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Coupled Fractional Fourier Transform On Sobolev Spaces Related To The Negative Definite Functions

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Abstract
The chief interest of this article is to discuss non-archimedean pseudo- differential operator connected to coupled fractional Fourier transform. In this article, we some classes of p-adic complete inner product spaces, Bφ, k(Qp Qp), 0 k < ∞, connected to negative definite, radial and continuous functions φ : Qp C. In this article, we also define the non- archimedean pseudo-differential operator A α1,α2 involving coupled fractional Fourier transform connected to negative definite functions.
Keywords: *Non-archimedean analysis, Pseudo-differential operators, Frac- tional Fourier transform, M-dissipative operators.*

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1 Introduction

The connections of non-archimedean pseudo-differential operators with certain p-adic pseudo-differential equations that describe certain physical models [1–5]. Therefore, non-archimedean pseudo-differential operators have received a lot of attention in two decades.Non-archimedean pseudo-differential operators have gained popularity in recent years due to their utility in studying certain equations associated with new physical models/Models in physical form [6–12]. The interest in pseudo-differential operators in the p-adic context has grown significantly in recent years as a result of their utility in modelling various types of physical phenomena.

For example, modelling geological processes (such as the formation of petroleum micro-scale reservoirs and fluid flows in porous media such as rock); the dynamics of complex systems such as macromolecules, glasses, and proteins; the study of Coulomb gases, etc. [13–17]. Nonlocal diffusion problems arise in a wide range of applications in the archimedean setting, including biology, image processing, particle systems, and coagulation models.

My research work is motivated/ inspried by the works of Ismael Gutiérrez García and Anselmo Torresblanca-Badillo [10, 16, 18].

The interaction of non-archimedean pseudo-differential operators and stochastic processes on p-adics has received a lot of attention in recent decades because of the connection of the p-adic pseudo-differential equations associated with certain physical models, see [19].

2 Mathematical background of coupled fractional Fourier analysis on $\mathbb{Q}_p \times \mathbb{Q}_p$

Definition 1. ***The field of p-adic numbers:*** *Let p be a prime number. Through out this manuscript p will denote a prime number. Firstly we define p-adic norm* $|.|_p$ *on* $\mathbb{Q}$ *as follows*

$$|\eta|_p = \begin{cases} 0, & \text{if } \eta = 0, \\ p^{-\tau}, & \text{if } \eta = p^{\tau}\frac{\rho}{\sigma}, \end{cases}$$

where ρ *and* σ *are integers coprime with p. The integer* $\tau := ord(\eta)$, *with* $ord(0) := +\infty$, *is called the p-adic order of* η.
The unique expansion of any p-adic number $\eta \neq 0$ *is of the form*

$$\eta = p^{ord(\eta)} \sum_{i=0}^{\infty} \eta_i p^i, \tag{1}$$

where $\eta_i \in \{0, 1, 2, ..., p-1\}$ *and* $\eta_0 \neq 0$. *Using (1), we define the fractional part of* $\eta \in \mathbb{Q}_p$, *denoted by* $\{\eta\}_p$, *as the rational number*

$$\{\eta\}_p = \begin{cases} 0, & \text{if } \eta = 0 \text{ or } ord(\eta) \geq 0, \\ p^{ord(\eta)} \sum_{i=0}^{-ord_p(\eta)-1} \eta_i p^i, & \text{if } ord(\eta) < 0. \end{cases}$$

Extention of the p-adic norm on $\mathbb{Q}_p$ *is given by*

$$||\eta||_p = |\eta|_p, \quad \forall \eta \in \mathbb{Q}_p.$$

Let $r_0 \in \mathbb{Z}$ *and* $a_0 \in \mathbb{Q}_p$. *We consider* $I_{r_0}(\eta_0) = \{ \eta \in \mathbb{Q}_p : || \eta - \eta_0||_p \leq p^{r_0} \}$. *The empty set and the points are the only connected subsets of* $\mathbb{Q}_p$. *Therefore, the topological space* $(\mathbb{Q}_p, ||\ ||_p)$ *is totally disconnected. The necesssary and sufficient condition for the compactness of a subset of* $\mathbb{Q}_p$ *is that bounded and closed subdet of* $\mathbb{Q}_p$.

