IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-0661, p-ISSN: 2278-8727, Volume 22, Issue 3, Ser. 1 (May. – June. 2026), PP 01-14
www.iosrjournals.org

Development Of Global Φ-Monotone System For Quantum Stochastic Differential Equations

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Abstract
This paper develops a global ϕ-monotone system for a class of quantum stochastic differential equations (QSDE) in a locally convex space whose topology is generated by a strong family of seminorms defined on the space of observables. We study the monotone system associated with the operator appearing at the right hand side of the equivalent form of the QSDE.This is estab-lished within the framework of the Hudson-Parthasarathy formulation of quantum stochastic calculus. In addition, we establish the non-expansiveness of the associated resolvent operator and some properties of the Yosida approximations with respect to the Hausdoff topology in the locally convex space.

Date of Submission: 26-04-2026 Date of Acceptance: 06-05-2026

I. Introduction

Quantum Stochastic Differential Equations (QSDEs) provide a fundamental mathematical frame-work for describing the evolution of operator-valued stochastic processes in quantum probability and open quantum systems. The modern theory of QSDEs originates from the pioneering work of Hudson and Parthasarathy [16], who developed a theory which generalizes the Ito calculus. Following this foundational development, the theory of QSDEs has been extensively studied and expanded in several directions. Comprehensive treatments of quantum stochastic calculus and its applications were presented by Parthasarathy [19]. Further investigations of the operator-valued stochastic processes and quantum dynamical systems were carried out by Ayoola [2, 3, 4] and Ekhaguere [11, 12, 13, 14] who examined structural properties of stochastic evolutions and their associated operator dynamics in various functional analytic settings.

Other authors have broadened the scope of QSDEs to include unbounded coefficients and more general classes of quantum dynamical generators. Notably, Fagnola and Wills [15] in-vestigated stochastic generators of quantum dynamical semigroups and established structural results linking QSDEs with completely positive semigroup dynamics. Speicher [20] and other researchers also strengthened the probabilistic foundations of noncommutative stochastic pro-cesses, thereby enriching the mathematical framework underlying quantum stochastic analysis. Prior to the work of Hudson and Parthasarathy, monotone operator theory has emerged as a powerful analytical tool for the study of nonlinear evolution equations, Hilbert, Banach and finite dimensional Euclidean spaces. The work of Minty [17] introduced the concept of maximal monotone operators which was later systematically developed by Browder [10]. These ideas became central in the analysis of nonlinear differential equations, variational inequalities, and differential inclusions, and were further developed in the viability theory of Aubin and Cellina. The theory of ϕ-monotone operators associated with QSDEs in locally convex spaces remains largely underdeveloped. Some progress in this direction was made by Ekhaguere [12], who investigated hypermaximal monotone properties of QSDE within a locally convex space with a weak topology. A comparable framework under strong topological systems capable of generating global ϕ-monotone structures remains largely absent. The present study is motivated by the need to extend the theory of global ϕ-monotone systems to QSDE in a strong locally convex space. In particular, we develop a locally convex operator framework in which the coefficients of QSDEs generate mappings possessing key topological properties such as homeomorphism, openness, and closed graph structures. Furthermore, we establish conditions ensuring maximal ϕ-monotonicity and full-range (surjectivity) properties, which play a crucial role in the analysis of quantum stochastic evolution equations.

The main contribution of this work is the extension of resolvent-based monotonicity methods, originally developed within classical Hilbert space operator theory, to a locally convex operator-valued setting relevant for quantum stochastic equation equations. In particular, we show that under suitable assumptions on the QSDE coefficients, certain operators associated with right hand side of te QSDE are bijective within the topology of the locally convex spaces. These results provide new analytical tools for establishing well-posedness of quantum stochastic evolution equations within the underlying locally convex space.

The rest part of the paper is organized as follows. Section 2 is devoted to the preliminary structures and notational framework required for the analysis, including the locally convex space and the associated ϕ-system. The main results are established in Section 3, concerning the monotonicity, maximality, and resolvent properties of the operator-valued mappings associated with the QSDE.

2 Preliminaries

In this section, we introduce the functional analytic framework required for the study of the properties of nonlinear operators associated with the quantum stochastic differential equation. Throughout this work, the operator-valued evolution is defined on a locally convex space endowed with global ϕ-system.

