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Properties On $S^{\#}$-Neighborhoods In Topological Spaces

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Abstract

The main objective of this paper is to further investigate $S^{\#}$-closed sets and introduce the concept of $S^{\#}$-neighborhoods in topological spaces along with their properties. We also define $S^{\#}$-interior and $S^{\#}$-closure operators and study their fundamental properties. Several theorems related to $S^{\#}$-derived sets, $S^{\#}$-closure, and $S^{\#}$-interior are established. Counterexamples are discussed to clarify the limitations of these concepts. This study contributes to the development of generalized topological structures and their relationships with existing topological concepts.

Keywords: *$S^{\#}$-closed set, $S^{\#}$-derived set, $S^{\#}$-closure, $S^{\#}$-interior, $S^{\#}$-neighborhood, Topological space.*

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I. Introduction

The concept of open sets plays a fundamental role in the study of topological spaces. The idea of regular open sets was introduced by Stone [9], which represents a stronger form of open sets. To generalize certain properties of open sets, Levine [3] introduced semi-open sets as a weaker form of open sets. Njastad [4] introduced α-open sets in topological spaces.

The concept of generalized closed sets (g-closed sets) was introduced by Levine in 1970. Later, Maki et al.,[6] developed generalized α-closed sets and related generalized closed concepts in topological spaces.

Motivated by these generalizations, we introduce a weaker form of generalized closed sets called **$S^{\#}$-closed sets** using the generalized closure operator. In this paper, we define **$S^{\#}$-neighborhoods, $S^{\#}$-derived sets, $S^{\#}$-closure, and $S^{\#}$-interior operators** and study their properties.

In this paper, the symbols (X, τ) and (Y, σ) (or simply X and Y) denote topological spaces, unless otherwise specified with additional separation axioms. For any subset $D \subseteq X$, the notations cl(D) and int(D) represent the closure and interior of D in X, respectively.

II. Preliminaries

In this section we recall definitions of semi-open sets, semi-closed sets, generalized closed sets and related concepts which are used throughout the paper.

Definition 2.1: A member K of a TS X is termed as

1. α-open set [7] wherever $K \subseteq \text{int}(\text{cl}(\text{int}(K)))$ also a α-closed set[5] wherever $\text{cl}(\text{int}(\text{cl}(K))) \subseteq K$.
2. semi-open set [3] whenever $K \subseteq \text{cl}(\text{int}(K))$ along with a semi-closed set [3] whenever $\text{int}(\text{cl}(K) \subseteq K$.

The complement of semi-open (resp. pre-open, semi-preopen) set is named as semi-closed [1] set of a space X. The collection of entire semi-open sets of X is represented as SO(X) and that of semi closed sets of X is labeled as SF(X).

Definition 2.2: For a member J of X

1. the intersection of entire semi-closed subsets of X involving K is termed as semi-closure of J [1] also it is represented as sCl(J).
2. The α-closure [5] of J is the least α-closed set including J also it is labelled as αcl(J).
3. semi-interior [2] of J labelled as sInt(J) is the union of entire semi-open sets included in J in X.

Definition 2.3: A member J of a TS X is known as

1. generalized closed (in short, g-closed) [4] wherever $\text{Cl}(J) \subseteq M$ with $J \subseteq M$ as M is O(X)

ii. α-generalized-closed (in short, αg-closed) set [6] whenever αCl(J) ⊆ N with the condition J ⊆ N and N as O(X).

iii. generalized α-closed(in short, gα-closed) set [6] if α Cl(J) ⊆ K whenever J⊆ K with K as αO(X).

iv. α-generalized semi-closed (briefly αgs-closed) [8] whenever αCl(J)⊆U with the condition that J⊆ U with U as SO(X).

III. $S^{\#}$-Neighborhoods.

Definition 3.1: A subset K of a TS X is termed as $S^{\#}$-neighborhood (in short, $S^{\#}$-nbd) of a point p of X whenever there occurs a $S^{\#}$-open set U so as $p \in U \subseteq K$.

Definition 3.2: Consider a member K of TS X. A subset N of X is termed as $S^{\#}$-nbd of K whenever there occurs an $S^{\#}$-open set G so as $K \subseteq G \subseteq N$.

