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On Some General Relationship Identities Of Pell And Pell-Lucas Numbers

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Abstract:
This study investigates the relationship between Pell numbers and Pell-Lucas numbers, which follow the same recurrence relation but differ in initial conditions. The goal of this study is to establish and prove general identities connecting the two sequences through the Principle of Mathematical Induction. Several key identities involving sums, products, squares, and linear combinations were derived and validated.

Keywords: *Pell numbers, Pell-Lucas numbers, relationship identities, principle of mathematical induction*

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I. Introduction

The Pell numbers P_n are defined by the recurrence relation $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$ with initial conditions $P_0 = 0$ and $P_1 = 1$. The first few terms of the sequence are 0, 1, 2, 5, 12, 29, 70, 169, … Pell numbers exhibit several interesting properties. One is that a Pell number can be prime if its index n is a prime number. For instance, when $n = 2$, the resulting Pell number is 5, which is a prime. Another notable property is that Pell numbers can form Pythagorean triples (a, b, c), where a and b differ by one. These triples take the form $(2P_nP_{n+1}, {P_{n+1}}^2 - {P_n}^2, {P_{n+1}}^2 + {P_n}^2)$ [1]. Also, the sum of the Pell numbers up to P_{4n+1} is always a perfect square. For instance, the sum of the Pell numbers up to P_5, that is, $1 + 2 + 5 + 12 + 29 = 49$, is the square of $P_2 + P_3 = 2 + 5 = 7$ [2].

On the other hand, Pell-Lucas numbers Q_n are defined by the linear recurrence relation $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$ with initial conditions $Q_0 = Q_1 = 2$. The first few terms of the sequence are 2, 2, 6, 14, 34, 82, 198, … Pell-Lucas numbers also possess some fascinating properties. Notably, every number in the sequence is even. Furthermore, for $Q_n/2$ the index *n* must either be a prime number or a power of 2 to be a prime number. The indices *n* for which $Q_n/2$ results in primes include 2, 3, 4, 5, 7, 8, 16, … The Pell-Lucas sequence also alternates between even and odd terms. The odd-numbered terms in the sequence are always divisible by 2, while the even-numbered terms tend to be more varied in divisibility [2].

Though defined by the same recurrence relation, the Pell and Pell-Lucas number sequences possess distinct initial conditions that give rise to different properties and behaviors. While various individual identities relating to these two sequences are known, a gap exists in systematically deriving and proving general relationship identities between them. Thus, this study seeks to address this gap by establishing such identities using the Principle of Mathematical Induction. By doing so, this study aims to prove some key summations and product identities that connect to the Pell and Pell-Lucas numbers.

II. Methods

The methodology of this study consists of two main phases: formulation of identities through inspection and application of the Principle of Mathematical Induction (PMI). First, the study centered on analyzing patterns of numbers between the Pell and Pell-Lucas sequences. These sequences have distinct initial conditions but share similar recursive structures. By examining these properties, the study aimed to derive general identities such as summation formulas, product identities, and linear combinations involving Pell and Pell-Lucas numbers. Second, the study applied the Principle of Mathematical Induction (PMI) to prove these derived identities. The process begins with the *base case*, where the identity is proven for an initial value of n, usually $n = 0$ or $n = 1$. Then, in the *inductive hypothesis*, it is assumed that the identity holds for a given $n = k$. Finally, the *inductive step* demonstrates that if the identity holds for $n = k$, it must also hold for $n = k + 1$ [3]. By completing these steps, the identity is established for positive integers n, providing a rigorous mathematical foundation for the relationships between the Pell and Pell-Lucas number sequences.