Notation 2.1. Let D represent a complex pre-Hilbert space, and H stands for its completion. We denote by $L^+(D, H)$ the linear space consisting of all linear maps X from D to H with the property that the domain of the operator adjoint X^* contains D. We follow the notation and framework as in Ekhaguere [11], Ayoola [2], and others. We refer the reader to Ayoola [2, 3] for background materials.
Throughout this work, D represents a pre-Hilbert space with completion R, E, E_t, E^t, where $t > 0$, denote the linear spaces generated by the exponential vectors in the Fock spaces Γ(Lγ2(R+)), Γ(Lγ2([0,t))), Γ(Lγ2([t,∞))), respectively. The linear space of operator observables is denoted by

$$B \equiv L^+(DE, \otimes\Gamma(L_\gamma^2(R_+))) \qquad (2.1)$$

where denotes algebraic tensor product and I_t (resp. I^t) denotes the identity map on

$\otimes\Gamma(L_\gamma^2([0, t)))$ (resp. $\otimes\Gamma(L\gamma2([t,\infty)))$), $t > 0$.

Notation 2.2. We endow on the space B the topology (τ_s) generated by the family

($\|\cdot\|_\xi$, $\xi \in DE$) of seminorms defined by $\|x\|_\xi = \|x\xi\|$, $x \in B$, where

$x: DE \quad \otimes\Gamma(L_\gamma^2(R_+))$.

We denote by the completion of the locally convex space B in the topology τ_s.

In this work, we study properties of an operator associated with quantum stochastic differential equations of the form

$$dx(t) = E(t, x(t))\, d\Lambda_\pi + F(t, x(t))\, dA_f(t) + G(t, x(t))\, dA_g^+ + H(t, x(t))\, dt$$
$$x(t_0) = x_0, \text{ almost all } t \in I. \qquad (2.2)$$

understood in the integral form
$x(t) = x_0 + .$ (2.3)

In the work of Ekhaguere (2007), the equivalent form of (2.2) has been shown to be of the form

$x(t)\xi = P(t, x(t))\xi$

$x(t_0) = x_0$, almost all $t \in [t_0, T]$, (2.4)
where
$P(t, x(t))\xi = []\xi.$

(2.5)

Definition 2.3 (Adapted Process)

An operator-valued process X(t) is said to be adapted if for each $t \geq 0$,

$$X(t) \in L^{+}(DE, \otimes\Gamma(L\gamma 2(R+)))\otimes I^{t}.$$

Definition 2.4 (Absolute Continuity in)
A process $x \in$ is called strongly absolutely continuous if for every $\xi \in DE$, the map t $X(t)\xi$, $t \in I$, is absolutely continuous.

2.1 Monotone and Maximal Monotone Operators
Definition 2.5 (Monotone Operator)

Let $P(t, \cdot)$: . The operator $P(t, \cdot)$ is said to be monotone, if for all x, y ∈

$$| \quad (x - y)\xi, (P(t, x) - P(t, y))\xi \quad | \geq 0,$$

for every $\xi \in D\otimes E$.

Definition 2.6 (Maximal Monotone Operator)

An operator $P(t, \cdot)$ is maximal monotone if it is monotone and

$$Rng(I + \lambda P(t, \cdot)) = , \qquad \text{for every } \lambda > 0.$$

Definition 2.7 (Resolvent Operator)

For $\lambda > 0$, the resolvent of $P(t, \cdot)$ is defined to be the map

$$J_{\lambda}(t,\cdot) := (I + \lambda P(t,\cdot))^{-1} \qquad (2.6)$$

whenever the inverse exists.

Definition 2.8 (Yosida Approximation)

The Yosida approximation of $P(t, \cdot)$ is defined to be the map:

$$P_{\lambda}(t, x) := (x - J_{\lambda}(t, x)). \qquad (2.7)$$

2.2 Lipschitz and Contractive Maps

Definition 2.9 (Locally Lipschitz Maps). A map F: will be called locally Lipschitz if for every bounded set B ⊂ and each seminorm $\|\cdot\|_\xi$, there exists $L_\xi(B) > 0$ such that

$$\|F(x) - F(y)\|_\xi \le L_\xi(B)\|x - y\|_\xi,\ x, y \in$$

Definition 2.10 (Non-expansive/Contractive Map)
The map F will be called non-expansive if

$$\|F(x) - F(y)\|_\xi \le \|x - y\|_\xi.$$

It is contractive if there exists $0 < K < 1$ such that

$$\|F(x) - F(y)\|_\xi \le K\|x - y\|_\xi.$$

2.3 Closed Operators
Definition 2.11 (Closed Operators)

An operator $P(t, \cdot)$ is closed whenever

$$x_n \to x,\ P(t, x_n) \to u, \text{ then } u = P(t, x).$$

2.4 Some Fundamental Results

We shall need the following results in what follows.

