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Test For Integer Roots Of Polynomial Equations

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***Abstract**:*
This paper presents a simple and time-saving method for finding integer roots of polynomial equations. Unlike the traditional approach, which requires testing all factors of the constant term, the proposed method limits the search to a few selected factors. This reduces computation effort and makes the process faster and more efficient.

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I. Derivation Of The Formula /Method

Let the polynomial equation of n^{th} degree be
$f(x) = Ax^n+Bx^{n-1}+Cx^{n-2}+Dx^{n-3}+......P=0$ .....(1)
where A,B,C,D,P etc. all are positive or negative integer coefficients of decreasing powers of x,P being the constant term or coefficient of x°. Let α a factor of P be an integer root of this equation then we can always write the equation as $(A_1x^{n-1}+B_1x^{n-2}+C_1x^{n-3}+D_1x^{n-4}+....P_1)(x-\alpha)=0$ ....(2),where the expression $(A_1x^{n-1}+....+P_1)$is obtained by dividing f(x) by (x-α)and $A_1,B_1,C_1,D_1.,..P_1$ are also positive or negative coefficients of powers of x. We shall call it residual expression or $f_1(x)$ which will be of(n-1)th degree. Similarly if β ,another factor of P is an integer root of the polynomial we can write the equation (1) as $(A_2x^{n-2} +B_2x^{n-3}+C_2x^{n-4}+........+P_2)(x-\alpha)(x-\beta)=0$ or $f_2(x)(x-\alpha)(x-\beta)=0$. Multiplying both sides of the equation (2) we get $A_1x^n-\alpha A_1x^{n-1}+B_1x^{n-1}-\alpha B_1x^{n-2}+C_1x^{n-2}.-\alpha C_1x^{n-3}+D_1x^{n-3}+........-\alpha P_1=0$ or $Ax^n+ (B_1-\alpha A_1)x^{n-1} +(C_1-\alpha B_1) x^{n-2}+(D_1-\alpha C_1)x^n-3 +........-\alpha P_1=0$.....(3) equating the coefficients of (1) and (3) we get
$A=A_1$
$B=B_1-\alpha A_1$ or $B_1=B+\alpha A_1$
$C=C_1-\alpha B_1$ or $C_1=C+\alpha B_1$
$D=D_1-\alpha C_1$ or $D_1=D+\alpha C_1$
$P = -\alpha P_1$ ....(4)
Adding both sides we have
$A+B+C+D+......P=A_1+B_1+C_1+D_1 ....+P_1-\alpha(A_1+B_1+C_1+D_1+..........P_1)$
or $\Sigma A=\Sigma A_1-\alpha\Sigma A_1=\Sigma A_1(1-\alpha)$
or $\Sigma A/(1-\alpha)=\Sigma A_1$
or $\Sigma A/-(\alpha-1)=\Sigma A_1$
from above we draw following conclusions .

II. Hints / Steps For Application Of The Process

1. $\Sigma A/-(\alpha-1)$ must be an integer (positive or negative) which becomes a necessary condition for α to become an integer root of the polynomial equation so if it is not an integer then α cannot be a root.

2.$\Sigma A/-(\alpha-1)$ should be equal to ΣA_1 then only α will be a confirmed root of the polynomial equation. In other words if $\Sigma A/-(\alpha-1)$ comes out to be an integer but not equal to ΣA_1 then α cannot be a confirmed root of the polynomial.

3.If 1 is a root of this equation then $f(x)=f(1)=A+B+C+D+......+P=0$ or $\Sigma A=0$ so in that case we cannot test for other roots by this method .In this case we shall have to find
the equation $f_1(x) =0$ for testing other integer roots, either by dividing the equation by(x-α) or by using (4) of I as will be shown in examples.

4.when testing for the integer 2 as a possible root $\Sigma A/- (\alpha- 1)= \Sigma A/- (2- 1)= \Sigma A/- 1$ which is always an integer but it is not necessary that 2 might be a possible root unless f(2)= 0.Since 2 being a small number we can easily find f(2).

5. We can also take help of Descartes rule of signs by observing the number of changes of sign in f(x) for positive roots and number of changes of signs in f(- x) for negative roots and then taking the positive and negative factors of the constant term P test for integer
roots accordingly.

