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A Study Of Crystallographic Groups And Their Symmetries

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Abstract
Frieze Groups are plane symmetry groups whose translations are isomorphic to Z. The word 'frieze' is derived from a Greek word which means decorative pattern repeated all the way around the building usually seen in Greek architecture. Similarly we have another set of groups which are an extension of frieze groups called crystallographic groups in which the translations are isomorphic to $Z \oplus Z$. Commonly referred as wallpaper groups, they have much more flexibility as compared to frieze groups. In this paper we shall aim to study about crystallographic groups and their symmetries.
Keywords – *Frieze, Crystallographic, Rotational Symmetry, Reflections.*

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I. Introduction

The five motions that we need to define are translations, horizontal reflection, vertical reflection, rotations and glide reflections. While in frieze groups there are only 7 patterns possible due to only horizontal and vertical reflection axis whereas in wallpaper groups there are multiple possibilities depending on the orientation of the figure. Moreover in frieze groups only one rotational symmetry is possible which is of $180°$, whereas in crystallographic groups, there are more than one possibilities which is the main reason why there are more wallpaper patterns as compared to frieze pattern. The different motions on which a wallpaper group depends can be seen as follows

Translations

A translation is the linear shift of some figure along the plane. In crystallographic group we have horizontal translations (t) and vertical translations (t').

Horizontal Reflection

Reflection along a horizontal line such that it preserves distance and symmetry is called horizontal reflection symmetry denoted by h.

Vertical Reflection

Reflection along a vertical line such that it preserves distance and symmetry is called vertical reflection symmetry denoted by v.

Rotations

Rotation symmetry is the non-trivial rotations around a particular point. It is denoted by r where $0^\circ < r < 360^\circ$. Thus crystallographic groups have more patterns than frieze groups

Glide Reflection

Glide reflection is the combination of a translation and a horizontal reflection. If we have a reflection on the same axis as a glide reflection, the it is called as the trivial one.

II. Why Only 5 Rotations In Crystallographic Groups

There are only 5 rotational symmetries possible in a wallpaper group which are $60°, 90°, 120°, 180°$ or no symmetry that is $360°$. To determine the reason behind only 5 symmetries, we shall look at some regular polygons whose internal angles divide 360. We only consider the angles dividing 360 as they are the only possible rotationally symmetric groups otherwise a group won't be formed. Thus we shall look at the internal angles of regular polygons represented by the following formula

$$\theta = \frac{n-2}{n}(180)$$

Now θ has to divide 360 and n has to be greater than 2. So we will start with the first value of $n = 3$ which gives us $60°$. Thus a rotational symmetry of $60°$ is possible. Similarly putting $n = 4,5,6, \ldots$ we will find the value of θ. Consider the following values for some value of n

$\boldsymbol{n}$	3	4	5	6	7	…
$\boldsymbol{angle}$	60	90	108	120	$128\frac{4}{7}$	…

From the above table, we will select those value of n, that divide 360 and such values of n are 3,4 and 6 only. Thus the corresponding values of θ are $60°, 90°, 120°, 180°$ and no rotational symmetry. Hence we get 5 rotational symmetries.

III. Construction Of Crystallographic Groups

Unlike frieze groups which have only seven patterns, crystallographic groups have 17 patterns. These groups are the symmetry groups of plane patterns whose subgroups of translations are isomorphic to $Z \oplus Z$. It is difficult to construct the symmetry of these groups unlike frieze groups thus we shall see geometrically.

No Rotational Symmetry

The first case is of the group having no rotational symmetry, so we will first see if they have a reflection axis. If the reflection axis is also not present, then we will check the glide reflection. If there is a glide reflection, then the group is called (pg). If the glide reflection is also not present, then the group is labelled as $(p1)$. If there exists a reflection axis then the groups is labelled (cm) if glide reflection is present otherwise (pm), that is with no glide reflections.