Theorem 2.12. Minty Surjectivity Theorem [Brezis][10]

Let $P(t, \cdot)$ be a maximal monotone operator on a locally convex space. Then for every $\lambda > 0$,

$$\mathrm{Rng}(I + \lambda P(t, \cdot)) = .$$

Proposition 2.13. Resolvent is Non-expansive [Bauschke][5]

If P is maximal monotone, then

$$\|J_\lambda x - J_\lambda y\|_\xi \le \|x - y\|_\xi,$$

where J_λ is given by equation (2.6).

Proposition 2.14. Properties of Yosida Approximation [Bauschke] [5]
Suppose that the map $P(t, \cdot)$ is maximal monotone. Then the Yosida approximation satisfies the following:

1. P_λ is single-valued.
2. P_λ is Lipschitz continuous.
3. $P_\lambda(t, x) \to P(t, x)$ as $\lambda \to 0$.

2.5 Resolvent–Yosida Regularisation [Nwigbo][18]
We approximate the operator $P(t, \cdot)$ by the single-valued Yosida approximation given by

$$P_\lambda(t, \cdot) = (I - (I + \lambda P(t, \cdot))^{-1}), \qquad \lambda > 0.$$

Each operator $P_\lambda(t, \cdot)$ is single-valued and Lipschitz continuous. It serves as a regularized representative of $P(t, \cdot)$ in the locally convex space.

3 Main Results

In this section, we assemble some topological properties of the operator associated with the quantum stochastic differential equations (QSDE) introduced in Section 2. We investigate conditions under which the coefficient operator induces a maximal monotone map on the locally convex space B~\widetilde{B}B. Such properties play an important role in the analysis of the associated stochastic equation and in the construction of Yosida approximations and resolvent operators. Throughout this section, we consider the QSDE introduced earlier and its equivalent form.

Definition 3.1. Global ϕ-System. Let be given by equation (2.1). Then we shall call the pair (ϕ,) a global ϕ-system associated with the locally convex space if the following conditions are satisfied:

For each $\xi \in DE$, there exist maps

$$\phi: \times$$

and

$$\phi(x, y) = \psi(x - y)$$

such that

$$\|\psi(x)\|_\xi \leq \|x\|_\xi$$

and

$$| \quad \psi(x)\xi, x\xi \quad | \geq C_{0,\xi}\|x\|_\xi^2.$$

For $t > 0$, $z \in$, $C_{0,\xi} > 0$, and a positive constant $C_{t,\xi} > 0$, we have

$$\psi(tz) = C_{t,\xi}(z)\psi(z).$$

Lemma 3.2. Given with a global ϕ-system. Suppose

Lemma 3.3

Let

$$F:$$

be hemicontinuous, strongly ϕ-monotone and coercive, satisfying

$| \quad \phi(x, y)\xi, (F(x) - F(y))\xi \quad | \geq C_\xi\|x - y\|_\xi^2, C_\xi > 0.$

Then

$$\text{Rng}(F) = .$$

Proof

Let $z \in$ be arbitrary and define

$$G(x) = F(x) - z.$$

Then from the hypotheses,

$$| \quad \phi(x, y)\xi, (G(x) - G(y))\xi \quad | \geq C_\xi\|x-y\|_\xi^2.$$

From coercivity of G,

$$\to \infty \text{ as } \|x\|_\xi \to \infty.$$

Then G is hemicontinuous, strongly ϕ-monotone, and coercive.