III. Results And Discussion

This section presents the general relationship between Pell and Pell-Lucas numbers, along with their proof using the Principle of Mathematical Induction. To establish these identities, some observable patterns were examined manually and mathematically. The number sequences are placed into the table to facilitate the analysis of the observable relationships and patterns. Table 1 presents the values of Pell and Pell-Lucas numbers for $n \geq 0$. The numbers are examined to determine possible associations between them. One key observation found is that the sum of Pell numbers before and after n is a Pell-Lucas number, that is, $Q_n = P_{n+1} + P_{n-1}$ for $n \geq 1$.

Table 1. Some Few Terms of Pell and Pell-Lucas Numbers for $n \geq 0$

n	P_n	Q_n	n	P_n	Q_n
0	0	2	7	169	478
1	1	2	8	408	1154
2	2	6	9	985	2786
3	5	14	10	2378	6726
4	12	34	11	5741	16238
5	29	82	12	13860	39202
6	70	198	13	33461	94642

To show that the relationship equation of Pell and Pell-Lucas numbers holds for all values of $n \geq 1$, the principle of mathematical induction was used. To prove this, we let $Q_n = P_{n+1} + P_{n-1}$ for $n \geq 1$. When $n = 1$, the right-hand side of the equation is $P_2 + P_0 = 2 + 0 = 2$ which matches the left-hand side of the equation, $Q_1 = 2$. Hence, the base case holds. Next, we assume that the statement is true for $n = k$, that is, the inductive hypothesis is $Q_k = P_{k+1} + P_{k-1}$. We must show that the same statement is also true for the next integer $n = k + 1$, that is, $Q_{k+1} = P_{k+2} + P_k$. Note that the recurrence relation for Pell-Lucas numbers $Q_{k+1} = 2Q_k + Q_{k-1}$. It follows that

$$\begin{aligned} Q_{k+1} &= 2(P_{k+1} + P_{k-1}) + (P_k + P_{k-2}) && \textit{Inductive hypothesis} \\ &= 2P_{k+1} + 2P_{k-1} + P_k + P_{k-2} && \textit{Simplify} \\ &= (2P_{k+1} + P_k) + (2P_{k-1} + P_{k-2}) && \textit{Regrouping} \\ Q_{k+1} &= P_{k+2} + P_k && \textit{Definition of } P_n \end{aligned}$$

Hence, the statement holds for $n = k + 1$. By the principle of mathematical induction, $Q_n = P_{n+1} + P_{n-1}$ is true for positive integers $n \geq 1$.

Another equation related to Pell and Pell-Lucas numbers, as observed from the table of values, is that the Pell-Lucas numbers equal half the difference between the Pell number two places ahead and the Pell number two places behind in the sequence. In symbols, this can be written in the form $Q_n = \frac{1}{2}(P_{n+2} - P_{n-2})$ for $n \geq 2$. By examining the values (presented in Table 1), it is observed that when $n = 2$, the right-hand side of the equation becomes $\frac{1}{2}(P_4 - P_0) = \frac{1}{2}(12 - 0) = 6$ which matches the left-hand side, $Q_2 = 6$. By substituting values for $n \geq 2$ until all possible values are used, it is hypothesized that this equation holds for all values of $n \geq 2$. Below is the proof of this identity, along with other identities.

Proof of General Relationship Identities of Pell and Pell-Lucas Numbers

Identity 1: $Q_n = \frac{1}{2}(P_{n+2} - P_{n-2})$ for $n \geq 2$.

Part 1: Base Case

Let $Q_n = \frac{1}{2}(P_{n+2} - P_{n-2})$ be the statement to be proven for all positive integers $n \geq 2$. If $n = 2$, the left-hand side of the equation is $Q_2 = 6$ while the right-hand side is $\frac{1}{2}(P_4 - P_0) = \frac{1}{2}(12 - 0) = 6$. Since we have shown that both sides of the equation are equal, the base case holds for $n = 2$.

Part 2: Inductive Hypothesis

Assume that the statement is true for $n = k$, that is, $Q_k = \frac{1}{2}(P_{k+2} - P_{k-2})$.