6. Since f(x) is a polynomial of n^{th} degree it can have maximum (n+ 1) number of coefficients of powers of x including P. Similarly $f_1(x)$ is a polynomial of $(n-1)^{th}$ degree so it can have maximum n number of coefficients. Some of the coefficients may be zero also.
In view of above points /conclusions some examples are being cited below to clearly understand the process.

III. Cited Examples

Example 1.Find integer roots of the polynomial equation $x^3 - 13x^2 + 56x - 80 = 0$
Solution. Here f(x) has 3 change of signs and f(- x) has no change of sign so it has no negative roots therefore we have to test for positive roots only.
factors of the constant term 80 are 1,2,4,5,8,10,16,20,40,80.
$\Sigma A = 1 - 13 + 56 - 80 = -36$
$f(1) = \Sigma A = -36$ and $f(2) = -12$ so 1& 2 are not the roots(see points 3&4 of II),for other factors we test as below

factor (α)	ΣA/- (α- 1)
4	- 36/- (4- 1)= 12 an integer
5	- 36/- (5- 1)= 9 an integer
8	- 36/- (8- 1)= 36/7 not an integer
10	- 36/- (10- 1)= 36/9not integer
16	- 36/- (16- 1)= 36/15 not integer
20	- 36/- (20- 1)= 36/ 19 not integer
40	- 36/- (40- 1)= 36/39 not integer
80	- 36/- (80- 1)= 36/79 not integer

so only 4 & 5 are possible integer roots .Substituting 4 and 5 in f(x) we find
that f(4) &f(5) both are = 0 so 4 &5 are confirmed roots. Though 4 is a repeated root of this polynomial equation .Alternatively if one possible root say 4 is found as above we can proceed to find A1, B1, C1 etc. with the help of......(4)of (I)to check whether $\Sigma A / -(\alpha - 1) = \Sigma . A_1$ and in this way find the equation $f_1(x) = 0$ to detect other roots.
here A= 1,B= - 13,C= 56,D= - 80,α= 4
so by ......(4) of (I)
$A_1 = A = 1$
$B_1 = B + \alpha A_1 = -13 + 4 \bullet 1 = -9$
$C_1 = C + \alpha B_1 = 56 + 4(-9) = 20$
$D1 = D + \alpha C_1 = -80 + 4 \bullet 20 = 0$
from above $\Sigma A_1 = 1 - 9 + 20 + 0 = -12$
hence $\Sigma A / -(\alpha - 1) = -36 / -(4 - 1) = -12 = \Sigma A_1$ and the equation $f_1(x)$ becomes
$x^2 - 9x + 20 = 0$ or $(x-4)(x-5) = 0$
so x= 4,5 and the 3 roots are 4,4,5.
Example 2. Find integer roots of the polynomial equation
$3x^4 - 123x^2 - 103x - 455 = 0$
Solution. Here A=3, B=0, C=-123, D=-103
E=-455, so ΣA=3+0-123-103-455=-678
factors of the constant term 455 are (±) 1,5,7,13,35,65,91,455.Since there is only one change of sign so it has one positive root only so we test for the positive root
first except 1 (because $f(1) = \Sigma A \neq 0$)
taking the factors of 455 as below

factor(α)	ΣA/-(α-1)
5	-678/-(5-1) =339/2 not an integer
7	-678/-(7-1) =113 an integer
13	-678/-(13-1) =339/6 not integer
35	-678/-(35-1) =339/17 not integer
65	-678/-(65-1) =339/32 not integer
91	-678/-(91-1) =113/15 not integer
455	-678/-(455-1) =339/227 not integer

so only 7 is a possible positive root ,
also $f(7) = 3 \bullet 7^4 - 123 \bullet 7^2 - 103 \bullet 7 - 455 = 0$
therefore 7 is a confirmed root. Alternatively deriving ΣA1 using..... (4) of l

as in example 1 we can confirm that 7 is a root as below
$A_1= A= 3$,
$B_1= B+\alpha A_1= 0+ 7(3)= 21$
$C_1= C+ \alpha B_1= - 123+ 7(21)= 24$
$D_1= D+ \alpha C_1= - 103+ 7(24)= 65$
$E_1= E+ \alpha D_1= - 455+ 7(65)= 0$
so $\Sigma A_1= 3+ 21+ 24+ 65=113$ also
$\Sigma A/- (\alpha- 1)= - 678/- (7- 1)= 113$ affirming 7 as a confirmed root. From above
$f_1(x) =3x^3+21x^2+24x+65=0$; we proceed for finding negative integer roots of this equation. $f1(-x)$ has three change of sign.
so this equation has either three negative roots or one negative and two imaginary roots. Negative factors(β) of constant term 65 are -1, -5, -13, -65
we test them as below(except -1 because-1 is not a root since $\Sigma A1=113$ which is not equal to zero)