By the locally convex Minty–Browder theorem (Brezis [8]), there exists $x \in$ such that

$$G(x) = 0.$$

Hence,

$$F(x) = z.$$

Because $z \in$ was arbitrary, it follows that

2.2 Lipschitz and Contractive Maps

Definition 2.9. (Locally Lipschitz Maps) A map $F : \tilde{B} \longrightarrow \tilde{B}$ will be called locally Lipschitz if for every bounded set $B \subset \tilde{B}$ and each seminorm $||\cdot||_\xi$ there exists $L_\xi(B) > 0$ such that

$$||F(x) - F(y)||_\xi \leq L_\xi(B)||x - y||_\xi \quad x, y \in \tilde{B}.$$

Definition 2.10. (Non-expansive/Contractive Map)
The map F will be called non-expansive if

$$||F(x) - F(y)||_\xi \leq ||x - y||_\xi$$

It is contractive if there exists $0 < K < 1$ such that

$$||F(x) - F(y)||_\xi \leq K||x - y||_\xi$$

2.3 Closed Operators

Definition 2.11. (Closed operators)
An operator $P(t, \cdot)$ is closed whenever

$$x_n \longrightarrow x, P(t, x_n) \longrightarrow u, \quad \text{then } u = P(t, x).$$

2.4 Some Fundamental Results

We shall need the following results in what follows:

Theorem 2.12. Minty Surjectivity Theorem[Brezis][10]
Let $P(t, \cdot)$ be a maximal monotone operator on a locally convex space. Then for every $\lambda > 0$,

$$Rng(I + \lambda P(t, \cdot)) = \tilde{B}.$$

Proposition 2.13. Resolvent is Non-expansive[Bauschke][5]
If P is maximal monotone, then

$$||J_\lambda x - J_\lambda y||_\xi \leq ||x - y||_\xi$$

where J_λ is given by equation (2.6).

Proposition 2.14. Properties of Yosida Approximation[Bauschke][5]
Suppose that the map $P(t, \cdot)$ is maximal monotone, then the Yosida approximation satisfies the following:

1. P_λ is single-valued
2. P_λ is Lipschitz continuous
3. $P_\lambda(t, x) \longrightarrow P(t, x)$ as $\lambda \longrightarrow 0$.

2.5 Resolvent-Yosida Regularisation[Nwigbo][18]

. We approximate the operator $P(t, \cdot)$ by single-valued Yosida approximation given by

$$P_\lambda(t, \cdot) = \frac{1}{\lambda}(I - (I + \lambda P(t, \cdot))^{-1}), \qquad \lambda > 0$$

Each operator $P_\lambda(t, \cdot)$ is single-valued and Lipschitz continuous. It serves as a regularized representative of $P(t, \cdot)$ in the locally convex space.

(H3) Coercivity

For every $\xi \in DE$, we have

$$\to \infty \text{ as } \|x\|_\xi \to \infty.$$

Then for each $t \geq 0$ and $\lambda > 0$, the following hold:

(a). The operator $I + \lambda P(t, \cdot)$ is surjective, that is,

$$Rng(I + \lambda P(t, \cdot)) = .$$

(b). The operator $I + \lambda P(t, \cdot)$ is injective.
(c). The operator $I + \lambda P(t, \cdot)$: $\to$ is bijective.
(d). The operator $P(t, \cdot)$ is maximal ϕ-monotone on .

(e). The resolvent $J_\lambda(t) = (I + \lambda P(t))^{-1}$ satisfies

$$\|J_\lambda(t)x - J_\lambda(t)y\|_\xi \leq \|x - y\|_\xi,$$

for all $x, y \in$.

(f). The Yosida approximation $P_\lambda(t)x = (x - J_\lambda(t)x)$ is single-valued and satisfies

$$\|P_\lambda(t)x - P_\lambda(t)y\|_\xi \leq \|x - y\|_\xi,$$

for all $x, y \in$.

Hence, $P_\lambda(t, \cdot)$ is globally Lipschitz continuous.

Proof. Given $t \in I$ and $\lambda > 0$, define the auxiliary operator

$$F_\lambda:$$

by

$$F_\lambda(x) = x + \lambda P(t, x).$$

We prove the result in the following order.