3 Few functional spaces

A function $f : Q_p \times Q_p \to C$ is called locally constant if for any $\eta, \xi \in Q_p$ there exists an integer $r(\eta, \xi) \in Z$
such that $f(\eta + \eta') = f(\eta)$ for all $\eta' \in I_{r(\eta)}$.
A function $f : Q_p \times Q_p \to C$ is called a test function (or a Bruhat-Schwartz function) if it is a compact support with locally constant. The set of all complex valued test functions on $Q_p \times Q_p$ is denoted by $D(Q_p \times Q)$ or simply D. The set of all distributions (all continous functionals) on D is denoted by $D'(Q_p \times Q_p)$ or simply D'. The mapping $U, \psi : D(Q) \times D(Q) \to C$ for $U \in D(Q_p \times Q_p)$ and $\psi \in D(Q_p \times Q_p)$ is defined as follows:

$$U, \psi = \int_{Q}\int_{Q_p} U(\zeta, \xi)\psi(\zeta, \xi)d\zeta d\xi.$$

Definition 2. ***Regular Distribution:*** *Let M be an arbitrary compact subset of $Q_p \times Q_p$. i.e. $M \subset Q_p \times Q_p$. Then $L^1_{loc}(Q_p \times Q_p) = \{\varphi \,|\, \varphi : Q_p \times Q_p \to C$ such that $\varphi \in L^1(M)\}$. A distribution $\varphi \in D(Q_p \times Q_p)$ is defined by every function $\varphi \in L^1_{loc}(Q_p \times Q_p)$ according to the formula*

$$\varphi, \psi = \int_{Q}\int_{Q_p} \varphi(\zeta, \eta)\psi(\zeta, \eta)d\zeta d\eta.$$

This type of distributions is known as regular distributions.
Let $\sigma \in [0, \infty)$. Then the set
$L^\sigma(Q_p \times Q_p, dxdy) = \{h : Q_p \times Q_p \to C$ such that $\int_{Q_p}\int_{Q_p} |h(x, y)|^\sigma dxdy < \infty\}$, the set $L^\infty(Q_p \times Q_p, dxdy) = \{h : Q_p \times Q_p \to C$ such that essential sup. $|h| < \infty\}$, the set $C(Q_p \times Q_p, C) = \{h : Q_p \times Q_p \to C$ and h is a continous function$\}$, and the set

$$C_0(Q_p \times Q_p, C) = \{h : Q_p \times Q_p \to C \text{ and } \lim_{(||\zeta||_p, ||\eta||_p) \to (\infty,\infty)} h(\zeta, \eta) = 0\}$$

are complex vector space under the binary operation vector addition (+) and scalar multiplication (.). It also implies that $C_0(Q_p \times Q_p, C), ||.||_{L^\infty}$ is a Banach space.

4 Coupled fractional Fourier transform

Fourier Transform:-
The set $\chi_p(y) = e^{2\pi i y}$ for $y \in Q_p$. The map $\chi_p(.)$ is an additive character on Q_p, i.e. a continuous map from $(Q_p, +)$ into S (the unit circle considered as multiplicative group) satisfying $\chi_p(x_\circ + x_1) = \chi_p(x_\circ)\chi_p(x_1)$, $x_\circ, x_1 \in Q_p$. The additive characters of Q_p from an Abelian group which is isomorphic to $(Q_p, +)$, the isomorphism is given by $\xi \to \chi_p(\xi x)$. Given $x, \xi \in Q_p$, if $f \in L_1(R)$ its Fourier transform is defined by

$$(Ff)(\xi) = \int_{Q^p} \chi_p(\xi x)f(x)dx \quad \text{for} \quad \xi \in Q_p.$$

Fractional Fourier Transform on $\mathbb{Q}_p$:-

In this chapter, we introduce the definition of fractional Fourier transform on the field of p-adic numbers $\mathbb{Q}_p$. Firstly, the map $\chi_p^{\vartheta}(.,.)$ is defined on $\mathbb{Q}_p$ as follows:

$$\forall \zeta,\ \eta \in \mathbb{Q}_p,\ \chi_p^{\vartheta}(\zeta,\eta) = \begin{cases} C^{\vartheta} e^{\frac{i(\zeta^2+\eta^2)\cot\vartheta}{2} - i\zeta\eta\csc\vartheta}, & \vartheta \neq n\pi, n \in Z \\ \sqrt{\frac{1}{2\pi}}\, e^{-i\zeta\eta} & \vartheta = \frac{\pi}{2} \end{cases}$$

$$C^{\vartheta} = \sqrt{\frac{1-i\cot\vartheta}{2\pi}}$$

If $\psi \in L^1(\mathbb{Q}_p)$, its fractional Fourier transform of one dimension [20,21] is defined as follows:

$$(F_{\vartheta}\psi)(\eta) = \hat{\psi}_{\vartheta}(\eta) = \int_{\mathbb{Q}_p} \chi_p^{\vartheta}(\zeta,\eta)\psi(\zeta)d\zeta, \quad for\ \eta \in \mathbb{Q}_p. \tag{2}$$