Part 3: Inductive Step

Show that the same statement holds for the next integer $n = k + 1$, that is, $Q_{k+1} = \frac{1}{2}(P_{k+3} - P_{k-1})$. Note that the recurrence relation for Pell-Lucas numbers $Q_{k+1} = 2Q_k + Q_{k-1}$ and substituting the inductive hypothesis, it follows that

Q_{k+1}	$=$	$2\left[\frac{1}{2}(P_{k+2}-P_{k-2})\right]+\frac{1}{2}(P_{k+1}-P_{k-3})$	*Inductive hypothesis*
	$=$	$P_{k+2}-P_{k-2}+\frac{1}{2}P_{k+1}-\frac{1}{2}P_{k-3}$	*Simplify*
	$=$	$(P_{k+2}+\frac{1}{2}P_{k+1})-(P_{k-2}+\frac{1}{2}P_{k-3})$	*Regrouping*
	$=$	$\frac{1}{2}(2P_{k+2}+P_{k+1})-\frac{1}{2}(2P_{k-2}+P_{k-3})$	*Simplify*
	$=$	$\frac{1}{2}P_{k+3}-\frac{1}{2}P_{k-1}$	*Definition of* P_n
Q_{k+1}	$=$	$\frac{1}{2}(P_{k+3}-P_{k-1})$.	*Simplify*

Thus, the statement holds for $n=k+1$. By the principle of mathematical induction, $Q_n=\frac{1}{2}(P_{n+2}-P_{n-2})$ is true for all positive integers $n\geq 2$.

Identity 2: $P_n=\frac{1}{8}(Q_{n+1}+Q_{n-1})$ for $n\geq 1$.
Part 1: Base Case
Let $P_n=\frac{1}{8}(Q_{n+1}+Q_{n-1})$ be the statement to be proven for all positive integers $n\geq 1$. If $n=1$, the left-hand side of the equation is $P_1=1$ while the right-hand side is $\frac{1}{8}(Q_2+Q_0)=\frac{1}{8}(6+2)=1$. We have shown that both sides of the equation are equal for $n=1$. When $n=2$, the left-hand side of the equation is $P_2=2$ while the right-hand side is $\frac{1}{8}(Q_3+Q_1)=\frac{1}{8}(14+2)=2$. Again, we have shown that both sides of the equation are equal for $n=2$. Hence, the base case holds.

Part 2: Inductive Hypothesis
Assume that the statement is true for $n=k$, that is, $P_k=\frac{1}{8}(Q_{k+1}+Q_{k-1})$.

Part 3: Inductive Step
Show that the same statement holds for the next integer $n=k+1$, that is, $P_{k+1}=\frac{1}{8}(Q_{k+2}+Q_k)$. Note that the recurrence relation for Pell numbers $P_{k+1}=2P_k+P_{k-1}$ and substituting the inductive hypothesis, then

P_{k+1}	$=$	$2\left[\frac{1}{8}(Q_{k+1}+Q_{k-1})\right]+\frac{1}{8}(Q_k+Q_{k-2})$	*Inductive hypothesis*
	$=$	$\frac{2}{8}Q_{k+1}+\frac{2}{8}Q_{k-1}+\frac{1}{8}Q_k+\frac{1}{8}Q_{k-2}$	*Simplify*
	$=$	$\left(\frac{2}{8}Q_{k+1}+\frac{1}{8}Q_k\right)+\left(\frac{2}{8}Q_{k-1}+\frac{1}{8}Q_{k-2}\right)$	*Regrouping*
	$=$	$\frac{1}{8}(2Q_{k+1}+Q_k)+\frac{1}{8}(2Q_{k-1}+Q_{k-2})$	*Simplify*
	$=$	$\frac{1}{8}Q_{k+2}+\frac{1}{8}Q_k$	*Definition of* Q_n
P_{k+1}	$=$	$\frac{1}{8}(Q_{k+2}+Q_k)$.	*Simplify*

Thus, the statement holds for $n=k+1$. By the principle of mathematical induction, $P_n=\frac{1}{8}(Q_{n+1}+Q_{n-1})$ is true for positive integers $n\geq 1$.