(A) Injectivity of F_λ

Let $x, y \in B$ and suppose that

$$F_\lambda(x) = F_\lambda(y).$$

Then

$$x + \lambda P(t, x) = y + \lambda P(t, y),$$

so that

$$x - y + \lambda(P(t, x) - P(t, y)) = 0.$$

Applying $\phi(x, y)$, we obtain

and by assumption (H2),

$$| \quad \phi(x, y)\xi, (P(t, x) - P(t, y))\xi \quad | \geq m_\xi \|x - y\|_\xi^2.$$

Hence,
$0 \geq (C_{0,\xi} + \lambda m_\xi)\|x - y\|_\xi^2$.

Since $C_{0,\xi} + \lambda m_\xi > 0$, it follows that

$$\|x - y\|_\xi = 0.$$

Therefore,

$$x = y.$$

Lemma 3.3. Let

$$F : \tilde{B} \longrightarrow \tilde{B}$$

be hemicontinuous, strongly ϕ-monotone and coercive.

$$|\langle \phi(x, y)\xi, (F(x) - F(y))\xi \rangle| \geq C_\xi ||x - y||_\xi^2, \quad C_\xi > 0$$

Then,

$$Rng(F) = \tilde{B}.$$

Proof. Let $z \in \tilde{B}$ be arbitrary and define

$$G(x) = F(x) - z$$

Then from the hypotheses,

$$|\langle \phi(x, y)\xi, (G(x) - G(y))\xi \rangle| \geq C_\xi ||x - y||_\xi^2.$$

From coercivity of G,

$$\frac{|\langle \psi(x)\xi, G(x)\xi \rangle|}{||x||_\xi} \longrightarrow \infty \qquad \text{as } ||x||_\xi \longrightarrow \infty.$$

Then G is hemicontinuous, strong ϕ-monotone, and coercive. By the locally convex Minty-Browder theorem (Brezis [8]), there exists $x \in \tilde{B}$ such that

$$G(x) = 0.$$

Hence,

$$F(x) = z.$$

Because $z = \tilde{B}$ was arbitrary, it follows that

$$Rng(F) = \tilde{B}$$

□

Theorem 3.4. Let $\tilde{B}$ be locally convex space defined in equation (2.1) endowed with the global ϕ-system.
For each $t \in [0, T]$, consider the operator

$$P(t, \cdot) : \tilde{B} \longrightarrow \tilde{B}, \quad t \geq 0.$$

Assume that for each fixed $t \in [0, T]$ the following conditions hold:

(H_1) Hemicontinuity.
For every $x, y \in \tilde{B}$ the mapping

$$s \longrightarrow P(t, x + sy)$$

is continuous on [0, 1].

(H_2) Strong ϕ-monotonicity.
There exists a constant $m_\xi > 0$ for semi-norm $|| \cdot ||_\xi$ such that

$$|\langle \phi(x, y)\xi, (P(t, x) - P(t, y))\xi \rangle| \geq m_\xi ||x - y||_\xi^2,$$

$\forall x, y \in \tilde{B}, \xi \in \mathbb{D} \otimes \mathbb{E}.$

(E) Surjectivity

Since F_λ is hemicontinuous, strongly ϕ-monotone and coercive, the locally convex Minty–Browder theorem implies that

$\bar{D}$

$$Rng(F_\lambda) = .$$

Therefore,

$$Rng(I + \lambda P(t, \cdot)) = .$$

Thus, F_λ is surjective.

(H_3) Coercivity.
For every $\xi \in \mathbb{D}\otimes\mathbb{E}$

$$\frac{|\langle \psi(x)\xi, P(t,x)\xi\rangle|}{||x||_\xi} \longrightarrow \infty \quad \text{as } ||x||_\xi \longrightarrow \infty$$

Then for each $t \geq 0$ and $\lambda > 0$ the following holds:

(a). The operator $I + \lambda P(t,\cdot)$ is surjective, that is

$$Rng(I + \lambda P(t,\cdot)) = \tilde{\mathcal{B}}$$

(b). The operator $I + \lambda P(t,\cdot) = \tilde{\mathcal{B}}$ is injective.

(c). The operator $I + \lambda P(t,\cdot) : \tilde{\mathcal{B}} \longrightarrow \tilde{\mathcal{B}}$ is bijective.

(d). The operator $P(t,\cdot)$ is maximal ϕ-monotone on $\tilde{\mathcal{B}}$ for each $t \geq 0$.