The inverse fractional Fourier transform of a map $\varphi \in L^1(\mathbb{Q}_p)$ is

$$(F_{\vartheta}^{-1}\varphi)(\zeta) = \int_{\mathbb{Q}_p} \chi_p^{-\vartheta}(\zeta,\eta)\varphi(\eta)d\eta, \quad for\ \zeta \in \mathbb{Q}_p. \tag{3}$$

The fractional Fourier transform is an isomorphism, continuous and linear map of $D(\mathbb{Q}_p)$ onto itself holding

$$(F_{\vartheta}(F_{\vartheta}^{-1}\varphi))(\zeta) = (F_{\vartheta}^{-1}(F_{\vartheta}\varphi))(\zeta) = \varphi(\zeta), \tag{4}$$

for every $\varphi \in D(\mathbb{Q}_p)$.

Exploiting the tensor product of n copies of the one-dimensional fractional Fourier transform each of order ϑ_l, $l = 1, 2, 3, \ldots, n$, the fractional Fourier transform has been extended to the higher-dimensional transform.

We assume that $\vartheta = (\vartheta_1,\vartheta_2)$, $\mathbf{x} = (x,\eta)$, $\mathbf{y} = (y,\zeta)$, $\chi_p^{\vartheta}(\mathbf{x},\mathbf{y}) = \chi_p^{\vartheta_1}(x,\eta).\chi_p^{\vartheta_2}(y,\zeta) = \chi_p^{\vartheta_1,\vartheta_2}(x,y,\eta,\zeta)$, where $\chi_p^{\vartheta_1}(x,\eta)$ and $\chi_p^{\vartheta_2}(y,\zeta)$ defined as above.

Coupled Fractional Fourier Transform on $\mathbb{Q}_p \times \mathbb{Q}_p$:-

The coupled fractional Fourier transform is defined as follows

$$[F_{\vartheta}\varphi](\eta,\zeta) = [F_{\vartheta_1,\vartheta_2}\varphi](\eta,\zeta) = \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \chi_p^{\vartheta}(\mathbf{x},\mathbf{y})\varphi(x,y)dxdy$$

$$= \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \chi_p^{\vartheta_1}(x,\eta)\chi_p^{\vartheta_2}(y,\zeta)\varphi(x,y)dxdy$$

$$= \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \chi_p^{\vartheta_1,\vartheta_2}(x,y,\eta,\zeta)\varphi(x,y)dxdy. \tag{5}$$

The corresponding inversion formula of (5) is defined as follows

$$\varphi(x,y) = \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \overline{\chi_p^{\vartheta_1,\vartheta_2}(x,y,\eta,\zeta)}[F_{\vartheta_1,\vartheta_2}\varphi](\eta,\zeta)d\eta d\zeta. \tag{6}$$

It is easy to observe that for $\vartheta_1 = \vartheta_2 = \frac{\pi}{2}$, the two-dimensional fractional Fourier transform $F_{\vartheta_1,\vartheta_2}$ becomes a classical two-dimensional Fourier transform.

5 Non-archimedean pseudo-differential operators with coupled fractional Fourier Transform and negative definite functions on Sobolev spaces

Definition 3. *[12] A mapping $\phi : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{C}$ is said to be positive definite, if*

$$\sum_{l,\, m=1}^{n} \phi\left(\eta_l - \eta_m, \xi_l - \xi_m\right) z_l \overline{z_m} \geq 0 \tag{7}$$

for all $n \in \mathbb{N}$, $\eta_1, \ldots\ldots\ldots, \eta_n \in \mathbb{Q}_p$ and $\xi_1, \ldots\ldots\ldots\ldots, \xi_n \in \mathbb{Q}_p$, $z_1, \ldots\ldots\ldots\ldots\ldots\ldots, z_n \in \mathbb{C}$.