Theorem 3.3: Consider member D of a TS X. Thereupon D is $S^{\#}$-open iff A contains an $S^{\#}$-nbd of each of its points.

Proof: Consider D be an $S^{\#}O(X)$. Allow $x \in A$, which imparts $x \in A \subseteq A$. Thus A is $S^{\#}$-nbd of x. Hence A includes an $S^{\#}$-nbd of each of its points.

Conversely, D includes a $S^{\#}$-nbd of each of its points. Being each $x \in A$ their arises a neighborhood N_x of x so as $x \in N_x \subseteq D$. Referring to the definition of $S^{\#}$-nbd of x, there is a $S^{\#}$-open set G_x thereby $x \in G_x \subseteq N_x \subseteq D$. Now we prove that $D = \cup \{G_x: x \in D\}$. Allow $x \in D$. Accordingly there exist $S^{\#}$-open set G_x so as $x \in G_x$. Thereupon, $x \in \cup\{G_x: x \in D\}$ which implies $D \subseteq \cup\{G_x: x \in D\}$. Now consider $y \in \{G_x: x \in D\}$ so that $y \in$ some G_x for some $x \in D$ and hence $y \in D$. Hence, $\cup\{G_x: x \in D\} \subseteq D$. So $A = \cup\{G_x: x \in D\}$. Also each G_x is a $S^{\#}$-open set. So D is a $S^{\#}$-open set.

Theorem 3.4: Whenever D is a $S^{\#}$-closed subset of X and $x \in X – D$, then there exists a $S^{\#}$-nbd N of x thereby $N \cap D = \phi$.

Proof: When D is a $S^{\#}F(X)$, thereupon X – D is a $S^{\#}O(X)$. By the Theorem 2.3.3, X – D contains a $S^{\#}$-nbd of each of its points. Which implies that, there exist a $S^{\#}$-nbd N of x so as $N \subseteq X\text{-}D$. That is, no point of N belongs to D and hence $N \cap D = \phi$.

Theorem 3.5: Consider X be a TS. When D is a $S^{\#}F(X)$ along with $x \in X – D$ then there arises a $S^{\#}$-nbd N of x as long as $D \cap N \neq \phi$.

Proof: Since D is $S^{\#}F(X)$, X – D is $S^{\#}O(X)$. In view of Theorem 2.3.3, X – D contains a $S^{\#}$-nbd of each of its points. Accordingly, there arises a $S^{\#}$-nbd of N of x thereby $N \subseteq X\text{-}D$. So $N \cap D \neq \phi$.

Definition 3.6: Consider a subset J of a TS X. Accordingly a point $p \in X$ is named as $S^{\#}$-limit point of J iff each $S^{\#}$-nbd of x includes a point of J unlike from p. That is $[N – \{p\}] \cap J \neq \phi$, for every $S^{\#}$-nbd N of p. Also equivalently iff each $S^{\#}$-open set W including p comprise a point of J apart from p.
Collection of entire $S^{\#}$-limit points of J is termed as $S^{\#}$-derived set of J and is labeled as $S^{\#}$-d(J).

Theorem 3.7: Consider J and K be subsets of X and $J \subseteq K$ implies $S^{\#}\text{-d}(J) \subseteq S^{\#}\text{-d}(K)$.

Proof: Allow $i \in S^{\#}\text{-d}(J)$ implies i is a $S^{\#}$-limit point of J that is every $S^{\#}$-nbd of i comprise a point of J apart from i. As $J \subseteq K$, every $S^{\#}$-nbd of i contains a point of K other than i. Consequently, i is a $S^{\#}$-limit point of K. That is $i \in S^{\#}\text{-d}(K)$. Thereupon $S^{\#}\text{-d}(J) \subseteq S^{\#}\text{-d}(K)$.

Theorem 3.8: A subset P of X is $S^{\#}$-closed iff $S^{\#}\text{-d}(P) \subseteq P$.