factors(β)	$\Sigma A1/-(\beta-1)$
-5	113/-(-5-1) =113/6 not an integer
-13	113/-(-13-1) =113/14 not integer
-65	113/-(-65-1) =113/66 not integer

so the equation $f_1(x)$ has no negative
Integer root. We could also directly check from the original equation $f(x) =0$ that it has no negative integer roots.
Example 3. show that the equation
$7x^3-11x^2+ 45= 0$ has no integer roots.
here $\Sigma A= 7- 11+ 45= 41$ which is a prime
number and is divisible by 1 and itself, also
$f(1)= \Sigma A= 41$ so 1 is not a root. Other factors of the constant tem 45 are- 1,
$\pm3,\pm5,\pm9,\pm15,\pm45$. none of these factors
make $\Sigma A/- (\alpha- 1)$ equal to an integer hence overall there are no integer roots of this equation.

IV. Relation Between Any Two Roots Of Cubic Equations

Let the cubic equation be $x^3+ ax^2+ bx+ c= 0$.........(1),and α,β are any two roots of this equation ,substituting these values in (1) we have

$\alpha^3+ a\alpha^2+ b\alpha+ c= 0$............(2)

$\beta^3+ a\beta^2+ b\beta+ c= 0$............(3)

subtracting(3) from (2) we get
$(\alpha^3- \beta^3)+ a(\alpha^2- \beta^2)+ b(\alpha- \beta)= 0$ dividing it by $(\alpha- \beta)$ we get $(\alpha^2+ \beta^2+ \alpha\beta)+ a(\alpha+ \beta)$
$+ b= 0$ or$(\alpha+ \beta)^2- \alpha\beta+ a(\alpha+ \beta)+ b= 0$ or
$(\alpha+ \beta)^2+ a(\alpha+ \beta)= \alpha\beta- b$ or
$(\alpha+ \beta)(\alpha+ \beta+ a)= \alpha\beta- b$..........(4)
this is the relation between any two roots of any cubic equation .So if any one root of the cubic is known we can check whether another number is a root or not.
This is true for all type of roots whether real or imaginary.

Example 4.given that 2 is a root of the equation.$x^3- 9x^2+ 26x- 24= 0$ show that
3&4 are also its roots .but 6,8 and 12 are not its roots.
solution. here $a = - 9$, $b= + 26$
let the given root $2= \alpha$ we take $\beta= 3$ then
$(\alpha+ \beta)(\alpha+ \beta+ a)$ should be equal to $\alpha\beta- b$
so LHS= $(2+ 3)(2+ 3- 9)= 5(- 4)= - 20$
RHS=$(2\bullet3)- 26= 6- 26= - 20$ so LHS= RHS hence 3 is also a root. If $\beta= 4$ then
LHS= $(2+ 4)(2+ 4- 9)= 6\bullet (- 3)= - 18$;
RHS= $2\bullet4- (26)= 8- 26= - 18$ so here also
LHS= RHS therefore 4 is also a root.
if $\beta= 6$ then LHS= $(2+ 6)(2+ 6- 9)= - 8$
RHS= $2\bullet6- 26= 12- 26= - 14$ so here LHS $\neq$ RHS hence 6 is not a root. If $\beta= 8$
LHS= $(2+ 8)(2+ 8- 9)= 10$;RHS= $2\bullet8- 26= - 10$ so LHS$\neq$RHS hence 8 is not a root
if $\beta= 12$;LHS= $(2+ 12)(2+ 12- 9)= 70$;
RHS= $2\bullet12- 26= - 2$ so here also LHS$\neq$RHS hence 12 is also not a root.

V. Conclusion

This paper presents a simple and effective shortcut method for testing and determining integer roots of polynomial equations. The proposed approach reduces computational effort and can significantly save time. It is expected to be particularly useful for secondary- and higher-level students.

References

[1]. Hall, H. S. And Knight, S. R., *Higher Algebra* And General Books On Theory Of Equations.