Identity 3: $P_0+P_1+P_2+\cdots+P_n=\frac{Q_{n+1}-2}{4}$ for $n\geq 0$.
Part 1: Base Case
Let $P_0+P_1+P_2+\cdots+P_n=\frac{Q_{n+1}-2}{4}$ be the statement to be proven for all positive integers $n\geq 0$. If $n=0$, the left-hand side of the equation is $P_0=0$ while the right-hand side is $\frac{Q_1-2}{4}=\frac{2-2}{4}=0$. Since we have shown that both sides of the equation are equal, the base case holds for $n=0$.

Part 2: Inductive Hypothesis
Assume that the statement holds for $n=k$, that is, $P_0+P_1+P_2+\cdots+P_k=\frac{Q_{k+1}-2}{4}$.

Part 3: Inductive Step
Show that the same statement holds for the next integer $n=k+1$, that is,

$$P_0+P_1+P_2+\cdots+P_k+P_{k+1}=\frac{Q_{k+2}-2}{4}.$$

Starting from the inductive hypothesis:

$$P_0 + P_1 + P_2 + \cdots + P_k = \frac{Q_{k+1} - 2}{4}$$

Now, we add P_{k+1} to both sides of the equation:

$$P_0 + P_1 + P_2 + \cdots + P_k + P_{k+1} = \frac{Q_{k+1} - 2}{4} + P_{k+1}$$

From Identity 2, $P_k = \frac{1}{8}(Q_{k+1} + Q_{k-1})$, it follows that $P_{k+1} = \frac{1}{8}(Q_{k+2} + Q_k)$. Substituting this into the right-hand side of the equation gives

$$\begin{aligned}
[P_0 + P_1 + P_2 + \cdots + P_k] + P_{k+1} &= \frac{Q_{k+1} - 2}{4} + P_{k+1} && \textit{Inductive hypothesis}\\
&= \frac{Q_{k+1} - 2}{4} + \frac{1}{8}(Q_{k+2} + Q_k) && \textit{Recalling Identity 2}\\
&= \frac{2Q_{k+1} - 4 + Q_{k+2} + Q_k}{8} && \textit{Simplify}\\
&= \frac{(2Q_{k+1} + Q_k) + Q_{k+2} - 4}{8} && \textit{Regrouping}\\
&= \frac{Q_{k+2} + Q_{k+2} - 4}{8} && \textit{Definition of } Q_n\\
&= \frac{2Q_{k+2} - 4}{8} && \textit{Simplify}\\
P_0 + P_1 + P_2 + \cdots + P_k + P_{k+1} &= \frac{Q_{k+2} - 2}{4}. && \textit{Simplify}
\end{aligned}$$

We have shown that the equation holds for $n = k$. It also holds for $n = k + 1$, and the base case is verified. Therefore, by mathematical induction, we conclude that $P_0 + P_1 + P_2 + \cdots + P_n = \frac{Q_{n+1} - 2}{4}$ holds for $n \geq 0$.

Identity 4: $Q_n{}^2 + Q_{n+1}Q_{n-1} = 16P_n{}^2$ for all $n \geq 1$.

Part 1: Base Case

Let $Q_n{}^2 + Q_{n+1}Q_{n-1} = 16P_n{}^2$ be the statement to be proven for all positive integers $n \geq 1$. If $n = 1$, the left-hand side of the equation becomes $Q_1{}^2 + Q_2Q_0 = (2)^2 + (6)(2) = 16$ while the right-hand side of the equation is $16P_1{}^2 = 16(1)^2 = 16$. Since we have shown that both sides are equal, the base case holds for $n = 1$.