$180°$ Rotational Symmetry

Same as above, we will start by checking if there are any reflection axis. If not, then we will go to glide reflections. The group with glide reflections is called (pgg). Because of $180°$ rotational symmetry, the reflections will be perpendicular geometrically. Thus there will be two glide reflections (hence gg). If the group doesn't have any glide reflection then it will be called $(p2)$. Now we will check if the group has reflection axis. If there is only one reflection axis and a perpendicular glide reflection axis, then the group is called (pmg). If the group has two reflection axis perpendicular to each other with the rotational centre on the axis of reflection, then the group is called (pmm) otherwise (cmm).

$120°$ Rotational Symmetry

We will start again by checking if there are any reflection axis. With rotational symmetry we can visualize by imagining a triangle. In a triangle there are three possible reflection axis. If there are no reflections then glide reflections cannot exist also thus the only group left is the trivial group $(p3)$ with rotational symmetry only. If the reflection axis exists, then we have to check whether each of the rotational vertices are on the reflection axis or not. Thus we get $(p3m1)$ otherwise $(p31m)$.

$90°$ Rotational Symmetry

Again we will start by checking the reflection axis. If there doesn't exist any reflection axis then we are left with the trivial group having only rotational symmetry that is $(p4)$. If reflection axis exists, then we have two cases whether they exist in four directions or they exist in only two directions. The group with 4 reflection axis is labelled $(p4m)$ and the group with 2 reflection axis are called $(p4g)$ (2 reflection axis and 2 glide reflection).

$60°$ Rotational Symmetry

The high order of rotation in this case limits other symmetries therefore we only have two groups in this case – one is the trivial group with only rotational symmetries $(p6)$ and other is the group with reflections $(p6m)$ We can see the above mentioned groups in the following figure (1) :-

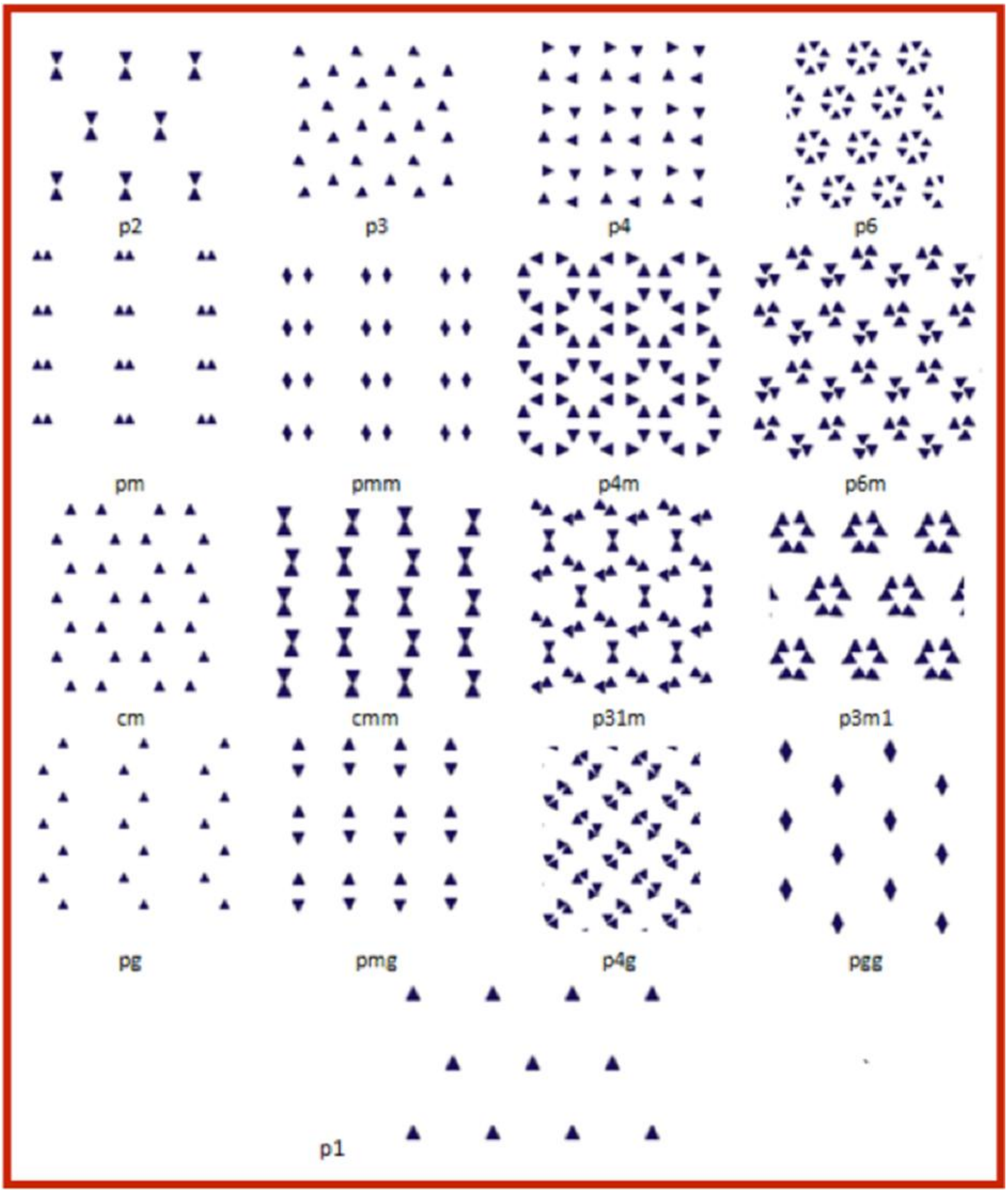
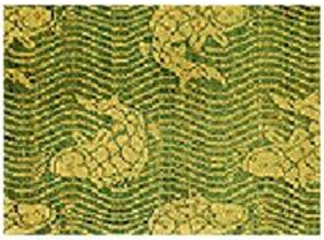