(e). The resolvent $J_\lambda(t) = (I + \lambda P(t))^{-1}$ satisfies

$$||J_\lambda(t)x - J_\lambda(t)y||_\xi \leq ||x - y||_\xi$$

for all $x, y \in \tilde{\mathcal{B}}$.

(f). The Yosida approximation is well defined and non-expansive and the Yosida approximation $P_\lambda(t)x = \frac{1}{\lambda}(x - J_\lambda(t)x)$ is single-valued and satisfies

$$||P_\lambda(t)x - P_\lambda(t)y||_\xi \leq \frac{1}{\lambda}||x - y||_\xi$$

for all $x, y \in \tilde{\mathcal{B}}$.
Hence, $P_\lambda(t)$ is globally Lipschitz continuous.

Proof. Given $t \in I$ and $\lambda > 0$.
Define the auxiliary operator

$$F_\lambda : \tilde{\mathcal{B}} \longrightarrow \tilde{\mathcal{B}}$$

by

$$F_\lambda(x) = x + \lambda P(t,x).$$

We prove the result in the following order;

(A). Injectivity of F_λ
Let $x, y \in \tilde{\mathcal{B}}$ and suppose that

$$F_\lambda(x) = F_\lambda(y).$$

Then

$$x + \lambda P(t,x) = y + \lambda P(t,y)$$

so that

$$x - y + \lambda(P(t,x) - P(t,y)) = 0.$$

Applying $\phi(x,y)$, we have

$$|\langle \phi(x,y)\xi, (x-y)\xi\rangle| + \lambda|\langle \phi(x,y)\xi, (P(t,x) - P(t,y))\xi\rangle| = 0.$$

By global ϕ-system, we have

$$|\langle \phi(x,y)\xi, (x-y)\xi\rangle| \geq C_{0,\xi}||x - y||^2_\xi,$$

Corollary 3.5. Let $P(t, \cdot)$:   satisfy the hypotheses of Theorem 3.4. Then for every $\lambda > 0$, the resolvent

$$J_\lambda(t, \cdot) = (I + \lambda P(t, \cdot))^{-1}$$

satisfies the following assertions.

(a). The resolvent $J_\lambda(t, \cdot)$ is well defined on .

(b). The resolvent $J_\lambda(t, \cdot)$ is non-expansive:

$$\|J_\lambda(t)x - J_\lambda(t)y\|_\xi \le \|x - y\|_\xi,$$

for all $x, y \in$, $\xi \in DE$.

(c). The Yosida approximation

$P_\lambda(t)x = (x - J_\lambda(t)x)$, $x \in$ is single-valued and locally Lipschitz continuous.

Proof. (a) Since $P(t, \cdot)$ is maximal monotone on , we obtain from Theorem 3.4 that

$$\mathrm{Rng}(I + \lambda P(t, \cdot)) = \ , \lambda > 0.$$

Hence, for every $x \in$, there exists $u \in$ such that

$$x = u + \lambda P(t, u).$$

This implies that

$$x = (I + \lambda P(t, \cdot))u,$$

and therefore

$$u = (I + \lambda P(t, \cdot))^{-1}x.$$

That is,

$$u = J_\lambda(t, \cdot)x.$$

Therefore, the inverse mapping

$$J_\lambda(t, \cdot) = (I + \lambda P(t, \cdot))^{-1}$$

is well defined on .

(b). Let $u = J_\lambda(t)x$, $v = J_\lambda(t)y$.

Then

$$x = u + \lambda P(t, u), \quad y = v + \lambda P(t, v).$$

Subtracting gives

$$x - y = (u - v) + \lambda(P(t, u) - P(t, v)).$$

Applying $\phi(u, v)$, we obtain

$$| \ \phi(u, v)\xi, (x - y)\xi \ | = | \ \phi(u, v)\xi, (u - v)\xi \ | + \lambda | \ \phi(u, v)\xi, (P(t, u) - P(t, v))\xi \ |.$$

By monotonicity,

$$| \ \phi(u, v)\xi, (P(t, u) - P(t, v))\xi \ | \ge 0.$$

Hence,

Using the seminorm compatibility of the global ϕ-system,

$$C_{0,\xi}\|u - v\|_\xi^2 \le | \ \phi(u, v)\xi, (x - y)\xi \ |.$$

By the Cauchy–Schwarz inequality,

$\overline{D}$

$$| \ \phi(u, v)\xi, (x - y)\xi \ | \le \|\phi(u, v)\|_\xi \|x - y\|_\xi.$$

Since

$$\|\phi(u, v)\|_\xi = \|u - v\|_\xi,$$

we obtain

(E). Surjectivity

Since F_λ is hemicontinuous, strongly ϕ-monotone and coercive, the locally convex Minty-Browder theorem implies that

$$Rng(F_\lambda) = \tilde{B}.$$

Therefore,

$$(I + \lambda P(t, \cdot)) = \tilde{B}.$$

Thus, F_λ is surjective.