Definition 4. *[12] A mapping $\varphi : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{C}$ is said to be negative definite, if*

$$\sum_{l,\, m=1}^{n} \left(\varphi(\zeta_l, \xi_l) + \overline{\varphi(\zeta_m, \xi_m)} - \varphi(\zeta_l - \zeta_m, \xi_l - \xi_m)\right) z_l \overline{z_m} \geq 0 \tag{8}$$

for all $n \neq 0 \in \mathbb{N}$, $\zeta_1, \ldots\ldots\ldots, \zeta_n \in \mathbb{Q}_p$, and $\xi_1, \ldots\ldots\ldots, \xi_n \in \mathbb{Q}_p$, $z_1, \ldots\ldots\ldots\ldots\ldots\ldots, z_n \in \mathbb{C}$.

The set of all negative definite functions on $\mathbb{Q}_p \times \mathbb{Q}_p$ is denoted by $N(\mathbb{Q}_p \times \mathbb{Q}_p)$ and the set of all continuous negative definite functions on $\mathbb{Q}_p \times \mathbb{Q}_p$ is $CN(\mathbb{Q}_p \times \mathbb{Q}_p)$, throughout this manuscript.

Example 1. *We consider the set $\mathbb{R}_+ = \{\eta \in \mathbb{R} : \eta \geq 0\}$ and define a continuous function $\boldsymbol{J} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\boldsymbol{J}(\eta, \xi) = \boldsymbol{J}(||\eta||_p, ||\xi||_p)$ for $\eta, \xi \in \mathbb{Q}_p$. Then $\boldsymbol{J}(\eta, \xi)$ is a radial function on $\mathbb{Q}_p \times \mathbb{Q}_p$. In addition, we assume that $\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} J(\eta, \xi)d\eta d\xi = 1$. This motivates that the copuled fractional Fourier transform of a continuous radial function is also a continuous radial function. The coupled fractional Fourier transform of $J(||\eta||_p, ||\xi||_p)$ i.e. $[F_{\vartheta_1,\vartheta_2}\boldsymbol{J}](||\eta||_p, ||\xi||_p)$ is a positive definite function and the coupled fractional Fourier transform of $\boldsymbol{J}(0,0) - \boldsymbol{J}(||\eta||_p, ||\xi||_p)$, i.e. $[F_{\vartheta_1,\vartheta_2}\boldsymbol{J}](0,0) - [F_{\vartheta_1,\vartheta_2}\boldsymbol{J}](||\eta||_p, ||\xi||_p)$ is a negative definite function.*

Definition 5. *Let $\varphi : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{C}$ be a negative definite and continuous radial function. We consider $\psi \in D(\mathbb{Q}_p \times \mathbb{Q}_p)$ and a real number $k \geq 0$. Now, the norm is defined as follows:*

$$^{\vartheta_1,\vartheta_2}||\psi||_{\varphi,\,k} = \left(\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \left\{\max\left(1, |\varphi(||\eta||_p, ||\xi||_p)|\right)\right\}^{2k} |\hat{\psi}_\vartheta(\eta)|^2 d\eta d\xi\right)^{\frac{1}{2}}.$$

One verifies that the function space $B_{\varphi,\,k}(\mathbb{Q}_p \times \mathbb{Q}_p)$, $k \geq 0$ is the completion of $D(\mathbb{Q}_p \times \mathbb{Q}_p)$ with respect to the norm $^{\vartheta_1,\vartheta_2}||\psi||_{\varphi,\,k}$.

Remark 1. *If $k' \geq k \geq 0$ then $^{\vartheta_1,\vartheta_2}||.||_{\varphi,\,k} \leq {}^{\vartheta_1,\vartheta_2}||.||_{\varphi,\,k'}$ and $B_{\varphi,\,k'}(\mathbb{Q}_p \times \mathbb{Q}_p) \subseteq B_{\varphi,\,k}(\mathbb{Q}_p \times \mathbb{Q}_p)$. Therefore $B_{\varphi,\,k'}(\mathbb{Q}_p \times \mathbb{Q}_p) \longleftrightarrow B_{\varphi,\,k}(\mathbb{Q}_p \times \mathbb{Q}_p)$ if $k' \geq k \geq 0$.*