Proof: Whenever P is $S^{\#}$-closed set. That is X-P is $S^{\#}$-open set. Now we prove that $S^{\#}\text{-d}(P) \subseteq A$. Make $x \in S^{\#}\text{-d}(P)$ implies x is a $S^{\#}$-limit point of P, that is each $S^{\#}$-nbd of x includes a point of P separate from x. Now imagine $x \notin P$ so that $x \in X\text{-}P$, which is $S^{\#}$-open and by definition of $S^{\#}$-open sets, there occurs an $S^{\#}$-nbd N of x thereby $N \subseteq X\text{-} P$. From this we conclude that N includes no point of P, which is a negation. Accordingly $x \in A$ and hence $S^{\#}\text{-d}(P) \subseteq P$.

Conversely assume that S $S^{\#}$-d(P) $\subseteq$ P and we will prove P is a $S^{\#}F(X)$ or X-P is $S^{\#}O(X)$. Enable x $\in$ X-P. Enable x be an arbitrary point of X-P, so that x$\notin$ P which implies that x$\notin$ $S^{\#}$-d(A). That is there exists a $S^{\#}$-nbd N of x which consists of only points of X – P. This means that X-P is $S^{\#}O(X)$. And hence P is $S^{\#}F(X)$.

Theorem 3.9: Consider X be a TS, each $S^{\#}$-derived set in X is $S^{\#}$-closed set.

Proof: Permit R be a subset of X and $S^{\#}$-d(R) is $S^{\#}$-derived set of R. Referring to Theorem 2.3.8, R is $S^{\#}$-closed iff $S^{\#}$-d(R) $\subset$ R. Hence $S^{\#}$-d(R) is $S^{\#}$-closed iff $S^{\#}$-d ($S^{\#}$-d(R)) $\subseteq$ $S^{\#}$-d(R). That is each $S^{\#}$-limit point of $S^{\#}$-d(R) belongs to $S^{\#}$-d(R).

Now allow q be an $S^{\#}$-limit point of $S^{\#}$-d(R). That is q $\in$ $S^{\#}$-d($S^{\#}$-d(R)). In case there exist an $S^{\#}$-open set W comprising x so as $\{W - \{q\}\} \cap$ $S^{\#}$-d(R) $\neq \phi$ which implies $\{W - \{q\}\} \cap R \neq \phi$, because each $S^{\#}$-nbd of an element of $S^{\#}$-d(R) has at least one point of R. Subsequently, q is a $S^{\#}$-limit z point of R. That is q belongs to $S^{\#}$-d(R). So q $\in$ $S^{\#}$-d($S^{\#}$-d(R)) implies q $\in$ $S^{\#}$-d(R). Thereupon, $S^{\#}$-d(R) is $S^{\#}$ F(X).

IV. $S^{\#}$-Interior And $S^{\#}$-Closure Operators.

Definition 4.1: Considering a member K of X, $S^{\#}$-closure of K, labeled as $S^{\#}Cl(K)$ and is defined as $S^{\#}Cl(K) = \cap\{Q: K \subseteq Q, Q \text{ is } S^{\#}F(X)\}$.

Theorem 4.2: For any p $\in$ X, p $\in$ $S^{\#}Cl(R)$ iff $R \cap V \neq \phi$ being each $S^{\#}$-open set V including p.

Proof: Make p $\in$ $S^{\#}Cl(R)$. Imagine there arises an $S^{\#}$-open set V including p so as $V \cap R = \phi$. Subsequently R $\subseteq$ X–V. As X–V is $S^{\#}$-closed, $\alpha gCl(R) \subseteq X–V$. This indicates x $\notin$ $S^{\#}Cl(R)$ which is a negation. Henceforth $V\cap R \neq \phi$ for each $S^{\#}$-open set V including p.

Conversely, assuming $R \cap V \neq \phi$ for each $S^{\#}$-open set V including p. To prove that p $\in$ $S^{\#}Cl(R)$. Speculate p $\notin$ $S^{\#}Cl(R)$. At that time there is an $S^{\#}$-closed set W including R so as p $\notin$ W. Like so p $\in$ X – W and X – W is $S^{\#}$-open. Also $(X– W) \cap R = \phi$ which is a negation. Henceforth p $\in$ $S^{\#}Cl(R)$.