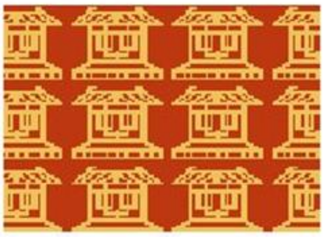
Figure (1)

IV. Use Of Crystallographic Groups In Art

Wallpaper groups are not only significant in mathematics but in art also. We can see them in different patterns drawn by many famous painters such as Makoto Nakamura and others. Many wallpaper designs can be created using the above symmetries hence the name wallpaper groups. Many artists create modern art using crystallographic symmetries. They are not only used on walls but also used to create decorative patterns on tiles. Some of the wallpaper patterns using crystallographic groups can be seen in figure (2)

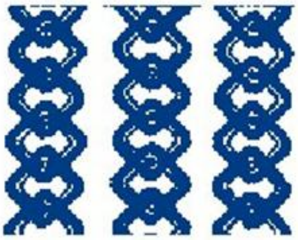

p1


p2


pm


pg

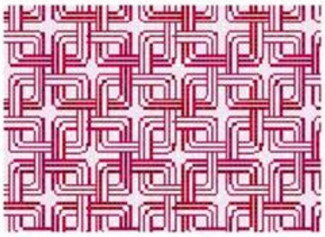

cm

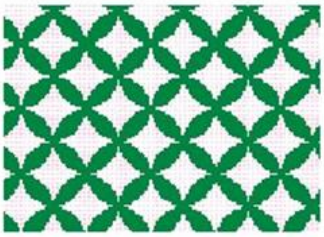

pmm

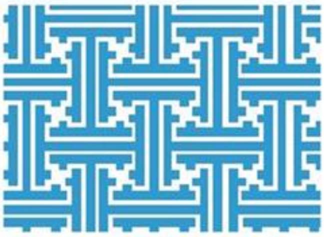

pmg


pgg

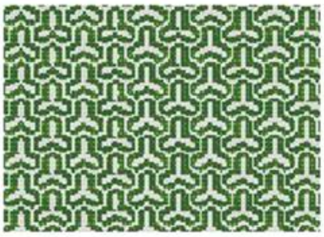

cmm


p4


p4m


p4g


p3


p31m


p3m1


p6


p6m

V. Conclusion

After looking at different wallpaper groups, we can clearly see some unique applications of their patterns. Once we understand their construction it's not hard to notice how vastly they are used in real life. This is the beauty of mathematics and how mathematics has been used even for decorative purposes since ancient times. We also saw that only 17 distinct patterns of wallpaper groups can exist and not more than that.

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