Part 2: Inductive Hypothesis

Assume that the statement holds for $n = k$, that is, $Q_k{}^2 + Q_{k+1}Q_{k-1} = 16P_k{}^2$.

Part 3: Inductive Step

We must show that the statement holds for the next integer $n = k + 1$, that is, $Q_{k+1}{}^2 + Q_{k+2}Q_k = 16P_{k+1}{}^2$. Using the recurrence relations of Pell-Lucas numbers $Q_{k+2} = 2Q_{k+1} + Q_k$, we express $Q_{k+1}{}^2 + Q_{k+2}Q_k$ in terms of its recurrence relation, and relate it to $16P_{k+1}{}^2$. The equation now becomes,

$$\begin{aligned}
Q_{k+1}{}^2 + Q_{k+2}Q_k &= Q_{k+1}{}^2 + (2Q_{k+1} + Q_k)Q_k && \textit{Definition of } Q_n\\
&= Q_{k+1}{}^2 + 2Q_{k+1}Q_k + Q_k{}^2 && \textit{Simplify}\\
Q_{k+1}{}^2 + Q_{k+2}Q_k &= (Q_{k+1} + Q_k)^2. && \textit{Applying factoring}
\end{aligned}$$

Simplifying further the equation, since it has been established that $Q_k = P_{k+1} + P_{k-1}$, it also follows that $Q_{k+1} = P_{k+2} + P_k$. Using the recurrence relation for the Pell number $P_{k+1} = 2P_k + P_{k-1}$, it now follows that

$$\begin{aligned}
Q_{k+1}{}^2 + Q_{k+2}Q_k &= (Q_{k+1} + Q_k)^2\\
&= [(P_{k+2} + P_k) + (P_{k+1} + P_{k-1})]^2 && \textit{Recalling Identity}\\
&= [(2P_{k+1} + P_k) + P_k + (P_{k+1} + P_{k-1})]^2 && \textit{Definition of } P_n\\
&= (2P_{k+1} + P_k + P_k + P_{k+1} + P_{k-1})^2 && \textit{Simplify}\\
&= (3P_{k+1} + 2P_k + P_{k-1})^2 && \textit{Combining like terms}\\
&= [3P_{k+1} + (2P_k + P_{k-1})]^2 && \textit{Regrouping}\\
&= (3P_{k+1} + P_{k+1})^2 && \textit{Definition of } P_n\\
&= (4P_{k+1})^2 && \textit{Simplify}\\
Q_{k+1}{}^2 + Q_{k+2}Q_k &= 16P_{k+1}{}^2. && \textit{Simplify}
\end{aligned}$$

We have shown that the equation holds for $n = k$. Further, it also holds for $n = k + 1$, and the base case is verified. Thus, by mathematical induction, we conclude that ${Q_n}^2 + Q_{n+1}Q_{n-1} = 16{P_n}^2$ holds for $n \geq 1$.

Identity 5: ${P_n}^2 + P_{n+1}P_{n-1} = \frac{1}{4}{Q_n}^2$ for all $n \geq 1$.

Part 1: Base Case

Let ${P_n}^2 + P_{n+1}P_{n-1} = \frac{1}{4}{Q_n}^2$ be the statement to be proven for all positive integers $n \geq 1$. If $n = 1$, the left-hand side of the equation becomes ${P_1}^2 + P_2P_0 = (1)^2 + (2)(0) = 1$ while the right-hand side of the equation is $\frac{1}{4}{Q_1}^2 = \frac{1}{4}(2)^2 = 1$. Since we have shown that both sides are equal, the base case holds for $n = 1$.

Part 2: Inductive Hypothesis

Assume that the statement holds for $n = k$, that is, ${P_k}^2 + P_{k+1}P_{k-1} = \frac{1}{4}{Q_k}^2$.