(F). Bijectivity

From (A) and (E)

$$I + \lambda P(t, \cdot)$$

is both injective and surjective.
Hence, it is bijective.

(G). Maximal ϕ-monotonicity

From Step (E), for every $\lambda > 0$ we have

$$Rng(I + \lambda P(t, \cdot)) = \tilde{B}.$$

By the Minty characterization of maximal ϕ-monotone operators in locally convex spaces, an operator $P(t, \cdot)$ is maximal ϕ-monotone if

$$Rng(I + \lambda P(t, \cdot)) = \tilde{B}$$

for every $\lambda > 0$. Therefore,

$$P(t, \cdot)$$

is maximal ϕ-monotone on $\tilde{B}$ for each $t \in [0, T]$.

(H). Resolvent properties

Define

$$J_\lambda(t) = (I + \lambda P(t, \cdot))^{-1}.$$

Since the inverse exists by bijectivity. $J_\lambda(t)$ is well defined.
From proposition (2.13),

$$||J_\lambda(t)x - J_\lambda(t)y||_\xi \leq ||x - y||_\xi.$$

Hence, the resolvent is non-expansive.

(I). Yosida Approximation

Define

$$P_\lambda(t)x = \frac{1}{\lambda}(x - J_\lambda(t)x).$$

Then

$$P_\lambda(t)$$

is a single-valued.
Moreover,

$$||P_\lambda(t)x - P_\lambda(t)y||_\xi \leq \frac{1}{\lambda}||x - y||_\xi$$

showing that $P_\lambda(t)$ is globally Lipschitz continuous.

Theorem 3.7

Let P: [0,T] × $\overline{D}$ be maximal monotone and given by equation (2.5).

Then, for every $\lambda > 0$ and each $t \in [0, T]$, the following conditions are satisfied.

(a).The map P(t, ·) is closed.

Corollary 3.5. Let $P(t,\cdot): \tilde{B} \longrightarrow \tilde{B}$ satify the hypotheses of Theorem (3.3). Then for every $\lambda > 0$. and the resolvent

$$J_\lambda(t,\cdot) = (I + \lambda P(t,\cdot))^{-1},$$

the following assertions hold:

(a). The resolvent $J_\lambda(t,\cdot)$ is well defined on $\tilde{B}$.

(b). The resolvent $J_\lambda(t,\cdot)$ is non-expansive;

$$||J_\lambda(t)x - J_\lambda(t)y||_\xi \leq ||x-y||_\xi, \quad x,y \in \tilde{B}, \quad \xi \in \mathbb{D}\otimes\mathbb{E}.$$

for all $x, y \in \tilde{B}$

(c). The Yosida approximation

$$P_\lambda(t)x = \frac{1}{\lambda}(x - J_\lambda(t)x), \; x \in \tilde{B}$$

is single-valued and locally Lipschitz continuous.

Proof. (a). Since $P(t,\cdot)$ is maximal monotone on $\tilde{B}$, we derived the following from Theorem (3.3).

$$Rng(I + \lambda P(t,\cdot)) = \tilde{B}, \quad \lambda > 0.$$

Hence, for every $x \in \tilde{B}$, there exists $u \in \tilde{B}$ such that

$$x = u + \lambda P(t,u),$$

this implies

$$x = (I + \lambda P(t,\cdot))u$$

and

$$u = (I + \lambda P(t,\cdot))^{-1}x$$

that is

$$u = J_\lambda(t,\cdot).$$

Therefore the inverse mapping

$$J_\lambda(t,\cdot) = (I + \lambda P(t,\cdot))^{-1},$$

is well defined on $\tilde{B}$.