Theorem 4.3: Wherever D $\subseteq$ X, thereupon D $\subseteq$ $S^{\#}Cl(D) \subseteq Cl(D)$.

Proof: As each C(X) is $S^{\#}F(X)$, the proof follows.

Theorem 4.4: Consider E and F be such sets of X.

a) $S^{\#}Cl(\phi) = \phi$

b) $S^{\#}Cl(X) = X$

c) $S^{\#}Cl(E)$ is $S^{\#}F(X)$.

d) If $E \subseteq F$, thereupon $S^{\#}Cl(E) \subseteq S^{\#}Cl(F)$.

e) $S^{\#}Cl(E \cup F) = S^{\#}Cl(E) \cup S^{\#}Cl(F)$.

f) $S^{\#}Cl[S^{\#}Cl(E)] = S^{\#}Cl(E)$.

Proof: The proof of a), b), c) along with d) pursue from the Definition 4.1

e). To prove that $S^{\#}Cl(E) \cup S^{\#}Cl(F) \subseteq S^{\#}Cl(E\cup F)$

We have $S^{\#}Cl(E) \subseteq S^{\#}Cl(E\cup F)$ and $S^{\#}Cl(F) \subseteq S^{\#}Cl(E\cup F)$.

Subsequently $S^{\#}Cl(E) \cup S^{\#}Cl(F) \subseteq S^{\#}Cl(E\cup F)$ (1).

Approve x be any point so as x $\notin$ $S^{\#}Cl(E) \cup S^{\#}Cl(F)$, then there exists $S^{\#}$-closed sets A and B thereby $E \subseteq A$ along with $F \subseteq B$, x $\notin$ A and x $\notin$ B. Thereupon x$\notin$ $A\cup B$, $E \cup F \subseteq A \cup B$ and $A \cup B$ is $S^{\#}$-closed set. Hence x $\notin$ $S^{\#}Cl(E\cup F)$. Like so, we obtain $S^{\#}Cl(E\cup F) \subseteq S^{\#}Cl(E) \cup S^{\#}Cl(F)$---(2). Henceforth from (1) and (2), $S^{\#}Cl(E \cup F) = S^{\#}Cl (E) \cup S^{\#}Cl(F)$.

f). Taking A be $S^{\#}$-closed set including E. At that time due to definition, $S^{\#}Cl(E) \subseteq A$. As A is $S^{\#}$-closed set and includes $S^{\#}Cl(E)$ and is included in each $S^{\#}$-closed set including E, it follows that $S^{\#}Cl[\, S^{\#}Cl(E)] \subseteq S^{\#}Cl(E)$. Thereupon, $S^{\#}Cl[\, S^{\#}Cl(E)] = S^{\#}Cl(E)$.

Theorem 4.5: $S^{\#}$-closure is a Kuratowski closure operator on X.

Proof: Follows from the Theorem 4.4

Remark 4.6: Whenever K is $S^{\#}$-closed iff $S^{\#}Cl(K) = K$.

Proof: Enable K be $S^{\#}F(X)$. As $K \subseteq K$ and K is $S^{\#}F(X)$, $K \in \{G: K \subseteq G, G \text{ is } S^{\#}F(X)\}$ which implies that $\cap \{G: K \subseteq G, G \text{ is } S^{\#}F(X)\} \subset A$. That is $S^{\#}Cl(A) \subset A$. Note that $K \subset S^{\#}Cl(K)$ is always true. Hence $A = S^{\#}Cl(A)$.

Conversely, guess $S^{\#}Cl(K) = K$. In view of $K \subseteq K$ and K is $S^{\#}$-closed set. Afterwards, K must be a closed set. Hence K is $S^{\#}$-closed.

Definition 4.7: For a subset K of X, $S^{\#}$-interior of A, labeled as $S^{\#}Int(K)$ and is defined as $S^{\#}Int(K) = \cup\{ W: W \subseteq K \text{ and } W \text{ is } S^{\#}O(X)\}$. That is $S^{\#}Int(K)$ is the union of all $S^{\#}$-open sets contained in K.