Part 3: Inductive Step

We must show that the same statement holds for the next integer $n = k + 1$, i.e., ${P_{k+1}}^2 + P_{k+2}P_k = \frac{1}{4}{Q_{k+1}}^2$. Using the recurrence relations of Pell numbers, $P_{k+1} = 2P_k + P_{k-1}$, we express ${P_{k+1}}^2 + P_{k+2}P_k$ in terms of its recurrence relation, and relate it to $\frac{1}{4}{Q_{k+1}}^2$. The equation now becomes,

$$\begin{aligned}
{P_{k+1}}^2 + P_{k+2}P_k &= {P_{k+1}}^2 + (2P_{k+1} + P_k)P_k && \textit{Definition of } P_n \\
&= {P_{k+1}}^2 + 2P_{k+1}P_k + {P_k}^2 && \textit{Simplify} \\
{P_{k+1}}^2 + P_{k+2}P_k &= (P_{k+1} + P_k)^2. && \textit{Applying factoring}
\end{aligned}$$

Since it has been established that $P_k = \frac{1}{8}(Q_{k+1} + Q_{k-1})$, it also follows that $P_{k+1} = \frac{1}{8}(Q_{k+2} + Q_k)$. Using the recurrence relations of Pell-Lucas numbers $Q_{k+1} = 2Q_k + Q_{k-1}$ and some formulations, the equation now becomes

$$\begin{aligned}
{P_{k+1}}^2 + P_{k+2}P_k &= (P_{k+1} + P_k)^2 \\
&= \left[\tfrac{1}{8}(Q_{k+2} + Q_k) + \tfrac{1}{8}(Q_{k+1} + Q_{k-1})\right]^2 && \textit{Recalling Identity 2} \\
&= \left[\tfrac{1}{8}(2Q_{k+1} + Q_k) + \tfrac{1}{8}Q_k + \tfrac{1}{8}(Q_{k+1} + Q_{k-1})\right]^2 && \textit{Definition of } Q_n \\
&= \left(\tfrac{2}{8}Q_{k+1} + \tfrac{1}{8}Q_k + \tfrac{1}{8}Q_k + \tfrac{1}{8}Q_{k+1} + \tfrac{1}{8}Q_{k-1}\right)^2 && \textit{Simplify} \\
&= \left(\tfrac{3}{8}Q_{k+1} + \tfrac{2}{8}Q_k + \tfrac{1}{8}Q_{k-1}\right)^2 && \textit{Combining like terms} \\
&= \left[\tfrac{3}{8}Q_{k+1} + \tfrac{1}{8}(2Q_k + Q_{k-1})\right]^2 && \textit{Applying factoring} \\
&= \left(\tfrac{3}{8}Q_{k+1} + \tfrac{1}{8}Q_{k+1}\right)^2 && \textit{Definition of } Q_n \\
&= \left(\tfrac{4}{8}Q_{k+1}\right)^2 && \textit{Combining like terms} \\
{P_{k+1}}^2 + P_{k+2}P_k &= \tfrac{1}{4}{Q_{k+1}}^2. && \textit{Simplify}
\end{aligned}$$

We have shown that the equation holds for $n = k$. Furthermore, it also holds for $n = k + 1$, and the base case is verified. Therefore, by mathematical induction, we conclude that ${P_n}^2 + P_{n+1}P_{n-1} = \frac{1}{4}{Q_n}^2$ holds for $n \geq 1$.

IV. Conclusion

This study has successfully investigated and established several general relationship identities between the Pell and Pell-Lucas number sequences. Through a detailed exploration of their recursive definitions, algebraic properties, and summation identities, it has uncovered significant associations that highlight the connection between these two number sequences. It has been demonstrated that Pell and Pell-Lucas numbers, despite their differing initial conditions, share fundamental structural similarities, which gave rise to some of the general identities of Pell and Pell-Lucas numbers. The study proved identities that link the squares, products, and summation by employing the principle of mathematical induction. These findings extend existing knowledge on recurrence relations and shed light on the algebraic and combinatorial properties inherent in both sequences.

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