueness of *q*, if possible suppose that q^* $(q \ne q^*)$ be another common fixed of *A,B,P* and *Q* . Then $Aq^* = Pq^* = q^* = Bq^* = Qq^*$.

From (2.1), $b^k S(q,q,q^*) = b^k S(Aq,Aq,Bq^*) \le M_b(q,q^*)$

and $M_b(q,q^*) = \max\left\{S(Pq,Pq,Qq^*), S(Aq,Aq,Pq), S(Bq^*,Bq^*,Qq^*), \frac{S(Aq,Aq,Qq^*)+S(Pq,Pq,Bq^*)}{2b}\right\}$

$$= \max\left\{S(q,q,q^*), S(q,q,q), S(q^*,q^*,q^*), \frac{S(q,q,q^*)+S(q,q,q^*)}{2b}\right\}$$

$= S(q,q,q^*)$ since $b > 1$.

Hence, $b^k S(q,q,q^*) \le S(q,q,q^*)$,

Since $b^k > b > 1$, this is a contradiction to our supposition $q \ne q^*$.

Therefore $q = q^*$,

Similarly the proof follows under assumption (ii). □

Corollory 2.2. *Let A and P be two self maps of an* S_b-metric space *(X, S) with* $b > 1$ *such that* $b^k S(Au,Au,Av) \le M_b(u,v)$ *for all* $u, v \in X$, *where* $k > 1$ *is a constant and*

$$M_b(u,v) = \max\left\{S(Pu,Pu,Pv), S(Au,Au,Pu), S(Av,Av,Pv), \frac{S(Au,Au,Pv)+S(Pu,Pu,Av)}{2b}\right\}.$$

If the pair (A, P) satisfy $CLR_P -$ *property and* $A(X) \subseteq P(X)$, *then* $C(A,P) \ne \phi$.

Furthermore, if the pair (A, P) is weakly compatible, then the maps A and P have a unique common fixed point.

Proof: The proof follows by taking *B* = *A* and *Q* = *P* in Theorem 2.1.

III. Conclusion

Common fixed point theorems for four weakly compatible self-maps in an S_b-metric space are established in this study under suitable contractive conditions. By employing the Common Limit Range property, which provides a more general framework than traditional assumptions, the existence and uniqueness of a common fixed point are demonstrated. Corresponding results are also presented, along with corollaries in which the conclusions are restricted to two self-maps. These findings contribute to the ongoing development of Fixed Point Theory in generalized metric settings by extending several well-known results from classical metric and related spaces to the broader context of S_b-metric Space.

References

[1]. Souayah, Nizar, And Nabil Mlaiki. "A Fixed Point Theorem In Sb-Metric Spaces." J. Math. Computer Sci 16.315 (2016): 131-139.

[2]. Radenovic, S., Et Al. "Common Fixed Point Of Four Maps In $ S_B $-Metric Spaces." Journal Of Linear And Topological Algebra 5.02 (2016): 93-104.

[3]. Bakhtin, IA0748. "The Contraction Mapping Principle In Quasimetric Spaces." Functionalanalysis 30 (1989): 26-37.

[4]. Czerwik, Stefan. "Contraction Mappings In B-Metric Spaces." Acta Mathematica Et Informatica Universitatis Ostraviensis 1.1 (1993): 5-11.

[5]. S. Sedghi, N. Shobe And A. Aliouche, "A Generalization Of Fixed Point Theorems In Smetric Spaces", Matemati˘cki Vesnik

64(249) (2012), 258 – 266.

[6]. Jungck, Gerald. "Compatible Mappings And Common Fixed Points." International Journal Of Mathematics And Mathematical Sciences 9.4 (1986) : 771-779.

[7]. Jungck, Gerald, And Billy E. Rhoades. "Fixed Points For Set Valued Functions Without Continuity." Indian Journal Of Pure And Applied Mathematics 29 (1998): 227-238.

[8]. Aamri, Mohamed, And D. El Moutawakil. "Some New Common Fixed Point Theorems Under Strict Contractive Conditions." Journal Of Mathematical Analysis And Applications 270.1 (2002): 181-188.

[9]. Sintunavarat, Wutiphol, And Poom Kumam. "Common Fixed Point Theorems For A Pair Of Weakly Compatible Mappings In Fuzzy Metric Spaces." Journal Of Applied Mathematics 2011 (2011). 1-14

[10]. Sedghi, Shaban, Abdolsattar Gholidahneh, And K. P. R. Rao. "Common Fixed Point Of Two R-Weakly Commuting Mappings In Sb-Metric Spaces." Math. Sci. Lett 6.3 (2017): 249-253.

[11]. Rohen, Yumnam, Tatjana Došenović, And Stojan Radenović. "A Note On The Paper" A Fixed Point Theorems In Sb-Metric Spaces"." Filomat 31.11 (2017): 3335-3346.

[12]. Stephen, Thounaojam, Et Al. "Fixed Points Of Generalized A-Meir-Keeler Contraction Mappings In Sb-Metric Spaces." J. Funct. Spaces 4684290 (2021).

[13]. Nihal, T. A. Ş., And Nihal ÖZGÜR. "New Generalized Fixed Point Results On $ S_ {B} $-Metric Spaces." Konuralp Journal Of Mathematics 9.1 (2016): 24-32.

[14]. Priyobarta, N., Yumnam Rohen, And Nabil Mlaiki. "Complex Valued Sb-Metric Spaces." J. Math. Anal 8.3 (2017): 13-24.