Theorem 4.8: Consider a subset J of X, then $S^{\#}Int(J)$ is the largest $S^{\#}$-open subset of X included in J, whenever J is $S^{\#}$-open.

Proof: Make $J \subseteq X$ be $S^{\#}$-open. Accordingly $S^{\#}Int(J) = \cup\{G: G \subseteq J \text{ and } G \text{ is } S^{\#}O(X)\}$. Since $J \subseteq J$ and J is $S^{\#}$-open, $J = S^{\#}Int(J)$ is the largest $S^{\#}$-open member of X included in J.

Theorem 4.9: Whenever $R \subseteq T$, thereupon $S^{\#}Int(R) \subseteq S^{\#}Int(T)$.

Proof: Speculate $R \subseteq T$, we know that $S^{\#}Int(R) \subseteq T$. Also we have $R \subseteq T$. which implies $S^{\#}Int(R) \subseteq T$, $S^{\#}Int(R)$ is an open set which is contained in T. Yet $S^{\#}Int(R)$ is the largest open set included in T. Accordingly, $S^{\#}Int(T)$ is larger than $S^{\#}Int(R)$. That is $S^{\#}Int(R) \subseteq S^{\#}Int(T)$.

Theorem 4.10: For any member JA, KB of X, succeeding results holds:
1) $S^{\#}Int(\phi) = \phi$
2) $S^{\#}Int(X) = X$
3) Whenever $J \subseteq K$ then $S^{\#}Int(J) \subseteq S^{\#}Int(K)$
4) $S^{\#}Int(J)$ is the largest $S^{\#}$-open set included in J
5) $S^{\#}Int(J \cap K) = S^{\#}Int(J) \cap S^{\#}Int(K)$
6) $S^{\#}Int(J \cup K) \supseteq S^{\#}Int(J) \cup S^{\#}Int(K)$
7) $S^{\#}Int[S^{\#}Int(J)] = S^{\#}Int(J)$

Proof: Proof follows from the Definition 4.7

Theorem 4.11: Whenever K is $S^{\#}$-open if and only if $S^{\#}Int(K) = K$.

Remark 4.12: For any member K of X, $Int(K) \subseteq S^{\#}Int(K) \subseteq K$.

Theorem 4.13: For any $A \subseteq X$, $[X - S^{\#}Int(A)] = S^{\#}Cl(X - A)$.

Proof: Enable $x \in X - S^{\#}Int(A)$. Thereupon $x \notin S^{\#}Int(A)$. In particular each $S^{\#}$-open set G including x is so as $G \not\subset A$. This implies, each $S^{\#}$-open set G including x intersects X–A. That is, $G \cap (X–A) \neq \phi$. Referring to Theorem 4.2, $x \in S^{\#}Cl(X–A)$ and therefore $[X– S^{\#}Int(A)] \subseteq S^{\#}Cl(X – A)$.

Contrarily, take $x \in S^{\#}Cl(X\text{-}A)$. Accordingly, each $S^{\#}$-open set G including x intersects X–A. That is, $G \cap (X\text{-}A) \neq \phi$. Particularly, each $S^{\#}$-open set G consisting x is so as $G \not\subset A$. With reference to Definition 4.7, $x \notin S^{\#}Int(A)$. Particularly, $x \in [X\text{- } S^{\#}Int(A)]$, and so $[S^{\#}Cl(X\text{-}A)] \subseteq [X\text{- } S^{\#}Int(A)]$. Thence $[X\text{- } S^{\#}Int(A)] = [S^{\#}Cl(X \text{ - } A)]$.

Remark 4.14: For any $A \subseteq X$. We obtain
(i) $[X – S^{\#}Cl(X – A)] = [S^{\#}Int(A)]$
(ii) $[X – S^{\#}Int(X – A)] = [S^{\#}Cl(A)]$
Taking complement in Theorem .4.13 as well as replacing A by X – A, results follows.

V. Conclusion

In this paper, we introduced $S^{\#}$-neighborhoods and studied their properties along with $S^{\#}$-derived sets, $S^{\#}$-closure and $S^{\#}$-interior operators. These results extend existing concepts in generalized topology and may be useful for further research in topological structures.

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