(b). Let $u = J_\lambda(t)x$, $v = J_\lambda(t)y$. Then

$$x = u + \lambda P(t,u), \; y = v + \lambda P(t,v)$$

subtract

$$x - y = (u - v) + \lambda(P(t,u) - P(t,v).$$

Applying $\phi(u,v)$ gives

$$|\langle\phi(u,v)\xi, (x-y)\xi\rangle| = |\langle\phi(u,v)\xi, (u-v)\xi\rangle| + \lambda|\langle\phi(u,v)\xi, (P(t,u) - P(t,v))\xi.$$

By monotonicity, we have

$$|\langle\phi(u,v)\xi, (P(t,u) - P(t,v))\xi\rangle| \geq 0,$$

so that

$$|\langle\phi(u,v)\xi, (x-y)\xi\rangle| \geq |\langle\phi(u,v)\xi, (u-v)\xi\rangle|,$$

Now suppose

$\overline{D}$

$$x + \lambda P(t)x = y + \lambda P(t)y.$$

Then

$$x - y + \lambda(P(t)x - P(t)y) = 0.$$

Applying $\phi(x, y)$, we obtain

using the seminorm compatibility, we have

$$C_{0,\xi}||u-v||_\xi^2 \le |\langle \phi(u,v)\xi, (x-y)\xi\rangle|.$$

By Cauchy inequality

$$|\langle \phi(u,v)\xi, (x-y)\xi\rangle| \le ||\phi(u,v)||_\xi ||x-y||_\xi.$$

Since $||\phi(u,v)||_\xi = ||u-v||_\xi$, we have

$$C_{0,\xi}||u-v||_\xi^2 \le ||u-v||_\xi ||x-y||_\xi$$

Hence,

$$||u-v||_\xi \le ||x-y||_\xi,$$

that is

$$||J_\lambda(t)x - J_\lambda(t)y||_\xi \le ||x-y||_\xi$$

$$|\langle \phi(u,v)\xi, (x-y)\xi\rangle| \ge C_{0,\xi}||u-v||_\xi^2.$$

Thus $J_\lambda(t)$ is non-expansive.

(c). For $x, y \in \tilde{B}$ and the Yosida approximation $\lambda P(t)x = (x - J_\lambda(t)x)$.

$$\lambda(P_\lambda(t)x - P_\lambda(t)y) = ((x - J_\lambda(t)x) - (y - J_\lambda(t)y)) \qquad = ((x-y) - (J_\lambda(t)x - J_\lambda(t)y))$$

If, in addition, the resolvent $J_\lambda(t,\cdot)$ is firmly nonexpansive in the sense that

$$||J_\lambda(t)x - J_\lambda(t)y||_\xi^2 \le |\langle \phi(J_\lambda(t)x, J_\lambda(t)y)\xi, (x-y)\xi\langle|,$$

then

$$||(x-y) - (J_\lambda(t)x - J_\lambda(t)y)||_\xi \le ||x-y||_\xi.$$

Therefore

$$\lambda||P_\lambda(t)x - P_\lambda(t)y||_\xi,$$

which implies

$$||P_\lambda(t)x - P_\lambda(t)y||_\xi \le \frac{1}{\lambda}||x-y||_\xi.$$

Hence, $P_\lambda(t,\cdot)$ is single-valued and locally Lipschitz continuous with constant $\frac{1}{\lambda}$. □

Definition 3.6. Define the operator

$$P : I \times \tilde{B} \longrightarrow \tilde{B}$$

by

$$P(t, x(t))\xi = \frac{d}{dt}\int_{t_0}^{t} (E(s)x(s)\, d\Lambda_\pi + F(s)x(s)\, dA_f(s) + G(s)x(s)\, dA_g^+ + H(s)x(s)\, ds). \tag{3.2}$$

Suppose that each coefficient map E, F, G, H is linear. Then for all $x, y \in \tilde{B}$

$$P(t, \alpha x + \beta y) = \alpha P(t,x) + \beta P(t,y).$$

Hence the map $x \longrightarrow P(t,x)$ is linear whenever each coefficient is linear. We therefore consider operator representation

$$P(t,x) = P(t)x$$

for some

$$P : I \longrightarrow \tilde{B}^*$$

where $\tilde{B}^*$ is the space of continuous linear operator on $\tilde{B}^*$.

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