Moreover, we notice that $B_{\varphi,0}(\mathbb{Q}_p \times \mathbb{Q}_p) = L^2(\mathbb{Q}_p \times \mathbb{Q}_p)$ *and for* $k > 0$ *we obtain that* $B_{\varphi,k}(\mathbb{Q}_p \times \mathbb{Q}_p) \subset L^2(\mathbb{Q}_p \times \mathbb{Q}_p)$.
By the Parseval-Steklov equality satisfy, we can verify $||g||_{L^2(\mathbb{Q}_p \times \mathbb{Q}_p)} \leq {}^{\vartheta_1,\vartheta_2}||g||_{\varphi,k}$, $\forall g \in B_{\varphi,k}(\mathbb{Q}_p \times \mathbb{Q}_p)$ *and for* $k \geq 0$, *so that we obtain that*

$$B_{\varphi,k}(\mathbb{Q}_p \times \mathbb{Q}_p),\ {}^{\vartheta_1,\vartheta_2}||.||_{\varphi,k} \longleftrightarrow L^2(\mathbb{Q}_p \times \mathbb{Q}_p),\ ||.||_{L^2(\mathbb{Q}_p \times \mathbb{Q}_p)}.$$

Using the density property of $D(\mathbb{Q}_p \times \mathbb{Q}_p)$ *in* $B_{\varphi,k}(\mathbb{Q}_p \times \mathbb{Q}_p)$ *and* $L^2(\mathbb{Q}_p \times \mathbb{Q}_p)$, *it implies that* $B_{\varphi,k}(\mathbb{Q}_p \times \mathbb{Q}_p)$ *is dense in* $L^2(\mathbb{Q}_p \times \mathbb{Q}_p)$.

Definition 6. *We consider* $\phi : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{C}$ *is a negative definite and continous radial function. We define for* $k \geq 0$

$$L_2(d_k) = \left\{ g \in L^1_{loc} : d_k g \in L^2(\mathbb{Q}_p \times \mathbb{Q}_p) \right\},$$

where $d_k(\eta) = \max\left\{1, |\phi(||\eta||_p)|\right\}^k$, $\eta \in \mathbb{Q}_p$. *One can verify that* $L_2(d_k)$ *is a Hilbert space with the inner product*

$$\langle g, h\rangle_{L_2(d_k)} = \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} d_k(\zeta,\eta)g(\zeta,\eta)\overline{d_k(\zeta,\eta)h(\zeta,\eta)}\,d\zeta d\eta = \langle d_k g, d_k h\rangle_{L^2(\mathbb{Q}_p \times \mathbb{Q}_p)}. \tag{9}$$

Theorem 1. *We consider* $\phi : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{C}$ *is a negative definite and continous radial function. We define for* $k \geq 0$

$$L_2(d_k) = \left\{ g \in L^1_{loc} : d_k g \in L^2(\mathbb{Q}_p \times \mathbb{Q}_p) \right\},$$

where $d_k(\eta) = \max\left\{1, |\phi(||\eta||_p)|\right\}^k$, $\eta \in \mathbb{Q}_p$. *Then*

$$B_{\phi,k}(\mathbb{Q}_p \times \mathbb{Q}_p) = L_2(d_k).$$

Proof. Applying the Parseval-Steklov equality, we obtain that

$$\begin{aligned}
& g \in B_{\phi,k}(\mathbb{Q}_p \times \mathbb{Q}_p) \\
\Longleftrightarrow\ & \left(\int\int \max\left\{1, |\phi(||\eta||_p, ||\zeta||_p)|\right\}^{2k} |[F_{\vartheta_1,\vartheta_2} g](\eta,\zeta)|^2 d\eta d\zeta\right)^{\frac{1}{2}} < \infty \\
\Longleftrightarrow\ & \left(\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \left|\max\left\{1, |\phi(||\eta||_p, ||\xi||_p)|\right\}^k [F_{\vartheta_1,\vartheta_2} g](\eta,\xi)\right|^2 d\eta d\xi\right)^{\frac{1}{2}} < \infty \\
\Longleftrightarrow\ & ||\max\left\{1, |\phi(||\eta||_p)|\right\}^k g||_{L_2(d_k)} < \infty \\
\Longleftrightarrow\ & ||d_k g||_{L_2(d_k)} < \infty \\
\Longleftrightarrow\ & g \in L_2(d_k).
\end{aligned}$$

It implies that $B_{\phi,k}(\mathbb{Q}_p \times \mathbb{Q}_p) = L_2(d_k)$.
Hence the theorem is proved. □

Remark 2. *Since every function $g \in L^1_{loc}$ defines a regular distribution $g \in D'(\mathbb{Q}_p \times \mathbb{Q}_p)$, we obtain that*

$$B_{\phi,k}(\mathbb{Q}_p \times \mathbb{Q}_p) = \{g \in D'(\mathbb{Q}_p \times \mathbb{Q}_p) : ||g||_{\phi,k} < \infty\}.$$

From a Gel'fand triple the spaces $D(\mathbb{Q}_p \times \mathbb{Q}_p) \subset B_{\phi,k}(\mathbb{Q}_p \times \mathbb{Q}_p) \subset D'(\mathbb{Q}_p \times \mathbb{Q}_p)$.

Theorem 2. *If $\frac{1}{[max\{1, |\phi(||\zeta||_p, ||\xi||_p)|\}]^k} \in L^1(\mathbb{Q}_p \times \mathbb{Q}_p)$ for some $k \in \mathbb{N}$ then $B_{\phi,k}(\mathbb{C} \times \mathbb{C}) \hookrightarrow C_0(\mathbb{Q}_p \times \mathbb{Q}_p)$.*

Proof. We assume that the fixed $k \in \mathbb{N}$ such that $\frac{1}{[max\{1, |\phi(||\zeta||_p, ||\xi||_p)|\}]^k} \in L^1(\mathbb{Q}_p \times \mathbb{Q}_p)$. Using the Hölder inequality, for $g \in B_{\phi,k}(\mathbb{C} \times \mathbb{C})$ we obtain that

$$\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} |F_{\vartheta_1,\vartheta_2} g](\eta, \xi)| d\zeta d\xi$$

$$= \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \frac{1}{[max\{1, |\phi(||\zeta||_p, ||\xi||_p)|\}]^{\frac{k}{2}}} \times [max\{1, |\phi(||\zeta||_p, ||\xi||_p)|\}]^{\frac{k}{2}} |F_{\vartheta_1,\vartheta_2} g](\eta, \xi)| d\zeta d\xi$$

$$\leq \left(\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \frac{d\zeta d\xi}{[max\{1, |\phi(||\zeta||_p, ||\xi||_p)|\}]^k}\right)^{\frac{1}{2}} \times \left(\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} [max\{1, |\phi(||\zeta||_p, ||\xi||_p)|\}]^k |[F_{\vartheta_1,\vartheta_2} g](\eta, \xi)|^2 d\zeta d\xi\right)^{\frac{1}{2}}$$

$$\leq K||g||_{\phi,k} < \infty,$$

Where K is a constant. It implies that $F_{\vartheta_1,\vartheta_2} g \in L^1(\mathbb{Q}_p \times \mathbb{Q}_p)$. Applying Riemann-Lebesgue theorem, we obtain that $g \in C_0(\mathbb{Q}_p \times \mathbb{Q}_p)$. We also obtain that

$$||g||_{L^\infty} \leq ||\hat{g}_\vartheta||_{L^1} \leq ||g||_{\phi,k}.$$

It implies that $B_{\phi,k'}(\mathbb{C}) \leftrightarrow C_0(\mathbb{Q}_p \times \mathbb{Q}_p)$.
Hence the theorem is proved. □

Remark 3. *From $k \in \mathbb{N}$ in the Theorem2 we get that $B_{\phi,k'}(\mathbb{C} \times \mathbb{C}) \hookrightarrow C_0(\mathbb{Q}_p \times \mathbb{Q}_p)$, $\forall\ k' > k$. We recall that $D'(\mathbb{Q}_p \times \mathbb{Q}_p)$ is dense in $C_0(\mathbb{Q}_p \times \mathbb{Q}_p)$, we get that $B_{\phi,k'}(\mathbb{C} \times \mathbb{C})$ is dense in $C_0(\mathbb{Q}_p \times \mathbb{Q}_p)$, $\forall\ k' \geq k$.*

Example 2. *(a) The mapping $\phi(||\zeta||_p, ||\xi||_p) = \delta(||\zeta||_p^\gamma + ||\zeta||_p^\gamma)$, $\delta, \gamma > 0$ holds*

$$\frac{1}{[max\{1, |\phi(||\zeta||_p + ||\xi||_p)|\}]^k} \in L^1(\mathbb{Q}_p \times \mathbb{Q}_p), \quad \forall\ k > \frac{1}{\gamma}.$$

(b) Let $g(\zeta, \xi)$ be an elliptic polynomial of degree n_1. Then, we consider $\phi(\zeta, \xi) = |g(\zeta, \xi)|^\delta$, $\zeta, \xi \in \mathbb{Q}_p$ and for any fixed $\delta > 0$. We obtain that

$$\frac{1}{[max\{1, |\phi(||\zeta||_p, ||\xi||_p)|\}]^k} \in L^1(\mathbb{Q}_p \times \mathbb{Q}_p), \quad \forall\ k > \frac{1}{n_1\gamma}.$$

Examples 1 and 2, in this manuscript we assume that subclassses of non-constant negative definite functions, motivated by .
There are three types of negative definite function.
• A negative definite function $\phi : Q_p \times Q_p \to C$ is said to be type I, if $\exists$ a positive constant K=K(ϕ) such that

$$|\phi(||\zeta||_p, ||\xi||_p)| \leq K, \quad \forall\, \zeta,\ \xi \in Q_p.$$

• A negative definite function $\phi : Q_p \times Q_p \to C$ is said to be type II, if$\exists$ a positive constant $K_1 = K_1(\phi)$, $K_2 = K_2(\phi)$, $\delta_1 = \delta_1(\phi)$ and $\delta_2 = \delta_2(\phi)$, $0 < \delta_1 \leq \delta_2$, $K_1 \leq K_2$, such that

$$K_1[\max\{1,\ ||\zeta||_p + ||\xi||_p\}]^{\delta_1} \leq \max\{1,\ |\phi(||\zeta||_p, ||\xi||_p)|\}$$
$$\leq K_2[\max\{1,\ ||\zeta||_p + ||\xi||_p\}]^{\delta_2}, \quad \forall\, \zeta,\ \xi \in Q_p.$$

• A negative definite function $\phi : Q_p \times Q_p \to C$ is said to be type III, if $\forall$ constant $\delta_3 > 0$, $\exists$ a positive constant $K_3 = K_3(\phi, \delta_3)$ such that

$$K_3[\max\{1,\ ||\zeta||_p + ||\xi||_p\}]^{\delta_3} < \max\{1,\ |\phi(||\zeta||_p, ||\xi||_p)|\}, \quad \forall\, \zeta,\ \xi \in Q_p.$$

Now we introduce for $g \in B_{\varphi, k}(Q_p \times Q_p)$

$$A_{\varphi,k}(g)(\eta,\mu) = -F^{-1}_{\vartheta_1,\vartheta_2} \max\{1,\ |\varphi(||\zeta||_p, ||\xi||_p)|\}^{-\frac{k}{2}}[F_{\vartheta_1,\vartheta_2} g](\zeta,\xi)$$
$$= -\int_{Q_p}\int_{Q_p} \chi_p^{\vartheta_1,-\vartheta}(\zeta,\xi,\eta,\mu) \max\{1,\ |\varphi(||\zeta||_p, ||\xi||_p)|\}^{-\frac{k}{2}} \times [F_{\vartheta_1,\vartheta_2} g](\zeta,\xi)\, d\zeta d\xi.$$

We find the norm of $A_{\varphi,k}(g)(\eta,\mu)$ as follows:

$$||A_{\varphi,k}(g)(\eta,\mu)||_{\varphi,k}$$
$$= \left(\int_{Q_p}\int_{Q_p} \max\{1,\ |\varphi(||\zeta||_p, ||\xi||_p)|\}^{k} |[F_{\vartheta_1,\vartheta_2} g](\zeta,\xi)|^2 d\zeta d\xi\right)^{\frac{1}{2}}$$
$$\leq \left(\int_{Q_p}\int_{Q_p} \max\{1,\ |\varphi(||\zeta||_p, ||\xi||_p)|\}^{2k} |[F_{\vartheta_1,\vartheta_2} g](\zeta,\xi)|^2 d\zeta d\xi\right)^{\frac{1}{2}}$$
$$= ||g||_{\varphi,k} < \infty.$$

The mapping $\max\{1,\ |\varphi(||\zeta||_p, ||\xi||_p)|\}^{-\frac{k}{2}}$, $\zeta,\ \xi \in Q_p$ is known as the symbol of $A_{\varphi,k}$. Hence, $A_{\varphi,k} : B_{\varphi,k}(Q_p \times Q_p) \to B_{\varphi,k}(Q_p \times Q_p)$ is a well-defined non-archimedean pseudo-differential operator.

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